

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Physics Department

Physics 8.286: The Early Universe  
Prof. Alan Guth

October 23, 2022

**PROBLEM SET 6**

**DUE DATE:** Saturday, October 29, 2022, at 12:00 noon.

**READING ASSIGNMENT:** Steven Weinberg, *The First Three Minutes*, Chapter 7 (*The First One-Hundredth Second*), and also Barbara Ryden, *Introduction to Cosmology*, Chapter 9 (*Nucleosynthesis and the Early Universe*). For now we are skipping Chapter 8 (*The Cosmic Microwave Background*), because I would like you to read about nucleosynthesis in Ryden not too long after you've read about it in Weinberg. We will come back to Ryden's Chapter 8. Unfortunately, there is some confusion here that we will have to deal with. Ryden's treatment in Sections 9.2 and 9.3 of particle number densities in thermal equilibrium is not quite correct, as she leaves out the role of chemical potentials. Her statement in footnote 3 on p. 170 that "chemical potentials are small enough to be safely neglected" is in fact wildly incorrect. Equations for number densities such as Eqs. (9.11) and (9.12) are incorrect, but nonetheless all of the equations for ratios of number densities (such as (9.13), (9.25), or (9.26)) are correct, with the chemical potential factors canceling. We will come back to the subject of chemical potentials, but for now you should just accept Eqs. (9.13), (9.25), or (9.26) as equations that we will justify later. There are some references in Chapter 9 to equations from Chapter 8, but for now you can ignore these references. They all pertain to the derivations of Eqs. (9.13), (9.25), and (9.26), to which we will return.

**CALENDAR FOR THE NEXT FEW WEEKS:**

OCTOBER/NOVEMBER 2022					
MON	TUES	WED	THURS	FRI	SAT
October 24 Class 13	25	26 Class 14	27	28	29 PS6 due
31 Class 15	November 1	2 Class 16	3	4	5 PS7 due
7 Class 17	8	9 Class 18 <b>Quiz 2</b>	10	11 Veteran's Day	12

**PROBLEM 1: GEODESICS IN A SIMPLY TWO-DIMENSIONAL SPACE***(25 points)*

Consider a two dimensional non-Euclidean space, with coordinates  $(x, y)$ , and a metric given by

$$ds^2 = dx^2 + f(x) dy^2 , \quad (\text{P1.1})$$

where  $f(x)$  is an unspecified function of  $x$ , with  $f(x) > 0$ .

(a) *(10 points)* Using the geodesic equation in the form of Eq. (5.52) of Lecture Notes 5,

$$\frac{d}{ds} \left[ g_{ij} \frac{dx^j}{ds} \right] = \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \frac{dx^j}{ds} \frac{dx^k}{ds} , \quad (5.52)$$

derive explicitly the equations for geodesic functions  $x(s)$  and  $y(s)$ . Your answer should take the form of two differential equations that must be obeyed by  $x(s)$  and  $y(s)$  for them to describe a geodesic, and should not include the symbol  $g$ . That is, any component of  $g_{ij}$  should be replaced by its value according to the metric of Eq. (P1.1).

(b) *(15 points)* The geodesic equations should lead to two conserved quantities (i.e., quantities independent of  $s$ ) that can be written in terms of some or all of  $x(s)$ ,  $y(s)$ ,  $f(x)$ , and possibly first derivatives of these functions. Find these two conserved quantities. [*Hint: one should be completely obvious from the differential equations. The second can be found by a method analogous to Eq. (5.62) in Lecture Notes 5.*]

**PROBLEM 2: GEODESICS IN A FLAT UNIVERSE** *(25 points)*

According to general relativity, in the absence of any non-gravitational forces a particle will travel along a spacetime geodesic. In this sense, gravity is reduced to a distortion in spacetime.

Consider the case of a flat (i.e.,  $k = 0$ ) Robertson–Walker metric, which has the simple form

$$ds^2 = -c^2 dt^2 + a^2(t) [dx^2 + dy^2 + dz^2] .$$

Since the spatial metric is flat, we have the option of writing it in terms of Cartesian rather than polar coordinates. Now consider a particle which moves along the  $x$ -axis. (Note that the galaxies are on the average at rest in this system, but one can still discuss the trajectory of a particle which moves through the model universe.)

(a) *(8 points)* Use the geodesic equation to show that the coordinate velocity computed with respect to proper time (i.e.,  $dx/d\tau$ ) falls off as  $1/a^2(t)$ .

(b) *(8 points)* Use the expression for the spacetime metric to relate  $dx/dt$  to  $dx/d\tau$ .

- (c) (9 points) The physical velocity of the particle relative to the galaxies that it is passing is given by

$$v = a(t) \frac{dx}{dt} .$$

(Note that this is just a generalization of what we have previously said for photons,  $dx/dt = c/a(t)$ .) Show that the momentum of the particle, defined relativistically by

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

falls off as  $1/a(t)$ . (This implies, by the way, that if the particle were described as a quantum mechanical wave with wavelength  $\lambda = h/|\vec{p}|$ , then its wavelength would stretch with the expansion of the universe, in the same way that the wavelength of light is redshifted.)

**PROBLEM 3: METRIC OF A STATIC GRAVITATIONAL FIELD** (25 points)

In this problem we will consider the metric

$$ds^2 = - [c^2 + 2\phi(\vec{x})] dt^2 + \sum_{i=1}^3 (dx^i)^2 ,$$

which describes a static gravitational field. Here  $i$  runs from 1 to 3, with the identifications  $x^1 \equiv x$ ,  $x^2 \equiv y$ , and  $x^3 \equiv z$ . The function  $\phi(\vec{x})$  depends only on the spatial variables  $\vec{x} \equiv (x^1, x^2, x^3)$ , and not on the time coordinate  $t$ .

- (a) (5 points) Suppose that a radio transmitter, located at  $\vec{x}_e$ , emits a series of evenly spaced pulses. The pulses are separated by a proper time interval  $\Delta T_e$ , as measured by a clock at the same location. What is the coordinate time interval  $\Delta t_e$  between the emission of the pulses? (I.e.,  $\Delta t_e$  is the difference between the time coordinate  $t$  at the emission of one pulse and the time coordinate  $t$  at the emission of the next pulse.)
- (b) (5 points) The pulses are received by an observer at  $\vec{x}_r$ , who measures the time of arrival of each pulse. What is the **coordinate** time interval  $\Delta t_r$  between the reception of successive pulses?
- (c) (5 points) The observer uses his own clocks to measure the proper time interval  $\Delta T_r$  between the reception of successive pulses. Find this time interval, and also the redshift  $z$ , defined by

$$1 + z = \frac{\Delta T_r}{\Delta T_e} .$$

First compute an exact expression for  $z$ , and then expand the answer to lowest order in  $\phi(\vec{x})$  to obtain a weak-field approximation. (This weak-field approximation is in fact highly accurate in all terrestrial and solar system applications.)

- (d) (5 points) A freely falling particle travels on a spacetime geodesic  $x^\mu(\tau)$ , where  $\tau$  is the proper time. (I.e.,  $\tau$  is the time that would be measured by a clock moving with the particle.) The trajectory is described by the geodesic equation

$$\frac{d}{d\tau} \left( g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) = \frac{1}{2} (\partial_\mu g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} ,$$

where the Greek indices ( $\mu, \nu, \lambda, \sigma$ , etc.) run from 0 to 3, and are summed over when repeated. Calculate an explicit expression for

$$\frac{d^2 x^i}{d\tau^2} ,$$

valid for  $i = 1, 2$ , or  $3$ . (It is acceptable to leave quantities such as  $dt/d\tau$  or  $dx^i/d\tau$  in the answer.)

- (e) (5 points) In the weak-field nonrelativistic-velocity approximation, the answer to the previous part reduces to

$$\frac{d^2 x^i}{dt^2} = -\partial_i \phi ,$$

so  $\phi(\vec{x})$  can be identified as the Newtonian gravitational potential. Use this fact to estimate the gravitational redshift  $z$  of a photon that rises from the floor of this room to the ceiling (say 4 meters). (One significant figure will be sufficient.)

#### PROBLEM 4: CIRCULAR ORBITS IN A SCHWARZSCHILD METRIC (30 points)

**READ THIS:** This problem was Problem 16 of Review Problems for Quiz 2 of 2011, and the solution is posted as <http://web.mit.edu/8.286/www/quiz11/ecqr2-1.pdf>. Like Problem 4 of Problem Set 3, but unlike all other homework problems so far, in this case you are encouraged to look at the solutions and benefit from them. When you write your solution, you can even copy it verbatim from these solutions if you wish, although obviously you will learn more if you think about the solution and write your own version.

The Schwarzschild metric, which describes the external gravitational field of any spherically symmetric distribution of mass (including black holes), is given by

$$ds^2 = -c^2 d\tau^2 = - \left( 1 - \frac{2GM}{rc^2} \right) c^2 dt^2 + \left( 1 - \frac{2GM}{rc^2} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 ,$$

where  $M$  is the total mass of the object,  $0 \leq \theta \leq \pi$ ,  $0 \leq \phi < 2\pi$ , and  $\phi = 2\pi$  is identified with  $\phi = 0$ . We will be concerned only with motion outside the Schwarzschild horizon  $R_S = 2GM/c^2$ , so we can take  $r > R_S$ . (This restriction allows us to avoid the complications of understanding the effects of the singularity at  $r = R_S$ .) In this problem we will use the geodesic equation to calculate the behavior of circular orbits in this metric. We will assume a perfectly circular orbit in the  $x$ - $y$  plane: the radial coordinate  $r$  is fixed,  $\theta = 90^\circ$ , and  $\phi = \omega t$ , for some angular velocity  $\omega$ .

- (a) (7 points) Use the metric to find the proper time interval  $d\tau$  for a segment of the path corresponding to a coordinate time interval  $dt$ . Note that  $d\tau$  represents the time that would actually be measured by a clock moving with the orbiting body. Your result should show that

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}} .$$

Note that for  $M = 0$  this reduces to the special relativistic relation  $d\tau/dt = \sqrt{1 - v^2/c^2}$ , but the extra term proportional to  $M$  describes an effect that is new with general relativity—the gravitational field causes clocks to slow down, just as motion does.

- (b) (7 points) Show that the geodesic equation of motion (Eq. (5.65)) for one of the coordinates takes the form

$$0 = \frac{1}{2} \frac{\partial g_{\phi\phi}}{\partial r} \left( \frac{d\phi}{d\tau} \right)^2 + \frac{1}{2} \frac{\partial g_{tt}}{\partial r} \left( \frac{dt}{d\tau} \right)^2 .$$

- (c) (8 points) Show that the above equation implies

$$r \left( \frac{d\phi}{d\tau} \right)^2 = \frac{GM}{r^2} \left( \frac{dt}{d\tau} \right)^2 ,$$

which in turn implies that

$$r\omega^2 = \frac{GM}{r^2} .$$

Thus, the relation between  $r$  and  $\omega$  is exactly the same as in Newtonian mechanics. [Note, however, that this does not really mean that general relativity has no effect. First,  $\omega$  has been defined by  $d\phi/dt$ , where  $t$  is a time coordinate which is not the same as the proper time  $\tau$  that would be measured by a clock on the orbiting body. Second,  $r$  does not really have the same meaning as in the Newtonian calculation, since it is not the measured distance from the center of motion. Measured distances, you will recall, are calculated by integrating the metric, as for example in Problem 1 of Problem Set 5, *A Circle in a Non-Euclidean Geometry*. Since the angular ( $d\theta^2$  and  $d\phi^2$ ) terms in the Schwarzschild metric are unaffected by the mass, however, it

*can be seen that the circumference of the circle is equal to  $2\pi r$ , as in the Newtonian calculation.]*

- (d) (8 points) Show that circular orbits around a black hole have a minimum value of the radial coordinate  $r$ , which is larger than  $R_S$ . What is it?

**Total points for Problem Set 6: 105.**