MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 20, 2022
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# PROBLEM SET 4 SOLUTIONS

## PROBLEM 1: PHOTON TRAJECTORIES AND HORIZONS IN A FLAT UNIVERSE WITH $a(t) = bt^{1/2}$ *(20 points)*

(a) The defining equation for $a(t)$ is

$$\ell_{\text{phys}} = a(t)\ell_c \ ,$$

so

$$[a(t)] = \frac{[\ell_{\text{phys}}]}{[\ell_c]} = \boxed{\frac{\text{meter}}{\text{notch}}} \ ,$$

and

$$[b] = \frac{[a(t)]}{[t^{1/2}]} = \boxed{\frac{\text{meter}}{\text{notch} \cdot \text{second}^{1/2}}} \ .$$

(b) *(2 points)*

$$H(t) = \frac{\dot{a}}{a} = \frac{\frac{1}{2}bt^{-1/2}}{bt^{1/2}} = \boxed{\frac{1}{2t}} \ .$$

(c) According to Eq. (4.7) of Lecture Notes 4, the physical horizon distance is given by

$$\begin{aligned}\ell_{p,\text{hor}} &= a(t)\int_0^t \frac{c}{a(t')}\,\mathrm{d}t' \\ &= bt^{1/2}\int_0^t \frac{c}{bt'^{1/2}}\,\mathrm{d}t' = ct^{1/2}\left[2t'^{1/2}\right]_0^t \\ &= \boxed{2ct} \ .\end{aligned}$$

(d) The physical distance between $A$ and $B$ at any time $t$ is given by $\ell_{p,AB}(t) = bt^{1/2}\ell_c$, so its rate of change is

$$\frac{\mathrm{d}\ell_{p,AB}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[bt^{1/2}\ell_c\right] = \frac{b\ell_c}{2t^{1/2}} \ ,$$

and therefore, at time $t_A$,

$$\boxed{\left.\frac{\mathrm{d}\ell_{p,AB}(t)}{\mathrm{d}t}\right|_{t=t_A} = \frac{b\ell_c}{2t_A^{1/2}} \ .}$$

$\mathrm{d}\ell_{p,AB}(t)/\mathrm{d}t$ is positive, so $\boxed{\text{the physical distance between } A \text{ and } B \text{ is increasing}}$ with time. As $t_A \to 0$,

$$\boxed{\lim_{t_A\to 0} \left.\frac{\mathrm{d}\ell_{p,AB}(t)}{\mathrm{d}t}\right|_{t=t_A} = \infty \ .}$$

So, while the physical distance between $A$ and $B$ approaches zero as $t_A \to 0$, the relative recession velocity approaches infinity. So even though $A$ and $B$ are very close at early times, it is not easy for a photon to travel from one to the other.

(e) The coordinate speed of light is $c/a(t)$, so if a photon travels a coordinate distance $\ell_c$ during the interval from $t_A$ to $t_B$, then

$$\int_{t_A}^{t_B} \frac{c}{a(t)}\mathrm{d}t = \ell_c \ .$$

Evaluating the integral,

$$\int_{t_A}^{t_B} \frac{c}{a(t)}\mathrm{d}t = \int_{t_A}^{t_B} \frac{c}{bt^{1/2}}\mathrm{d}t = \frac{2c}{b}\left(t_B^{1/2} - t_A^{1/2}\right) \ ,$$

so

$$\boxed{t_B = \left(t_A^{1/2} + \frac{b\ell_c}{2c}\right)^2 \ .}$$

Clearly as $t_A \to 0$,

$$\boxed{\lim_{t_A\to 0} t_B = \left(\frac{b\ell_c}{2c}\right)^2 \ ,}$$

which is a $\boxed{\text{nonzero finite number.}}$

You were not asked to do this, but we can check that our answer is consistent with what we said earlier about the horizon distance. If the photon leaves $A$ at time zero, we expect it to arrive at $B$ precisely when the physical distance between $A$ and $B$ is equal to the horizon distance. If $t_B$ is given by the above formula, we see that the physical distance between the two pieces of matter at time $t_B$ is given by

$$\ell_{p,AB}(t_B) = bt_B^{1/2}\ell_c = \left[bt_B^{-1/2}\ell_c\right] t_B = \left[b\left(\frac{2c}{b\ell_c}\right)\ell_c\right] t_B = 2ct_B \ ,$$

which is just what we expect — the horizon distance.

(f) To be explicit, we can assume that $A$ is at the origin of the comoving coordinate system, and $B$ is at $(\ell_c, 0, 0)$. Then, at time $t$, the photon will be at $(x_\gamma(t), 0, 0)$, where

$$\begin{aligned} x_\gamma(t) &= \int_{t_A}^{t} \frac{c}{a(t')}\mathrm{d}t' \\ &= \int_{t_A}^{t} \frac{c}{bt'^{1/2}}\mathrm{d}t' = \frac{2c}{b}(t^{1/2} - t_A^{1/2}) \ . \end{aligned}$$

The coordinate separation between the photon and $B$ is then $\ell_c - x_\gamma(t)$, so the physical separation is given by

$$\ell_{p,\gamma B}(t) = a(t)[\ell_c - x_\gamma(t)] = \boxed{bt^{1/2}\ell_c - 2c\left(t - t^{1/2}t_A^{1/2}\right) \ .}$$

You were not asked to do so, but it is a useful check to make sure that this expression has the expected values at the two endpoints, $t_A$ and $t_B$:

$$\begin{aligned} \ell_{p,\gamma B}(t_A) &= bt_A^{1/2}\ell_c = \ell_{p,AB}(t_A) \ , \\ \ell_{p,\gamma B}(t_B) &= b\left(t_A^{1/2}\ell_c + \frac{b\ell_c}{2c}\right) - 2c\left[\left(t_A^{1/2} + \frac{b\ell_c}{2c}\right)^2 - t_A^{1/2}\left(t_A^{1/2} + \frac{b\ell_c}{2c}\right)\right] \\ &= \left(t_A^{1/2} + \frac{b\ell_c}{2c}\right)\left\{b\ell_c - 2c\left[\left(t_A^{1/2} + \frac{b\ell_c}{2c}\right) - t_A^{1/2}\right]\right\} = 0 \ . \end{aligned}$$

(g) Starting with the answer from part (f),

$$\frac{\mathrm{d}\ell_{p,\gamma B}(t)}{\mathrm{d}t} = \frac{b\ell_c}{2t^{1/2}} - 2c\left(1 - \frac{t_A^{1/2}}{2t^{1/2}}\right) \ ,$$

so

$$\boxed{\left.\frac{\mathrm{d}\ell_{p,\gamma B}(t)}{\mathrm{d}t}\right|_{t=t_A} = \frac{b\ell_c}{2t_A^{1/2}} - c \ .}$$

(h) Given the answer above, we see that $\mathrm{d}\ell_{p,\gamma B}(t)/\mathrm{d}t$ will vanish when $t_A = t_A^0$, where

$$\boxed{t_A^0 = \left(\frac{b\ell_c}{2c}\right)^2 .}$$

$\boxed{\text{If } t_A < t_A^0, \text{ then } \mathrm{d}\ell_{p,\gamma B}(t)/\mathrm{d}t > 0,}$ which means that if the photon is emitted early, $\boxed{\text{the physical distance between it and } B \text{ is initially increasing.}}$ That is, it is initially getting further from $B$, rather than approaching it. As $t_A \to 0$,

$$\boxed{\lim_{t_A \to 0}\left[\left.\frac{\mathrm{d}\ell_{p,\gamma B}(t)}{\mathrm{d}t}\right|_{t=t_A}\right] = \infty .}$$

So, if the photon is emitted very early, its initial recession velocity relative to $B$ can be arbitrarily large.

## PROBLEM 2: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(35 points)*

(a) Using the chain rule, the standard formula for the Hubble expansion rate can be rewritten as

$$H(\theta) = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} .$$

The parametric equations for $a$ and $t$ for an open, matter-dominated universe are given by

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) .$$

Recall that the hyperbolic trigonometric functions are defined by

$$\sinh\theta \equiv \frac{e^{\theta} - e^{-\theta}}{2} ,$$

$$\cosh\theta \equiv \frac{e^{\theta} + e^{-\theta}}{2} ,$$

and they are differentiated as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sinh\theta = \cosh\theta ,$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\cosh\theta = \sinh\theta .$$

So, differentiating the parametric equations,

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = \alpha\sqrt{k}\sinh\theta \ ,$$
$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\alpha}{c}(\cosh\theta - 1) = \frac{1}{\mathrm{d}\theta/\mathrm{d}t} \ .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{\kappa}\alpha(\cosh\theta - 1)}\right]\left[\alpha\sqrt{\kappa}\sinh\theta\right]\left[\frac{c}{\alpha(\cosh\theta - 1)}\right]$$
$$= \boxed{\frac{c\sinh\theta}{\alpha(\cosh\theta - 1)^2} \ .}$$

(b) This problem can be attacked by at least three different methods. While you were expected to use only one, we will show all three.

(i) One way to find $\rho$ is to use

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ .$$

This is usually the safest method to find $\rho$ for a cosmological model, since the above equation is one of the general Friedmann equations. The equation requires that the universe be homogeneous and isotropic, but it is valid for any form of matter. By contrast, the two other methods that will be shown below are valid only for "matter-dominated" universes (i.e., universes that are dominated by nonrelativistic matter, for which the pressure is always negligible). One can rewrite this equation as

$$\frac{8\pi}{3}G\rho = H^2 + \frac{kc^2}{a^2} \ .$$

Recalling that we described open universes by using $\kappa \equiv -k$, this can be rewritten as

$$\frac{8\pi}{3}G\rho = H^2 - \frac{\kappa c^2}{a^2} \ .$$

Replacing $H$ by the answer in part (a) and $a$ by its parametric equation, one finds

$$\frac{8\pi}{3}G\rho = \frac{c^2\sinh^2\theta}{\alpha^2(\cosh\theta - 1)^4} - \frac{\kappa c^2}{\alpha^2\kappa(\cosh\theta - 1)^2}$$
$$= \frac{c^2}{\alpha^2(\cosh\theta - 1)^4}\left[\sinh^2\theta - (\cosh\theta - 1)^2\right] \ .$$

Now make use of the hypertrigonometric identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

to simplify:

$$\begin{aligned} \sinh^2\theta - (\cosh\theta - 1)^2 &= \sinh^2\theta - \cosh^2\theta + 2\cosh\theta - 1 \\ &= 2(\cosh\theta - 1)\ , \end{aligned}$$

so

$$\frac{8\pi}{3} G\rho = \frac{2c^2}{\alpha^2(\cosh\theta - 1)^3}\ .$$

Dividing both sides of the equation by $(8\pi/3)G$, one finds

$$\boxed{\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3}\ .}$$

(ii) Use the definition of $\alpha$,

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2}\ ,$$

from Eq. (4.17) of Lecture Notes 4, with Eq. (4.39),

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{\kappa}}\ .$$

One can then solve for $\rho$, finding

$$\rho = \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{Ga^3}\ .$$

By substituting for $a(\theta)$ by using the parametric equation, one finds the final result:

$$\begin{aligned} \rho &= \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{G}\frac{1}{\alpha^3\kappa^{3/2}(\cosh\theta - 1)^3} \\ &= \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3}\ . \end{aligned}$$

(iii) $\rho$ can also be found from $\ddot{a} = -(4\pi/3)G\rho a$, as long as we know that the universe is matter-dominated. (Be careful, however, about applying this formula in other situations: if the pressure cannot be neglected, then this equation has to be modified.) To evaluate $\ddot{a}$, again use the chain rule. Starting with $\dot{a}$,

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \alpha\sqrt{\kappa}\sinh\theta\frac{c}{\alpha(\cosh\theta - 1)} = \frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1} \ .$$

Then

$$\begin{aligned}\ddot{a} = \frac{\mathrm{d}\dot{a}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\right]\frac{c}{\alpha(\cosh\theta - 1)} \\ &= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)}\left[\frac{\cosh\theta}{\cosh\theta - 1} - \frac{\sinh^2\theta}{(\cosh\theta - 1)^2}\right] \\ &= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}\left[\cosh\theta(\cosh\theta - 1) - \sinh^2\theta\right] \\ &= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}(1 - \cosh\theta) = -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} \ .\end{aligned}$$

So

$$\ddot{a} = -\frac{4\pi}{3}G\rho a \quad \Longrightarrow \quad -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} = -\frac{4\pi}{3}G\rho\alpha\sqrt{\kappa}(\cosh\theta - 1) \ ,$$

and

$$\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3} \ .$$

(c) The critical mass density satisfies the cosmological evolution equations for $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c \ .$$

Then

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} \ .$$

Now replace $H$ by the answer to part (a), and $\rho$ by the answer to part (b):

$$\begin{aligned}\Omega &= \frac{8\pi G}{3}\left[\frac{3}{4\pi}\frac{c^2}{G\alpha^2(\cosh\theta - 1)^3}\right]\left[\frac{\alpha^2(\cosh\theta - 1)^4}{c^2\sinh^2\theta}\right] \\ &= 2\frac{\cosh\theta - 1}{\sinh^2\theta} = 2\frac{\cosh\theta - 1}{\cosh^2\theta - 1} \\ &= 2\frac{\cosh\theta - 1}{(\cosh\theta + 1)(\cosh\theta - 1)} = \boxed{\frac{2}{\cosh\theta + 1}} \ .\end{aligned}$$

The answer can be written even more compactly, if one wishes, by using a further hypertrigonometric identity:

$$\Omega = \frac{2}{\cosh\theta + 1} = \frac{1}{\cosh^2 \frac{1}{2}\theta} = \mathrm{sech}^2 \frac{1}{2}\theta \ .$$

(d) The basic formula that determines the physical value of the horizon distance is given by Eq. (4.7) of the lecture notes:

$$\ell_{p,\mathrm{horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt' \ .$$

The complication here is that $a$ is given as a function of $\theta$, rather than $t$. The problem is handled, however, by a simple change of integration variables. One can change the integral over $t'$ to an integral over $\theta'$, provided that one replaces

$$\mathrm{d}t' \to \frac{dt'}{\mathrm{d}\theta'}\mathrm{d}\theta' = \frac{\alpha}{c}(\cosh\theta' - 1)\mathrm{d}\theta' \ .$$

One must also re-express the limits of integration in terms of $\theta$. So

$$\begin{aligned}
\ell_{p,\mathrm{horizon}}(t) &= a\big(\theta(t)\big) \int_0^{\theta(t)} \frac{c}{a(\theta')} \frac{dt'}{\mathrm{d}\theta'} \mathrm{d}\theta' \\
&= \alpha\sqrt{\kappa}\big(\cosh\theta(t) - 1\big) \int_0^{\theta(t)} \frac{c}{\alpha\sqrt{\kappa}(\cosh\theta' - 1)} \frac{\alpha}{c}(\cosh\theta' - 1)\,\mathrm{d}\theta' \ . \\
&= \alpha\big(\cosh\theta(t) - 1\big) \int_0^{\theta(t)} \mathrm{d}\theta' = \boxed{\alpha\,\theta(t)\,\big(\cosh\theta(t) - 1\big)\ .}
\end{aligned}$$

(e) The key to this problem is the use of power series expansions. When this problem appeared as a quiz problem in 1992, I was rather surprised to find that many of the students seemed very inexperienced in this technique. It is a very useful method of approximation, so I strongly urge you to learn it if you don't know it already. In general, any sufficiently smooth function $f(x)$ can be expanded about the point $x_0$ by the series

$$\begin{aligned}
f(x) = f(x_0) &+ \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 \\
&+ \frac{1}{3!}f'''(x_0)(x - x_0)^3 + \dots \ ,
\end{aligned}$$

where the prime is used to denote a derivative. In particular, the exponential, sinh, and cosh functions can be expanded about $\theta = 0$ by the formulas

$$\begin{aligned} e^\theta &= 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots \\ \sinh\theta &= \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^5}{7!} \cdots \\ \cosh\theta &= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots \ . \end{aligned}$$

For this problem, we expand the parametric equations for $a(\theta)$ and $t(\theta)$, keeping the first nonvanishing term in the power series expansions:

$$\begin{aligned} t &= \frac{\alpha}{c}(\sinh\theta - \theta) = \frac{\alpha}{c}\left(\frac{\theta^3}{3!} + \ldots\right) \\ a &= \alpha\sqrt{\kappa}(\cosh\theta - 1) = \alpha\sqrt{\kappa}\left(\frac{\theta^2}{2!} + \ldots\right) \ . \end{aligned}$$

The first expression can be solved for $\theta$, giving

$$\theta \approx \left(\frac{6ct}{\alpha}\right)^{1/3} ,$$

which can be substituted into the second expression to give

$$\boxed{a \approx \frac{1}{2}\alpha\sqrt{\kappa}\left(\frac{6ct}{\alpha}\right)^{2/3} .}$$

The power series expansions for the sinh and cosh are valid whenever the terms left out are much smaller than the last term kept, which happens when $\theta \ll 1$. Given the above relation between $\theta$ and $t$, this condition is equivalent to

$$t \ll \frac{\alpha}{6c} .$$

Thus,

$$\boxed{t^* \approx \frac{\alpha}{6c}, \text{ or } t^* \approx \frac{\alpha}{c} .}$$

Since there is no precise meaning to the statement that an approximation is valid, there is no precise value for $t^*$.

(f) From part (c), the expression for $\Omega$ is given by

$$\Omega = \frac{2}{\cosh\theta + 1} \ .$$

So,

$$1 - \Omega = 1 - \frac{2}{\cosh\theta + 1} = \frac{\cosh\theta - 1}{\cosh\theta + 1} \ .$$

Expanding numerator and denominator in power series,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots}{2 + \frac{\theta^2}{2!} + \ldots} \ .$$

Keeping only the leading terms,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2}}{2} = \frac{1}{4}\theta^2 \ ,$$

so

$$\boxed{1 - \Omega \approx \frac{1}{4}\left(\frac{6ct}{\alpha}\right)^{2/3} \ .}$$

This result shows that the deviation of $\Omega$ from 1 is amplified with time. This fact leads to a conundrum called the "flatness problem", which will be discussed later in the course.

A common mistake (very minor) was to keep extra terms, especially in the denominator. Keeping extra terms allows a higher degree of accuracy, so there is nothing wrong with it. However, one should always be sure to keep ***all*** terms of a given order, since keeping only a subset of terms may or may not increase the accuracy. In this case, an extra term in the denominator can be rewritten as a term in the numerator:

$$\frac{\frac{\theta^2}{2!}}{2 + \frac{\theta^2}{2!}} = \frac{1}{4}\frac{\theta^2}{1 + \frac{\theta^2}{4}} = \frac{1}{4}\theta^2\left(1 - \frac{\theta^2}{4} + \ldots\right)$$
$$= \frac{1}{4}\theta^2 - \frac{1}{16}\theta^4 + \ldots \ ,$$

where I used the expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots \ .$$

Thus, the extra term in the denominator is equivalent to a term in the numerator of order $\theta^4$, but other terms proportional to $\theta^4$ have been dropped. So, it is not worthwhile to keep the 2nd term in the expansion of the denominator.

## PROBLEM 3: THE CRUNCH OF A CLOSED, MATTER-DOMINATED UNIVERSE *(25 points)**

Dr. Niwde measures the two quantities $\Omega_0$ (in the range of $(1, \infty)$) and $H_0 < 0$. All of the quantities of interest (the time until the end of the universe $t_{\rm left}$, the minimum $z$, and the lookback time $t_{\rm lb,bluest}$) must all be stated in terms of the two physical observables $\Omega_0$ and $H_0$. The parametric form of the evolution of the closed universe is parameterized by the development angle $\theta$, which needs to be determined from the two physical observables, and the constant $\alpha$ which is a measure of the mass density of the universe. From class (or Eq. (4.33) in the Lecture Notes), we found

$$\alpha = \frac{c}{2|H|}\frac{\Omega}{(\Omega - 1)^{3/2}} \ . \tag{P3.1}$$

The value is a constant over the course of the universe, so it can be evaluated at any time (except $\theta = \pi$); therefore insert the values of $H_0$ and $\Omega_0$. We also have the relation (Eq. (4.35))

$$\cos\theta = \frac{2 - \Omega}{\Omega} \ . \tag{P3.2}$$

This needs to be solved for $\theta_0$, with the right-hand side evaluated for $\Omega = \Omega_0$. But the function $\cos\theta$ is not one to one, so the inverse is not unique. We could write

$$\theta_0 = \arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) \ , \tag{P3.3}$$

adding the words that $\theta_0$ is to be chosen in the interval $\theta \in [\pi, 2\pi]$. Such an answer is completely correct, but it is hard to use, since calculators are not capable of responding to such verbal instructions. Calculators normally return the "principal branch" of the $\arccos(x)$ function, which maps $x \in [-1, 1]$ to $\theta \in [0, \pi]$. (For the $\arcsin(x)$ function, the principal branch is conventionally taken to map $x \in [-1, 1]$ to $\theta \in [-\pi/2, \pi/2]$.) Note that $\cos\theta$ behaves monotonically during the contracting phase, as $\theta$ varies from $\pi$ to $2\pi$, while $\sin\theta$ varies from 0 to -1 and then back to 0. Thus $\theta$ is determined uniquely by $\cos\theta$ during the contracting phase, while for each value of $\sin\theta$ there are two values of $\theta$, which must be distinguished by an additional condition. To express $\theta_0$ in terms of the principal branch of $\arccos(x)$, note that $\cos\theta = \cos(2\pi - \theta)$. Using this, we can write

$$\theta_0 = 2\pi - \arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) \ , \tag{P3.4}$$

where $\arccos(x)$ is evaluated using the principal branch. That is, $\theta_0$ defined by Eq. (P3.4) satisfies Eq. (P3.2), and it lies in the range of $\pi$ to $2\pi$. For values of $\sin\theta_0$, one uses the

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$*$ Solution written by Leo Stein and Alan Guth.

identity $\sin\theta = \pm\sqrt{1-\cos^2\theta}$ (Eq. (4.37)). Since one knows that $\theta_0 \in [\pi, 2\pi]$, and $\sin\theta$ is negative on this interval, one takes the negative root:

$$\sin\theta_0 = -\frac{2\sqrt{\Omega_0 - 1}}{\Omega_0} \ . \qquad \text{(P3.5)}$$

Thus, the value of $t_0$, when Dr. Niwde makes his measurements, is given by

$$\begin{aligned} t_0 &= \frac{\alpha}{c}(\theta_0 - \sin\theta_0) \\ &= \frac{1}{2|H|}\frac{\Omega}{(\Omega-1)^{3/2}}\left[2\pi - \arccos\left(\frac{2-\Omega_0}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] \ . \end{aligned} \qquad \text{(P3.6)}$$

One is now ready to find $t_{\text{left}} = t_{\text{Crunch}} - t_0$, using $ct_{\text{Crunch}} = 2\pi\alpha$. Evaluating this, one finds

$$\boxed{t_{\text{left}} = \frac{\Omega_0}{2|H_0|(\Omega_0-1)^{3/2}}\left[\arccos\left(\frac{2-\Omega_0}{\Omega_0}\right) - \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] \ .} \qquad \text{(P3.7)}$$

(Alternatively, one could have taken $t_0$ directly from Eq. (4.38) of Lecture Notes 4, using the choices described in the table following the equation. Rewriting Eq. (4.38) explicitly for the contracting phase,

$$t_0 = \frac{1}{2|H_0|}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}\left\{\arcsin\left(-\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right\} \ , \qquad \text{(P3.8)}$$

where $\arcsin(x)$ is chosen between $\pi$ and $\frac{3}{2}\pi$ if $\infty \geq \Omega_0 \geq 2$, and between $\frac{3}{2}\pi$ and $2\pi$ if $2 \geq \Omega_0 \geq 1$. In terms of the principal branch of the $\arcsin(x)$ function, this can be written

$$t_0 = \frac{1}{2|H_0|}\frac{\Omega_0}{(\Omega_0-1)^{3/2}} \times \begin{cases} \left[\pi + \arcsin\left(\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] & \text{if } \infty \geq \Omega_0 \geq 2, \\ \left[2\pi - \arcsin\left(\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] & \text{if } 2 \geq \Omega_0 \geq 1 \ . \end{cases} \qquad \text{(P3.9)}$$

Finally, $t_{\text{left}} = 2\pi\alpha/c - t_0$ implies that

$$\boxed{t_{\text{left}} = \frac{1}{2|H_0|}\frac{\Omega_0}{(\Omega_0-1)^{3/2}} \times \begin{cases} \left[\pi - \arcsin\left(\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) - \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] & \text{if } \infty \geq \Omega_0 \geq 2, \\ \left[\arcsin\left(\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) - \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] & \text{if } 2 \geq \Omega_0 \geq 1 \ , \end{cases}}$$

(P3.10)

where $\arcsin(x)$ is evaluated using the principal branch. Note that the complexity of the if-construction above is avoided by using the arccos function, as in Eq. (P3.7))

Continuing, we are next asked to determine the bluest blueshift that Dr. Niwde can observe. Assume that the density of galaxies is high enough so that all possible distances (within the horizon distance) are well represented. Then there is always a galaxy whose light is just arriving at Dr. Niwde's observatory at $t_0$ for **any** $t_e$ in the range $0 < t_e < t_0$. We let $\theta_e \equiv \theta(t_e)$ and $a_e \equiv a(t_e)$ denote respectively the development angle and scale factor at time $t_e$. The bluest blueshift is then found by minimizing $1 + z = \frac{a_0}{a_e}$ over all the values of $a_e$ that are in the past of Dr. Niwde.

(As an aside, one may be concerned about whether some given value of $t_e$ might correspond to a distance beyond the horizon. This, however, can never happen, as the horizon distance corresponds to $t_e = 0$. As long as we don't consider negative values of $t_e$, the points we are considering are within the horizon.)

Returning to the question of minimization, $z$ is minimized when $a_e$ is maximized, which happens at $\theta_e = \pi$. Using $a/\sqrt{k} = \alpha(1 - \cos\theta)$, the value of $z_{\rm min}$ is found to be

$$1 + z_{\rm min} = \frac{a_0}{a_e} = \frac{1 - \cos\theta_0}{1 - \cos\theta_e} = \frac{1 - \cos\theta_0}{2} \ . \tag{P3.11}$$

Using the value of $\cos\theta_0$ from Eq. (P3.2), one finds

$$\boxed{z_{\rm min} = -\frac{1}{\Omega_0} \ .} \tag{P3.12}$$

Finally, the lookback time is simply $t_{\rm lb} = t_0 - t_e$, where $t_e = t(\theta = \pi) = \pi\alpha/c$. Using Eq. (P3.6) for $t_0$, this gives

$$\boxed{t_{\rm lb} = \frac{\Omega_0}{2|H_0|(\Omega_0 - 1)^{3/2}} \left[\pi - \arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right] .} \tag{P3.13}$$

Or, using Eq. (P3.9), one can write

$$\boxed{t_{\rm lb} = \frac{1}{2|H_0|}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \times \begin{cases} \left[\arcsin\left(\frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right] & \text{if } \infty \geq \Omega_0 \geq 2, \\ \left[\pi - \arcsin\left(\frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right] & \text{if } 2 \geq \Omega_0 \geq 1 \ . \end{cases}} \tag{P3.14}$$

It was not asked in the problem, but one may want to know the distance to a galaxy which is most blueshifted. The physical distance integral becomes simple when written in terms of $\theta$, giving

$$\ell_{p,\text{bluest}} = \alpha(1 - \cos\theta_0)(\theta_0 - \theta_e) \ . \tag{P3.15}$$

Inserting $\alpha$, $\theta_0$, and $\theta_e = \pi$, this is

$$\ell_{p,\text{bluest}} = \frac{c}{|H_0|\sqrt{\Omega_0 - 1}} \left[\pi - \arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right)\right] \ . \tag{P3.16}$$

## PROBLEM 4: THE AGE OF A MATTER-DOMINATED UNIVERSE AS $\boldsymbol{\Omega \to 1}$ *(15 points)*

To describe the limit as $\Omega \to 1$ from below, it is convenient to define

$$\Omega \equiv 1 - \epsilon \ , \tag{P4.1}$$

so we are now interested in the limit as $\epsilon \to 0$ from above. We can rewrite the expression for $|H|t$ as

$$|H|t = \frac{\Omega}{2(1-\Omega)^{3/2}} \left[\frac{2\sqrt{1-\Omega}}{\Omega} - \operatorname{arcsinh}\left(\frac{2\sqrt{1-\Omega}}{\Omega}\right)\right] \tag{P4.2}$$

$$= \frac{1-\epsilon}{2\epsilon^{3/2}} \left\{\frac{2\sqrt{\epsilon}}{1-\epsilon} - \operatorname{arcsinh}\left(\frac{2\sqrt{\epsilon}}{1-\epsilon}\right)\right\} \equiv f(\epsilon) \ . \tag{P4.3}$$

Using the power series for $\operatorname{arcsinh}(x)$ given in the problem statement, we can write

$$f(\epsilon) = \frac{1-\epsilon}{2\epsilon^{3/2}} \left\{\frac{2\sqrt{\epsilon}}{1-\epsilon} - \left[\frac{2\sqrt{\epsilon}}{1-\epsilon} - \frac{1}{6}\left(\frac{2\sqrt{\epsilon}}{1-\epsilon}\right)^3 + \mathcal{O}\left(\epsilon^{5/2}\right)\right]\right\} \ , \tag{P4.4}$$

where I have used the notation $\mathcal{O}(\epsilon^p)$ to denote a quantity for which the limit

$$\lim_{\epsilon \to 0} \frac{\mathcal{O}(\epsilon^p)}{\epsilon^p}$$

is finite. Recalling that

$$\frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \ldots = 1 + \mathcal{O}(\epsilon) \ , \tag{P4.5}$$

we can manipulate $f(\epsilon)$ to give

$$\begin{aligned} f(\epsilon) &= \frac{1-\epsilon}{2\epsilon^{3/2}} \left\{ \frac{1}{6} \left( \frac{2\sqrt{\epsilon}}{1-\epsilon} \right)^3 + \mathcal{O}(\epsilon^{5/2}) \right\} \\ &= \frac{1-\epsilon}{2} \left\{ \frac{1}{6} \left( \frac{2}{1-\epsilon} \right)^3 + \mathcal{O}(\epsilon) \right\} \\ &= \left\{ \frac{1}{2} + \mathcal{O}(\epsilon) \right\} \left\{ \frac{4}{3} + \mathcal{O}(\epsilon) \right\} = \frac{2}{3} + \mathcal{O}(\epsilon) \ . \end{aligned} \tag{P4.6}$$

Thus,

$$\lim_{\epsilon \to 0+} f(\epsilon) = \frac{2}{3} \ . \tag{P4.7}$$

**PROBLEM 5: ISOTROPY ABOUT TWO POINTS IN EUCLIDEAN SPACES** *(15 points extra credit)*

The solution to this problem will appear with the solutions to Problem Set 5.