MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe November 5, 2022
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PROBLEM SET 7 SOLUTIONS

PROBLEM 1: GAS PRESSURE AND ENERGY CONSERVATION

(a) From Eqs. (P1.2) and (P1.3) of the problem statement one has

$$U = V_{\rm phys} u = a^3(t) V_{\rm coord} u(t) \ .$$

If the change described by Eq. (P1.1) happens over a time interval dt, then

$$dU = -p dV \quad \Longrightarrow \quad \frac{dU}{dt} = -p \frac{dV}{dt} \ .$$

Remembering that $V_{\rm coord}$ does not vary with time, and using the chain rule for the differentiation of products of functions,

$$\frac{dU}{dt} = V_{\rm coord} \frac{d}{dt}\left(a^3 u\right) = V_{\rm coord} \frac{d}{dt}\left(a^3 \rho c^2\right)$$

and

$$\frac{dV}{dt} = V_{\rm coord} \frac{d}{dt}\left(a^3\right) \ .$$

So

$$\boxed{\frac{d}{dt}\left(a^3 \rho c^2\right) = -p \frac{d}{dt}\left(a^3\right) \ .}$$

Then, using the chain rule again,

$$3a^2 \dot{a} \rho c^2 + a^3 \dot{\rho} c^2 = -3pa^2 \dot{a} \ .$$

Dividing by $a^3 c^2$ and rearranging,

$$\boxed{\dot{\rho} = -3 \frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ .}$$

(b) We start by rewriting the Friedmann equation (Eq. (P1.6) of the problem statement),

$$\dot{a}^2 = \frac{8\pi}{3} G \rho a^2 - kc^2 \ .$$

Differentiating with respect to time, we have

$$2\dot{a}\ddot{a} = \frac{8\pi}{3}G\dot{\rho}a^2 + \frac{16\pi}{3}G\rho a\dot{a} \ .$$

Using the result of part (a) for $\dot{\rho}$ and dividing by $2\dot{a}$ yields an equation for $\ddot{a}$,

$$\ddot{a} = \frac{8\pi}{3}G\left[-\frac{3}{2}a\left(\rho + \frac{p}{c^2}\right)\right] + \frac{8\pi}{3}G\rho a \quad \Longrightarrow$$

$$\boxed{\ddot{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a \ .}$$

NOTE: Although we derived Eq. (P1.6) of the problem set in the context of Newtonian cosmology, the same equation holds exactly in the general relativistic treatment of the same problem. The equation above for $\ddot{a}$ also holds exactly in general relativity.

(c) To make use of the result of part (a), it would be helpful to eliminate a in favor of T. Using $aT = \text{const}$, note that

$$a = \frac{\text{const}}{T} \quad \Longrightarrow \quad \dot{a} = -\frac{\text{const}}{T^2}\dot{T} = -a\frac{\dot{T}}{T} \ ,$$

so Eq. (P1.5) becomes

$$\dot{\rho} = 3\frac{\dot{T}}{T}\left(\rho + \frac{p}{c^2}\right) \ .$$

Then using $\rho = aT^4$,

$$\dot{\rho} = 4aT^3\dot{T} = 3\frac{\dot{T}}{T}\left(aT^4 + \frac{p}{c^2}\right) \ .$$

Solving for p gives

$$\boxed{p = \frac{1}{3}aT^4c^2 = \frac{1}{3}\rho c^2 \ .}$$

Note that this is very different from ordinary non-relativistic gases. For the air in this room, $p \approx 10^{-12}\rho c^2$.

PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION

(a) This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by $dU = -p\,dV$. Using the fact that the energy density u is equal to ρc^2, the energy conservation relation can be written

$$\frac{dU}{dt} = -p\frac{dV}{dt} \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\rho c^2 a^3\right) = -p\frac{d}{dt}\left(a^3\right) \ .$$

Setting

$$\rho = \frac{\alpha}{a^6}$$

for some constant α, the conservation of energy formula becomes

$$\frac{d}{dt}\left(\frac{\alpha c^2}{a^3}\right) = -p\frac{d}{dt}\left(a^3\right) \ ,$$

which implies

$$-3\frac{\alpha c^2}{a^4}\frac{da}{dt} = -3pa^2\frac{da}{dt} \ .$$

Thus

$$p = \frac{\alpha c^2}{a^6} = \boxed{\rho c^2 \ .}$$

Alternatively, one may start from the equation for the time derivative of ρ,

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) .$$

Since $\rho = \frac{\alpha}{a^6}$, we take the time derivative to find $\dot{\rho} = -6(\dot{a}/a)\rho$, and therefore

$$-6\frac{\dot{a}}{a}\rho = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) ,$$

and therefore

$$p = \rho c^2 .$$

(b) For a flat universe, the Friedmann equation reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho \ .$$

Using $\rho \propto 1/a^6$, this implies that

$$\dot{a} = \frac{\beta}{a^2} \ ,$$

for some constant β. Rewriting this as

$$a^2 \, da = \beta \, dt \ ,$$

we can integrate the equation to give

$$\frac{1}{3} a^3 = \beta t + \text{const} \ ,$$

where the constant of integration has no effect other than to shift the origin of the time variable t. Using the standard big bang convention that $a = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$\boxed{a \propto t^{1/3} \ .}$$

The arbitrary constant of proportionality in this answer is consistent with the wording of the problem, which states that "You should be able to determine the function $a(t)$ up to a constant factor." Note that we could have expressed the constant of proportionality in terms of the constant α that we used in part (a), but there would not really be any point in doing that. The constant α was not a given variable. If the comoving coordinates are measured in "notches," then a is measured in meters per notch, and the constant of proportionality in our answer can be changed by changing the arbitrary definition of the notch.

(c) We start from the conservation of energy equation in the form

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) .$$

Substituting $\dot{\rho} = -n(\dot{a}/a)\rho$ and $p = (1/2)\rho c^2$, we have

$$-nH\rho = -3H\left(\frac{3}{2}\rho\right)$$

and therefore

$$\boxed{n = \frac{9}{2}.}$$

PROBLEM 3: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

(a) *(4 points)* The Friedmann equation in a flat universe is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.$$

Substituting $\rho = \text{const}/a^5$ and taking the square root of both sides gives

$$\frac{\dot{a}}{a} = \alpha a^{-5/2} \, ,$$

for some constant α. Rearranging to a form we can integrate,

$$da\, a^{3/2} = \alpha dt,$$

and therefore

$$\frac{2}{5}a^{5/2} = \alpha t.$$

Notice that once again we have eliminated the arbitrary integration constant by choosing the Big Bang boundary conditions $a = 0$ at $t = 0$. Solving for a yields

$$\boxed{a \propto t^{2/5}.}$$

(b) *(3 points)* The Hubble parameter is, from its definition,

$$\boxed{H = \frac{\dot{a}}{a} = \frac{2}{5t},}$$

where we have used the time dependence of $a(t)$ that we found in part (a). (Notice that we don't need to know the constant of proportionality left undetermined in part (a), as it cancels between numerator and denominator in this calculation.)

(c) *(4 points)* Recall that the horizon distance is the physical distance traveled by a light ray since $t = 0$,

$$\ell_{p,\text{horizon}}(t) = a(t) \int_0^t \frac{c\, dt'}{a(t')}.$$

Using $a(t) \propto t^{2/5}$, we find

$$\ell_{p,\text{horizon}}(t) = ct^{2/5} \int_0^t dt'\, t'^{-2/5}$$

or

$$\ell_{p,\text{horizon}}(t) = ct^{2/5}\left(\frac{5}{3}t^{3/5}\right) = \boxed{\frac{5}{3}ct.}$$

(d) *(4 points)* Since we know the Hubble parameter, we can find the mass density $\rho(t)$ easily from the Friedmann equation,

$$\rho(t) = \frac{3H^2}{8\pi G}.$$

Using the result from part (b), we find

$$\boxed{\rho(t) = \frac{3}{50\pi G}\frac{1}{t^2}.}$$

As a check on our algebra, since we found in (a) that $a \propto t^{2/5}$, and knew at the beginning of the calculation that $\rho \propto a^{-5}$, we should find, as we do here, that $\rho \propto t^{-2}$. Notice, however, that in this case we do not leave our answer in terms of some undetermined constant of proportionality; the units of ρ are not arbitrary, and therefore we care about its normalization.

PROBLEM 4: EFFECT OF AN EXTRA NEUTRINO SPECIES *(15 points)*

(a) The temperature vs. time relationship which holds during the radiation-dominated era was derived in class:

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 gG}\right)^{1/4}\frac{1}{\sqrt{t}} ,$$

or

$$t = \left(\frac{45\hbar^3 c^5}{16\pi^3 gG}\right)^{1/2}\frac{1}{(kT)^2} .$$

The quantity g receives a contribution of 2 from photons (two possible spin states), 7/2 from e^+e^- pairs (two spin states for e^- plus two spin states for e^+, times 7/8 for the Pauli exclusion principle) and 7/4 for each species of neutrino (one spin state for ν plus one spin state for $\bar{\nu}$, times 7/8). With three species of massless neutrinos,

$g = 10\frac{3}{4}$. So

$$t = \left[\frac{45 \times \left(1.055 \times 10^{-27} \text{ erg} \cdot \text{s}\right)^3 \times \left(2.998 \times 10^{10} \text{ cm} \cdot \text{s}^{-1}\right)^5}{16\pi^3 \times (10.75) \times (6.672 \times 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2})}\right]^{1/2}$$

$$\times \frac{1}{\left(10^6 \text{ eV}\right)^2} \times \left(\frac{1\,\text{eV}}{1.602 \times 10^{-12} \text{ erg}}\right)^2$$

$$= \boxed{.740 \text{ s.}}$$

By using $1 \text{ erg} = 1\,\text{g} \cdot \text{cm}^2/\text{s}^2$, one can see that the units work out right.

Note that this result is only an approximation, since $g = 10\frac{3}{4}$ is not correct for $kT > 100$ MeV, so the total time for the universe to evolve from the instant of the big bang to $kT = 1$ MeV would be influenced by the extra particles that contributed to g before kT fell to 100 MeV. But the correction would be small, since the formula above shows that $t \propto 1/T^2$; thus only a fraction $(1 \text{ MeV}/100 \text{ MeV})^2 = 10^{-4}$ of our calculation is influenced by these extra particles. The extra particles at high temperatures would cause the universe to evolve faster, so an upper limit on the correction would be to reduce the time to reach $kT = 100$ MeV to zero, which would reduce the answer to this question by one part in 10^4.

(b) With an additional neutrino species, g increases by 7/4 to 12.5. Thus, with four neutrino species, the time it takes for the universe to cool to $T = 1$ MeV decreases relative to the case of three neutrino species by a factor of

$$\left(\frac{12.5}{10.75}\right)^{1/2} = 1.0783 \ .$$

One then has $\boxed{t = .686 \text{ s.}}$ With an extra neutrino species, the universe would evolve faster — the extra mass acts to slow down the expansion faster.

(c) Under the standard assumptions, the energy density of matter in the universe is, at temperature T,

$$\rho = g\frac{\pi^2}{30}\frac{(kT)^4}{\hbar^3 c^5},$$

and $g = 10.75$. Plugging in numbers,

$$\rho = g\frac{\pi^2}{30}\frac{(kT)^4}{\hbar^3 c^5} = (10.75)\,\frac{\pi^2}{30}\,\frac{10^{24} \text{ eV}^4}{\left(1.055 \times 10^{-27}\right)^3 \text{ erg}^3 \cdot \text{s}^3}$$

$$\times \frac{1}{\left(2.998 \times 10^{10}\right)^5 \text{ cm}^5 \cdot \text{s}^{-5}} \times \frac{\left(1.602 \times 10^{-12}\right)^4 \text{ erg}^4}{1 \text{ eV}^4} \times \frac{1 \text{ g} \cdot \text{cm}^2 \cdot \text{s}^{-2}}{\text{erg}}$$

$$= \boxed{8.19 \times 10^5 \text{ g/cm}^3} = \boxed{8.19 \times 10^8 \text{ kg/m}^3 \ .}$$

The only change to this calculation in the case of an extra neutrino species is that we must now take $g = 12.5$. Then

$$\rho = \frac{12.5}{10.75}\, 8.19 \times 10^5 \text{ g/cm}^3 = \boxed{9.52 \times 10^5 \text{ g/cm}^3} = \boxed{9.52 \times 10^8 \text{ kg/m}^3 \ .}$$

PROBLEM 5: ENTROPY AND THE BACKGROUND NEUTRINO TEMPERATURE *(15 points)*

The entropy density of black-body radiation is given by

$$\begin{aligned} s &= g\left[\frac{2\pi^2}{45}\frac{k^4}{(\hbar c)^3}\right] T^3 \\ &= g\, C\, T^3 \ , \end{aligned}$$

where C is a constant. The neutrinos are decoupled, so their entropy is conserved. More precisely, the conserved quantity is $S_\nu \equiv a^3 s_\nu$, which indicates the entropy per cubic notch, i.e., entropy per unit comoving volume. We introduce the notation

Primed:	after e^+e^- annihilation
Unprimed:	before e^+e^- annihilation.

For the neutrinos,

$$S'_\nu = S_\nu \quad \Longrightarrow \quad g_\nu C\,(a'T'_\nu)^3 = g_\nu C\,(aT_\nu)^3 \quad \Longrightarrow$$

$$\boxed{a'T'_\nu = aT_\nu \ .}$$

For the photons,

Before e^+e^- annihilation,

$$T_\gamma = T_{e^+e^-} = T_\nu \ ; \qquad g_\gamma = 2,\ g_{e^+e^-} = 7/2 \ .$$

When e^+e^- pairs annihilate, their entropy is added to photons.

$$S'_\gamma = S_{e^+e^-} + S_\gamma \quad \Longrightarrow \quad 2C\left(a'T'_\gamma\right)^3 = \left(2 + \frac{7}{2}\right) C\,(aT_\gamma)^3 \quad \Longrightarrow$$

$$\boxed{a'T'_\gamma = \left(\frac{11}{4}\right)^{1/3} aT_\gamma \ ,}$$

so aT_γ increases by a factor of $(11/4)^{1/3}$.

Before e^+e^- annihilation the neutrinos were in thermal equilibrium with the photons, so $T_\gamma = T_\nu$. By considering the two boxed equations above, one has

$$\boxed{T'_\nu = \left(\frac{4}{11}\right)^{1/3} T'_\gamma\ .}$$

PROBLEM 6: FREEZE-OUT OF MUONS *(25 points)*

(a) The factors contributing to g from the muons are the following:

$$\begin{aligned} &2 \quad \text{since there are two particles, the muon and the antimuon}\\ &2 \quad \text{since there are two spin states for each particle}\\ &\frac{7}{8} \quad \text{since the } \mu^- \text{ and the } \mu^+ \text{ are fermions} \end{aligned}$$

Thus

$$\boxed{g_{\mu^+\mu^-} = 2 \times 2 \times \frac{7}{8} = \frac{7}{2}\ .}$$

(b) Besides the muons, the particles in thermal equilibrium when kT is just above 106 MeV are photons, neutrinos, and electron-positron pairs. As found in class

$$g_\gamma = 2 \qquad \text{(bosons, 2 spin states)}$$

$$g_\nu = \underbrace{3}_{\text{No. of species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{\frac{7}{8}}_{\text{Fermion factor}} = \frac{21}{4}\ .$$

$$g_{e^+e^-} = \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} \times \underbrace{\frac{7}{8}}_{\text{Fermion factor}} = \frac{7}{2}\ .$$

So, for kT just above 106 MeV, g is the sum of all of these contributions:

$$\boxed{g = g_{\mu^+\mu^-} + g_\gamma + g_\nu + g_{e^+e^-} = \frac{57}{4} = 14.25\ .}$$

(c) We know that entropy is to a high degree of accuracy conserved as the universe expands, although of course it is thinned by the expansion. The conservation of entropy means that the entropy contained within any comoving volume does not change. The entropy per comoving volume S is therefore constant, and can be written as

$$S = a^3(t)\, s(t) \ ,$$

where $s(t)$ is the entropy density. The expression for the entropy density of black body radiation is:

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3} \quad .$$

This formula describes the radiation of effectively massless particles, so it would not be valid when $kT \approx 106$ MeV, since for such temperatures the mass of the muons cannot be neglected. We can apply this formula, however, when $kT \gg 106$ MeV, when the muons are effectively massless, and we can also apply it when $kT \ll 106$ MeV, when the muons are essentially nonexistent; the power of conservation laws then allows us to set the two expressions for S equal. So let t_1 denote a time when kT is well ***above*** 106 MeV (but below the threshold for producing other particles), and let t_2 denote a time when kT is well ***below*** 106 MeV (but well above 0.5 MeV, when the electron-positron pairs will disappear). (Realistically the freeze-out of the muons will overlap the freezing out of the pions, with mass/energies of 135-140 MeV, but for the purpose of this problem we are ignoring the pions.)

The entropy per comoving volume is conserved, and by combining the two formulas above it can be written

$$S = C \times g(T) a^3 T^3 \quad \text{where} \quad C = \text{constant.}$$

Since $S(t_1) = S(t_2)$, we find

$$C \times g(t_1) \big[a(t_1) T(t_1) \big]^3 = C \times g(t_2) \big[a(t_2) T(t_2) \big]^3 \ ,$$

which implies that

$$\frac{(aT)|_{t_2}}{(aT)|_{t_1}} = \left[\frac{g(t_1)}{g(t_2)} \right]^{1/3} \quad .$$

We found $g(t_1)$ in part (b): $g(t_1) = 14.25$. After the muons disappear from the black body radiation they no longer contribute to the g in the expression for the entropy, so

$$g(t_2) = g_\gamma + g_\nu + g_{e^+e^-} = 2 + \frac{21}{4} + \frac{7}{2} = \frac{43}{4} = 10.75 \ .$$

Using these values in the expression above we obtain the increase in aT due to the annihilation of muon-antimuon pairs,

$$(aT)\big|_{t_2} = \left(\frac{14.25}{10.75}\right)^{1/3} (aT)\big|_{t_1} = \boxed{\left(\frac{57}{43}\right)^{1/3} (aT)\big|_{t_1}} \ .$$

Evaluating the cube root, we have

$$(aT)\big|_{t_2} \approx (1.10)\,(aT)\big|_{t_1} \ .$$