

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe
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PROBLEM SET 2 SOLUTIONS

PROBLEM 1: DISTANCE TO THE GALAXY (10 points)

This problem set is mainly an exercise in the use of comoving coordinates. Note that in the comoving coordinate system the trajectories of motion are simple: objects that move with the Hubble flow (*e.g.* galaxies, quasars, etc.) are stationary in these coordinates, and light rays move along straight lines with the time-varying speed $c/a(t)$. Thus, tracing the light ray from the galaxy to us, one finds:

$$\text{Coordinate velocity} = \frac{dx}{dt} = \frac{c}{a(t)} .$$

$$\text{Coordinate distance} = \ell_c = \int_{t_e}^{t_0} \frac{c}{a(t)} dt .$$

(Since the velocity is not constant, the distance traveled is given by the integral of v over time, rather than the product of v times the time interval.) Let $a(t) = bt^{2/3}$, where b is the arbitrary constant of proportionality. Then the coordinate distance, measured in notches, is given by

$$\ell_c = \frac{3c}{b} \left(t_0^{1/3} - t_e^{1/3} \right) .$$

The physical distance today is then found by multiplying the coordinate distance by the present value of the scale factor:

$$\ell_p(t_0) = a(t_0)\ell_c = bt_0^{2/3}\ell_c$$

$$= \boxed{3ct_0 \left[1 - \left(\frac{t_e}{t_0} \right)^{1/3} \right]} .$$

PROBLEM 2: TIME OF EMISSION (10 points)

From Lecture Notes 2, the redshift for light traveling through an expanding universe is related to the scale factor $a(t)$ through

$$1 + z = \frac{a(t_0)}{a(t_e)} .$$

(Note: one must **not** use the special relativity formula from Lecture Notes 1, because special relativity does not hold here. Special relativity does not include the effects of gravity, while the effects of gravity are significant in cosmology unless the mass density is very small compared to the critical density.) So

$$1 + z = \frac{bt_0^{2/3}}{bt_e^{2/3}} \implies \boxed{z = \left(\frac{t_0}{t_e}\right)^{2/3} - 1 .}$$

We can rearrange this expression and solve for t_e/t_0 :

$$\boxed{\frac{t_e}{t_0} = (1 + z)^{-3/2} .}$$

Applying this to the $z = 11.1$ record-holding galaxy discussed in the problem,

$$\boxed{\frac{t_e}{t_0} = (1 + 11.1)^{-3/2} = 0.0238 .}$$

So, roughly speaking, the universe was less than 1/40 of its present age when the light that we are seeing left the galaxy.

You were not asked to estimate the actual value of t_e , the time of emission, but it is very interesting to do that. If we use the current estimate of the age of the universe, $t_0 \approx 13.8$ billion years, then the universe was only $t_e \approx 328$ million years old when the light was emitted. According to more accurate calculations, which take into account that the universe has other constituents besides nonrelativistic matter, the galaxy emitted the light at about 410 million years after the big bang. Later in the term we will learn how to calculate this number more accurately. In any case, we can already conclude that the oldest galaxies formed amazingly quickly after the birth of the universe.

PROBLEM 3: DISTANCE IN TERMS OF REDSHIFT z (10 points)

Starting from the answer to Problem 1, the physical distance can be expressed in terms of t_e and t_0 :

$$\ell_p(t_0) = 3ct_0 \left[1 - \left(\frac{t_e}{t_0}\right)^{1/3} \right] .$$

We know from Problem 2 that the ratio t_e/t_0 is related to the redshift,

$$\left(\frac{t_e}{t_0}\right)^{1/3} = \frac{1}{\sqrt{1+z}} .$$

Since the problem asks for the answer in terms of H_0 , rather than t_0 , we need the relation between these two:

$$H = \frac{\dot{a}}{a} = \frac{\frac{2}{3}bt^{-1/3}}{bt^{2/3}} = \frac{2}{3t} ,$$

so at the present time,

$$t_0 = \frac{2}{3}H_0^{-1} .$$

Using these relations for $(t_e/t_0)^{1/3}$ and for t_0 in the expression for the physical distance $\ell_p(t_0)$, we find

$$\ell_p(t_0) = 2cH_0^{-1} \left[1 - \frac{1}{\sqrt{1+z}} \right] .$$

Notice that this now gives us the physical distance to the galaxy in terms of quantities we can go out and measure.

Numerically, if we take $H_0 \approx 67 \text{ km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}$, we get

$$\begin{aligned} \ell_p(t_0) &= 2 \times \left(2.998 \times 10^8 \frac{\text{m}}{\text{s}} \right) \times \frac{1 \text{ Mpc}}{67000 \left(\frac{\text{m}}{\text{s}} \right)} \left[1 - \frac{1}{\sqrt{1+11.1}} \right] \\ &= 8949 \text{ Mpc} \times .7125 \\ &= \boxed{6376 \text{ Mpc}} . \end{aligned}$$

The problem asked you to find the answer also in light-years, so we can use the relation

$$1 \text{ pc} = 3.2616 \text{ lt}\cdot\text{yr} ,$$

which assumes the International Astronomical Union (IAU) definition of a year (1 Julian year = 365.25 days). This gives the distance as $6376 \text{ Mpc} \times (3.2616 \times 10^6 \text{ lt}\cdot\text{yr}/\text{Mpc}) = \boxed{2.080 \times 10^{10} \text{ lt}\cdot\text{yr}} .$

PROBLEM 4: SPEED OF RECESSION (10 points)

The distant galaxy will remain at a **constant** coordinate distance ℓ_c , so its physical distance at any time t is given by

$$\ell_p(t) = a(t)\ell_c .$$

The rate of change of the physical distance is then given by Hubble's law:

$$v_p(t) \equiv \frac{d\ell_p}{dt} = \dot{a}\ell_c = H(t)\ell_p(t) ,$$

where

$$H(t) = \frac{\dot{a}}{a}$$

is the Hubble expansion rate. So

$$v_p(t_0) = H_0\ell_p(t_0) = 2c \left[1 - \frac{1}{\sqrt{1+z}} \right] ,$$

where in the last step we used the results of Problem 3. For $z = 11.1$, this gives

$$v_p = 1.425 c .$$

COMMENT:

In special relativity, it is impossible to accelerate matter to the speed of light or beyond. However, in the context of general relativity (which is what we are using in our description of the universe), we have just seen that “faster-than-light” travel is in some sense possible. However, the sense in which “faster-than-light” travel is possible is very restricted. Note that even in general relativity:

- a) In a race between a light pulse and a piece of matter, the light pulse will **always** win.
- b) The speed of light, when measured locally with clock and meter stick by any observer, will always have its usual value of $c = 2.99792458 \times 10^8$ m/s. By “locally”, I mean within a region that is small enough so that the curvature of space is not noticeable.

“Faster-than-light” travel occurs only in the sense that the space which separates us from the distant galaxy can **stretch**, as described by the growth of the scale factor $a(t)$. From such stretching, the distance from us to the galaxy can grow at a rate which exceeds the speed of light.

Incidentally, the Sloan Digital Sky Survey wrestled with the question of how to explain faster-than-light velocities to the public when they announced the discovery of a $z = 5.82$ quasar in a press release on April 13, 2000:

<http://classic.sdss.org/news/releases/20000413.qso.html> .

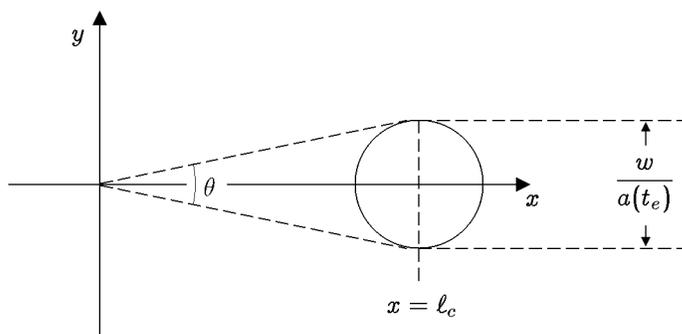
They published on the web a companion document,

<http://classic.sdss.org/news/releases/20000413.qso.q.html> ,

titled “How Can An Object We See Today be 27 Billion Light Years Away If the Universe is only 14 Billion Years Old?,” by Michael Turner and Craig Wiegert. They claimed that the speed of the quasar is 1.8 times the speed of light, somewhat faster than we found in this problem, even though the redshift was lower than the one we used. They were using a more complicated model of the universe than our $a(t) \propto t^{2/3}$ calculation, however, so it is not surprising that they got a different answer. Later in the course we will learn enough to repeat the calculation with the same model they used, so you will be able to check if they got it right.

PROBLEM 5: APPARENT ANGULAR SIZES (20 points)

Picture the universe in comoving coordinates, with the galaxy shown as it appeared at the time of emission. Choosing our own location as the center of coordinates, and choosing the x -axis to point in the direction of the galaxy, the picture looks like:



The coordinate diameter is $w/a(t_e)$, and the coordinate distance is ℓ_c . The angle subtended, in the small angle approximation, is then

$$\theta = \frac{w}{a(t_e)\ell_c} ,$$

where I have used $\theta \ll 1$. (Notice that we need to use the coordinate diameter, in order to compare with the coordinate distance. The conversion from the physical diameter to the coordinate diameter must be done *at the time of emission*, because we are interested in the light rays which leave the galaxy at the time of emission. Any evolution of the galaxy after t_e does not affect the light rays and is therefore invisible to us.) The light pulses travel along straight lines of the comoving coordinate system, so the light pulses from the edges of the galaxy arrive at the earth today along the dashed lines shown, separated by an angle θ . Note, by the way, that angles are not affected by the scale factor. For example, a map of the U.S. shows that Colorado is a rectangle, with borders that make a 90° angle at each corner. This feature appears on maps drawn at any scale, and one is confident that the actual state borders, if marked on the surface of the earth, would make a 90° angle at the corners.

We use the expression from Problem 1 for the coordinate distance ℓ_c ,

$$a(t_e)\ell_c = \left(bt_e^{2/3}\right) \left[\frac{3c}{b} \left(t_0^{1/3} - t_e^{1/3}\right)\right] ,$$

which can be rearranged to read

$$a(t_e)\ell_c = 3ct_0 \left(\frac{t_e}{t_0}\right) \left[\left(\frac{t_0}{t_e}\right)^{1/3} - 1\right] .$$

Using the expression $t_0 = \frac{2}{3}H_0^{-1}$ from Problem 3, and the expression for t_e/t_0 from Problem 2, this becomes

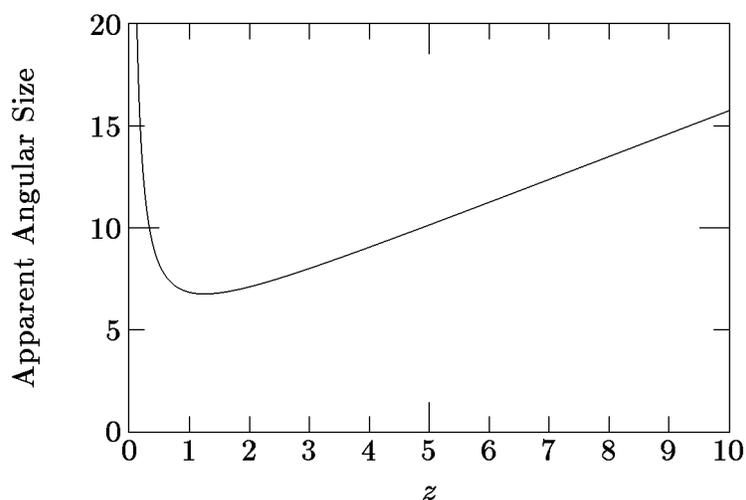
$$a(t_e)\ell_c = 2cH_0^{-1} \frac{1}{(1+z)^{3/2}} \left[\sqrt{1+z} - 1\right] .$$

So

$$\theta = \frac{H_0 w}{2c} \frac{(1+z)^{3/2}}{\sqrt{1+z} - 1} .$$

For a fixed physical diameter w , this function decreases as a function of redshift z until $z = 1.25$, and then it starts to increase! The usual rule that objects look smaller when they are further away does not apply— objects seen at very large redshifts are seen with light that left the object very early in the history of the universe, when the object was not that far away.

A plot of apparent angular size versus redshift looks like:



where θ is plotted in units of $H_0 w / 2c$. If one applies this relation to a galaxy with $w \approx 100,000$ light-years (the size of the disk of the Milky Way), again taking $H_0 \approx 67 \text{ km-s}^{-1}\text{-Mpc}^{-1}$, each unit on the vertical axis is equal to $H_0 w / 2c \approx 3.4 \times 10^{-6}$ radians ≈ 0.71 arcseconds. Thus the numbers on the vertical axis give roughly the apparent angular size of a typical galaxy measured in arcseconds (where 1 degree = 60 arcminutes = 3600 arcseconds).

The problem asked us to compare this with the apparent angular size of a circle of diameter w in an ordinary static, Euclidean space, at a distance equal to the present value of the physical distance to the galaxy. Using the distance found in Problem 3, the angular size in the static space would be

$$\theta_{\text{Static}} = \frac{w}{\ell_p(t_0)} = \frac{H_0 w}{2c} \frac{\sqrt{1+z}}{\sqrt{1+z}-1}.$$

Then

$$\frac{\theta}{\theta_{\text{Static}}} = 1 + z.$$

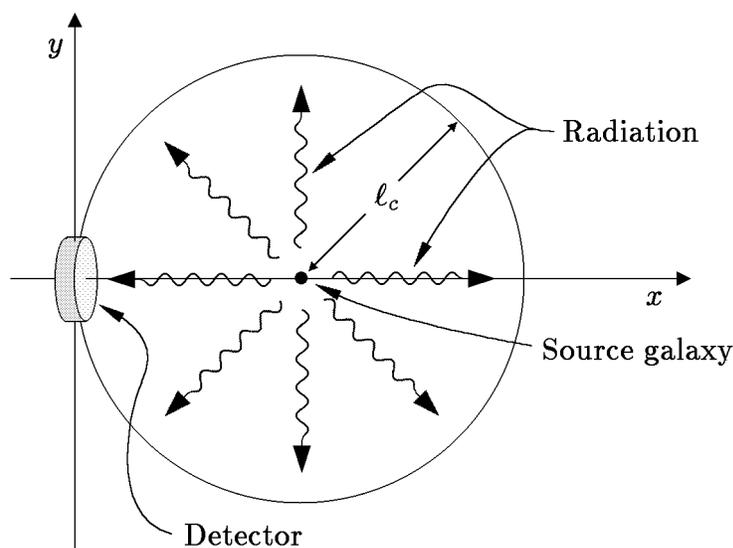
Thus, the image appears larger than it would if the universe were static and the galaxy were located at its present physical distance. On the other hand, one could also compare with the apparent angular size of a circle of diameter w in a static Euclidean space, at a distance equal to the physical distance to the galaxy **at the time of emission**. Then

$$\theta_{\text{Static, time of emission}} = \frac{w}{\ell_p(t_e)} = \frac{w}{a(t_e)\ell_c},$$

which is exactly the same as the answer that we obtained in the expanding universe calculation. Thus, the increase in the apparent angular size by the factor $1 + z$ is really just the effect of seeing the galaxy as it looked when the light was emitted, when the galaxy was closer than it is now.

PROBLEM 6: RECEIVED RADIATION FLUX (15 points)

Again we will view the problem in comoving coordinates, with our location at the origin. The distant galaxy is again on the x -axis. Draw a sphere centered on the galaxy, with radius equal to the galaxy coordinate distance ℓ_c . Also draw a detector at the origin (i.e., on the earth) facing the galaxy, with physical area A (measured at the present time).



Notice that energy from the galaxy will radiate uniformly on the sphere. The detector has a physical area A , so in the comoving coordinate picture its area in square notches would be $A/a(t_0)^2$. The detector therefore occupies a fraction of the sphere given by

$$\frac{[A/a(t_0)^2]}{4\pi\ell_c^2} = \frac{A}{4\pi\ell_p(t_0)^2},$$

so this fraction of the emitted photons will strike the detector.

Next consider the rate of arrival of the photons at the sphere. In lecture we figured out that if a periodic wave is emitted at time t_e and observed at time t_o , then the rate of arrival of the wave crests will be slower than the rate of emission by a redshift factor $1+z = a(t_o)/a(t_e)$. The same argument will apply to the rate of arrival of photons, so the rate of photon arrival at the sphere will be slower than the rate of emission by the factor $1+z$, reducing the energy flux by this factor. In addition, each photon is redshifted in frequency by $1+z$. Since the energy of each photon is proportional to its frequency, the energy flux is reduced by an additional factor of $1+z$. Thus, the rate at which energy reaches the detector is

$$\text{Power hitting detector} = \frac{A}{4\pi\ell_p(t_0)^2} \frac{P}{(1+z)^2}.$$

Once again, using $\ell_p(t_0)$ from Problem 3, we find

$$J = \frac{\text{Power hitting detector}}{A} = \frac{H_0^2 P}{16\pi c^2} \frac{1}{(1+z)(\sqrt{1+z}-1)^2} .$$

Here we have described the radiation in terms of photons, which gives a simple description. The same answer can also be obtained by using a classical description of the electromagnetic waves, but it is much harder. One would need to first understand how Maxwell's equations are modified by the expanding space, which would require elements of general relativity, and then one would have to solve these modified equations. We don't usually think of quantum mechanics as a method of simplifying classical calculations, but in this case that is what happens.