MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 12, 2022
Prof. Alan Guth

PROBLEM SET 8

DUE DATE: Saturday, November 19, 2022, at 12:00 noon.

READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology,* Chapter 11 *(Structure Formation: Gravitational Instability).* We are temporarily skipping Chapter 10 *(Inflation and the Very Early Universe)*, but we will come back to it shortly when we reach this topic in lecture.

CALENDAR THROUGH THE END OF THE TERM:

NOVEMBER/DECEMBER 2022					
MON	TUES	WED	THURS	FRI	SAT
November 14 Class 19	15	16 Class 20	17	18	19 PS8 due
21 Class 21	22	23 Class 22 DROP Date	24 Thanksgiving	25 Thanksgiving	26
28 Class 23	29	30 Class 24	December 1	2	3 PS9 due
5 Class 25	6	7 Class 26 Quiz 3	8	9	10
12 Class 27	13	14 Class 28 PS10 due Last Class!	15	16 Final Exams	17
19 Final Exams	20 Final Exams	21 Final Exams	22 Final Exams	23	24

PROBLEM 1: THE REDSHIFT OF THE COSMIC MICROWAVE BACKGROUND *(25 points)*

It was mentioned in Lecture Notes 6 that the black-body spectrum has the peculiar feature that it maintains its form under uniform redshift. That is, as the universe expands, even if the photons do not interact with anything, they will continue to be described by a black-body spectrum, although at a temperature that decreases as the universe expands. Thus, even though the cosmic microwave background (CMB) has not been interacting significantly with matter since 350,000 years after the big bang, the radiation today still has a black-body spectrum. In this problem we will demonstrate this important property of the black-body spectrum.

The spectral energy density $\rho_\nu(\nu, T)$ for the thermal (black-body) radiation of photons at temperature T was stated in Lecture Notes 6 as Eq. (6.75), which we can rewrite as

$$\rho_\nu(\nu, T) = \frac{16\pi^2\hbar\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1} \ , \tag{P1.1}$$

where $h = 2\pi\hbar$ is Planck's original constant. $\rho_\nu(\nu, T)\, d\nu$ is the energy per unit volume carried by photons whose frequency is in the interval $[\nu, \nu + d\nu]$. In this problem we will assume that this formula holds at some initial time t_1, when the temperature had some value T_1. We will let $\tilde{\rho}(\nu, t)$ denote the spectral distribution for photons in the universe, which is a function of frequency ν and time t. Thus, our assumption about the initial condition can be expressed as

$$\tilde{\rho}(\nu, t_1) = \rho_\nu(\nu, T_1) \ . \tag{P1.2}$$

The photons redshift as the universe expands, and to a good approximation the redshift and the dilution of photons due to the expansion are the only physical effects that cause the distribution of photons to evolve. Thus, using our knowledge of the redshift, we can calculate the spectral distribution $\tilde{\rho}(\nu, t_2)$ at some later time $t_2 > t_1$. It is not obvious that $\tilde{\rho}(\nu, t_2)$ will be a thermal distribution, but in fact we will be able to show that

$$\tilde{\rho}(\nu, t_2) = \rho\big(\nu, T(t_2)\big) \ , \tag{P1.3}$$

where in fact $T(t_2)$ will agree with what we already know about the evolution of T in a radiation-dominated universe:

$$T(t_2) = \frac{a(t_1)}{a(t_2)}\, T_1 \ . \tag{P1.4}$$

To follow the evolution of the photons from time t_1 to time t_2, we can imagine selecting a region of comoving coordinates with coordinate volume V_c. Within this comoving volume, we can imagine tagging all the photons in a specified infinitesimal range of frequencies, those between ν_1 and $\nu_1 + d\nu_1$. Recalling that the energy of each such photon is $h\nu$, the number dN_1 of tagged photons is then

$$dN_1 = \frac{\tilde{\rho}(\nu_1, t_1)\, a^3(t_1)\, V_c\, d\nu_1}{h\nu_1} \ . \tag{P1.5}$$

(a) We now wish to follow the photons in this frequency range from time t_1 to time t_2, during which time each photon redshifts. At the latter time we can denote the range of frequencies by ν_2 to $\nu_2 + d\nu_2$. Express ν_2 and $d\nu_2$ in terms of ν_1 and $d\nu_1$, assuming that the scale factor $a(t)$ is given.

(b) At time t_2 we can imagine tagging all the photons in the frequency range ν_2 to $\nu_2 + d\nu_2$ that are found in the original comoving region with coordinate volume V_c. Explain why the number dN_2 of such photons, on average, will equal dN_1 as calculated in Eq. (P1.5).

(c) Since $\tilde{\rho}(\nu, t_2)$ denotes the spectral energy density at time t_2, we can write

$$dN_2 = \frac{\tilde{\rho}(\nu_2, t_2)\, a^3(t_2)\, V_c\, d\nu_2}{h\nu_2} \ , \tag{P1.6}$$

using the same logic as in Eq. (P1.5). Use $dN_2 = dN_1$ to show that

$$\tilde{\rho}(\nu_2, t_2) = \frac{a^3(t_1)}{a^3(t_2)}\, \tilde{\rho}(\nu_1, t_1) \ . \tag{P1.7}$$

Use the above equation to show that Eq. (P1.3) is satisfied, for $T(t)$ given by Eq. (P1.4).

PROBLEM 2: BIG BANG NUCLEOSYNTHESIS *(20 points)*

The calculations of big bang nucleosynthesis depend on a large number of measured parameters. Below you are asked to qualitatively describe the effects of changing some of these parameters. Include a sentence or two to explain each of your answers. (These topics have not been discussed in class, but you are expected to be able to answer the questions on the basis of your readings in Weinberg's and Ryden's books.)

(a) *(5 points)* Suppose an extra neutrino species is added to the calculation. Would the predicted helium abundance go up or down?

(b) *(5 points)* Suppose the weak interactions were stronger than they actually are, so that the thermal equilibrium distribution between neutrons and protons were maintained until $kT \approx 0.25$ MeV. Would the predicted helium abundance be larger or smaller than in the standard model?

(c) *(5 points)* Suppose the proton-neutron mass difference were larger than the actual value of 1.29 MeV/c^2. Would the predicted helium abundance be larger or smaller than in the standard calculation?

(d) *(5 points)* The standard theory of big bang nucleosynthesis assumes that the matter in the universe was distributed homogeneously during the era of nucleosynthesis, but the alternative possibility of inhomogeneous big-bang nucleosynthesis has been discussed since the 1980s. Inhomogeneous nucleosynthesis hinges on the hypothesis that baryons became clumped during a phase transition at $t \approx 10^{-6}$ second, when the hot quark soup converted to a gas of mainly protons, neutrons, and in the early stages, pions. The baryons would then be concentrated in small nuggets, with a comparatively low density outside of these nuggets. After the phase transition but before nucleosynthesis, the neutrons would have the opportunity to diffuse away from these nuggets, becoming more or less uniformly distributed in space. The protons, however, since they are charged, interact electromagnetically with the plasma that fills the universe, and therefore have a much shorter mean free path than the neutrons. Most of the protons, therefore, remain concentrated in the nuggets. Does this scenario result in an increase or a decrease in the expected helium abundance?

PROBLEM 3: THE DEUTERIUM BOTTLENECK *(30 points)*

The "deuterium bottleneck" plays a major role in the description of big bang nucleosynthesis: all of the nuclear reactions involved in nucleosynthesis depend on deuterium forming at the start, but, due to its small binding energy, deuterium does not become stable until the temperature reaches a rather low value. In this problem we will explore the statistical mechanics of the deuterium bottleneck. Ryden discusses this in Section 9.3 *Deuterium Synthesis*, but the calculation here will be a little more complete.

As discussed in *Notes on Thermal Equilibrium* (which I will refer to as NTE), a dilute ideal gas of classical nonrelativistic particles of type X, in thermal equilibrium, has a number density given by

$$n_X = \bar{g}_X \left(\frac{m_X kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{m_X c^2}{kT}\right) \exp\left(\frac{\mu_X}{kT}\right) \,, \tag{P3.1}$$

where $\hbar = h/2\pi$, c, and k have their usual meanings: Planck's constant, the speed of light, and the Boltzmann constant. Here $\bar{g}_X$ is the number of spin degrees of freedom associated with the particle, m_X is the mass of the particle, T is the temperature, and μ_X is the chemical potential of the particle. Because the gas is dilute there are no corrections associated with fermions and the Pauli exclusion principle, since it is highly unlikely in any case for two particles to occupy the same quantum state. The dilute gas approximation is valid provided that

$$n_X \ll \left(\frac{m_X kT}{2\pi\hbar^2}\right)^{3/2} \,. \tag{P3.2}$$

Eq. (P3.1) is equivalent to Eq. (23) of NTE, or Eq. (8.26) of Ryden's 2nd edition.

(a) *(5 points)* I mentioned in NTE that Ryden's textbook sometimes writes Eq. (P3.1) incorrectly, omitting the chemical potential factor. See for example Eqs. (9.11) and (9.12). The author has a footnote about this (p. 170), which concludes that "in most cosmological contexts, as it turns out, the chemical potential is small enough to be safely neglected." We can check this statement by using the author's formula to calculate the proton density at 3 minutes into the big bang, at the time of Steven Weinberg's Fifth Frame, from chapter 5 of *The First Three Minutes*. At that time the temperature was $T = 10^9\,\mathrm{K}$. To find the right answer for comparison, we make use of the fact that the ratio of the number density n_b of baryons to the number density n_γ of photons is given by*

$$\eta \equiv \frac{n_b}{n_\gamma} = (6.15 \pm 0.35) \times 10^{-10} \quad \text{(95\% confidence)}. \tag{P3.3}$$

According to Weinberg, at that time 14% of the baryons were neutrons, with 86% protons. At the risk of appearing impertinent toward the author (but physicists are known for their impertinence), I will phrase the question this way: By how many kilo-orders of magnitude is the author's formula for n_p in error?† (Be prepared to have your calculators overflow — if they do, calculate the logarithm of the answer.)

(b) *(15 points)* For deuterium production, the relevant reaction is

$$n + p \longleftrightarrow D + \gamma\ . \tag{P3.4}$$

Recall that chemical potentials are defined initially in terms of conserved quantities, so the chemical potentials on both sides of any allowed process much match. Since the photon carries no conserved quantities, its chemical potential must vanish. It follows that $\mu_n + \mu_p = \mu_D$. This equality implies that if we form the ratio

$$\frac{n_D}{n_p\, n_n}\ , \tag{P3.5}$$

expressing each number density as in Eq. (P3.1), then the chemical potential factors will cancel out. (This is how the formula is normally used, and this is how Ryden uses it on pp. 175–176. From here on Ryden's treatment is correct, but we will proceed with slightly more detail.) To describe the bookkeeping for the reaction of Eq. (P3.4), we need to define our variables. I am using n_n, n_p, and n_D to mean the

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* Particle Data Group, Section 2: *Astrophysical Constants and Parameters* (2020), https://pdg.lbl.gov/2020/reviews/rpp2020-rev-astrophysical-constants.pdf.

† I exchanged email with Barbara Ryden about this after the first edition came out, and she said she would fix it in the next edition. She corrected Section 8.3, *The Physics of Recombination*, but did not follow through consistently.

number densities of free neutrons, free protons, and deuterium nuclei. n_b denotes the total baryon number density, so

$$n_b = n_n + n_p + 2n_D \ . \tag{P3.6}$$

Finally, I will use $n_n^{\rm TOT}$ and $n_p^{\rm TOT}$ to denote the total number densities of neutrons and protons respectively, whether free or bound inside deuterium. We assume that deuterium production happens fast enough so that there is no further change in the neutron-proton balance while deuterium is forming, so the ratio

$$f \equiv \frac{n_n^{\rm TOT}}{n_b} \tag{P3.7}$$

is fixed. We will describe the extent to which the reaction has proceeded by specifying the fraction x of neutrons that remain free,

$$x \equiv \frac{n_n}{n_n^{\rm TOT}} \ . \tag{P3.8}$$

Using these definitions, write the equation that equates the ratio $n_D/(n_p\, n_n)$ to a function of temperature, using Eq. (P3.1) for each of the number densities. The deuteron is spin-1, with $g = 3$, and the proton and neutron are each spin-$\frac{1}{2}$, with $g = 2$. Except in the exponential factor, you may approximate $m_n = m_p = m_D/2$. Manipulate this formula so that it has the form

$$F(\eta, f, x) = G(T) \ , \tag{P3.9}$$

where F and G are functions that you must determine. You will need the binding energy of deuterium,

$$B = (m_p + m_n - m_D)c^2 \approx 2.22 \text{ MeV}. \tag{P3.10}$$

Eq. (P3.9) determines x as a function of T, or vice versa, but we will not try to write $x(T)$ or $T(x)$ explicitly.

(c) *(5 points)* Using your result in part (b), and taking $f = 0.14$ from Weinberg's book, find the value of x, the fraction of neutrons that have been bound in deuterium, at the time of the Fifth Frame, when $T = 10^9$ K. You will probably want to solve the equation numerically. Two significant figures will be sufficient.

(d) *(5 points)* Again using your result from part (b), and assuming that $f = 0.14$ is still accurate, find the temperature at which $x = \frac{1}{2}$, i.e., the temperature for which half of the neutrons have become combined into deuterium. Again you will presumably find the answer numerically, and 2 significant figures will be sufficient. What is the value of kT at this temperature? Qualitatively, what feature of the calculation causes this number to be small compared to B?

PROBLEM 4: NEUTRINO NUMBER AND THE NEUTRON/ PROTON EQUILIBRIUM *(35 points)*

The following problem was 1998 Quiz 4, Problem 4. It was also posted in the 2018 Review Problems for Quiz 3, Problem 14. The solutions to these Review Problems are posted with the review problems on the class website, and you are invited to look at them. I urge you, however, to write your own solutions first, and then use the posted versions to check, and to revise if necessary.

In the standard treatment of big bang nucleosynthesis it is assumed that at early times the ratio of neutrons to protons is given by the Boltzmann formula,

$$\frac{n_n}{n_p} = e^{-\Delta E/kT} \ , \tag{P4.1}$$

where k is Boltzmann's constant, T is the temperature, and $\Delta E = 1.29$ MeV is the proton-neutron mass-energy difference. This formula is believed to be very accurate, but it assumes that the chemical potential for neutrons μ_n is the same as the chemical potential for protons μ_p.

(a) *(10 points)* Give the correct version of Eq. (P4.1), allowing for the possibility that $\mu_n \neq \mu_p$.

The equilibrium between protons and neutrons in the early universe is sustained mainly by the following reactions:

$$e^+ + n \longleftrightarrow p + \bar{\nu}_e$$

$$\nu_e + n \longleftrightarrow p + e^- \ .$$

Let μ_e and μ_ν denote the chemical potentials for the electrons (e^-) and the electron neutrinos (ν_e) respectively. The chemical potentials for the positrons (e^+) and the anti-electron neutrinos ($\bar{\nu}_e$) are then $-\mu_e$ and $-\mu_\nu$, respectively, since the chemical potential of a particle is always the negative of the chemical potential for the antiparticle.*

(b) *(10 points)* Express the neutron/proton chemical potential difference $\mu_n - \mu_p$ in terms of μ_e and μ_ν.

The black-body radiation formulas at the beginning of the quiz did not allow for the possibility of a chemical potential, but they can easily be generalized. For example, the formula for the number density n_i (of particles of type i) becomes

$$n_i = g_i^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{\mu_i/kT} \ .$$

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* This fact is a consequence of the principle that the chemical potential of a particle is the sum of the chemical potentials associated with its conserved quantities, while particle and antiparticle always have the opposite values of all conserved quantities.

(c) *(10 points)* Suppose that the density of anti-electron neutrinos $\bar{n}_\nu$ in the early universe was higher than the density of electron neutrinos n_ν. Express the thermal equilibrium value of the ratio n_n/n_p in terms of ΔE, T, and either the ratio $\bar{n}_\nu/n_\nu$ or the antineutrino excess $\Delta n = \bar{n}_\nu - n_\nu$. (Your answer may also contain fundamental constants, such as k, $\hbar$, and c.)

(d) *(5 points)* Would an excess of anti-electron neutrinos, as considered in part (c), increase or decrease the amount of helium that would be produced in the early universe? Explain your answer.

PROBLEM 5: IONIZATION OF HELIUM

(This problem is not required, but can be done for 20 points extra credit.)

This is Problem 8.3, p. 165, from Barbara Ryden's ***Introduction to Cosmology****.*

Imagine that at the time of recombination, the baryonic portion of the universe consisted entirely of ^{4}He (that is, helium with two protons and two neutrons in its nucleus). The ionization energy of helium (that is, the energy required to convert neutral He to He$^+$) is $Q_{\mathrm{He}} = 24.6$ eV. At what temperature would the fractional ionization of the helium be $X = \frac{1}{2}$. Assume that $\eta = 6 \times 10^{-10}$ and that the number density of He^{++} is negligibly small. [The relevant statistical weight factor for the ionization of helium is $g_{\mathrm{He}}/(g_e g_{\mathrm{He}^+}) = 1/4$.]

[Note added by Alan Guth: The g's that Ryden refers to here are what are called $\bar{g}$'s in the *Notes on Thermal Equilibrium.* They are not the same as either g or g^* as used in Lecture Notes 6. $\bar{g}_X$ simply counts the number of spin states of particle X, without any correction factor associated with fermions. Separating the three factors, $g_{\mathrm{He}} = 1$, $g_e = 2$, and $g_{\mathrm{He}^+} = 2$.]

Total points for Problem Set 8: 110, plus an optional 20 points of extra credit.