MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe November 19, 2022
Prof. Alan Guth

PROBLEM SET 9

DUE DATE: Saturday, December 3, 2022, at 12:00 noon.

READING ASSIGNMENT: *Inflation and the New Era of High-Precision Cosmology,* by Alan Guth, written for the MIT Physics Department annual newsletter, 2002. It is available at

https://physics.mit.edu/news/journal/physicsatmit_02_cosmology/

The data quoted in the article about the nonuniformities of the cosmic microwave background radiation has since been superceded by much better data, but the conclusions have not changed. They have only gotten stronger.

CALENDAR THROUGH THE END OF THE TERM:

NOVEMBER/DECEMBER 2022					
MON	TUES	WED	THURS	FRI	SAT
November 21 Class 21	22	23 Class 22 DROP Date	24 Thanksgiving	25 Thanksgiving	26
28 Class 23	29	30 Class 24	December 1	2	3 **PS9 due**
5 Class 25	6	7 Class 26 **Quiz 3**	8	9	10
12 Class 27	13	14 Class 28 **PS10 due** Last Class!	15	16 Final Exams	17
19 Final Exams	20 Final Exams	21 Final Exams	22 Final Exams	23	24

PROBLEM 1: BRIGHTNESS VS. REDSHIFT WITH A POSSIBLE COSMOLOGICAL CONSTANT *(25 points)*

In Lecture Notes 7, we derived the relation between the power output P of a source and the energy flux J, for the case of a closed universe:

$$J = \frac{P H_0^2 |\Omega_{k,0}|}{4\pi(1+z_S)^2 c^2 \sin^2 \psi(z_S)} \ ,$$

where

$$\psi(z_S) = \sqrt{|\Omega_{k,0}|} \int_0^{z_S} \frac{dz}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} \,.$$

Here z_S denotes the observed redshift, H_0 denotes the present value of the Hubble expansion rate, $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{vac},0}$ denote the present contributions to Ω from nonrelativistic matter, radiation, and vacuum energy, respectively, and $\Omega_{k,0} \equiv 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0}$.

(a) Derive the corresponding formula for the case of an open universe. You can of course follow the same logic as the derivation in the lecture notes, but the solution you write should be complete and self-contained. (I.e., you should **NOT** say "the derivation is the same as the lecture notes except for . . . .")

(b) Derive the corresponding formula for the case of a flat universe. Here there is of course no need to repeat anything that you have already done in part (a). If you wish you can start with the answer for an open or closed universe, taking the limit as $k \to 0$. The limit is delicate, however, because both the numerator and denominator of the equation for J vanish as $\Omega_{k,0} \to 0$.

PROBLEM 2: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF *(20 points)*

The following problem was Problem 8 of Review Problems for Quiz 3 of 2011, and the solution is posted as http://web.mit.edu/8.286/www/quiz11/ecqr3-1.pdf. You are invited to look at these solutions. I urge you, however, to write your own solutions first, and then use the posted versions to check, and to revise if necessary.

Consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and ***mysterious stuff.****** Mysterious stuff has the property that if it fills a container of volume V, which is allowed to expand, the mass density of the mysterious stuff falls off as $1/\sqrt{V}$. It follows that in an expanding universe the mass density of mysterious stuff falls off as $1/a^{3/2}(t)$.

Suppose that you are given the present value of the Hubble expansion rate H_0, and also the present values of the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\text{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe t_0 in terms of these quantities.

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* The same ***mysterious stuff*** was also the subject of Problem 7 of Review Problems for Quiz 3, from 2011, which you may want to look at. This problem, however, can be solved without any reference to that problem.

(a) *(10 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{a(t)}{a(t_0)}$$

for an arbitrary time t. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(b) *(10 points)* Write an integral expression for the age of the universe t_0. The expression should depend only on H_0 and the various contributions to Ω_0 listed above ($\Omega_{m,0}$, $\Omega_{r,0}$, etc.), but it might include x as a variable of integration.

PROBLEM 3: SHARED CAUSAL PAST *(20 points)*

Some years ago several of my colleagues published a paper (Andrew S. Friedman, David I. Kaiser, and Jason Gallicchio, "The Shared Causal Pasts and Futures of Cosmological Events," *Physical Review D,* Vol. 88, article 044038 (2013) [arXiv:1305.3943]), in which they investigated the causal connections in the standard cosmological model. In particular, they calculated the present redshift z of a distant quasar which has the property that a light signal, if sent from our own location at the instant of the big bang, would have just enough time to reach the quasar and return to us, so that we could see the reflection of the signal at the present time. They found $z = 3.65$, using $\Omega_{\text{matter},0} = 0.315$, $\Omega_{\text{rad},0} = 9.29 \times 10^{-5}$, $\Omega_{\text{vac},0} = 0.685 - \Omega_{\text{rad},0}$, and $H_0 = 67.3$ km-s^{-1}-Mpc^{-1}. Feel free to read their paper if you like. Your job, however, is to carry out an independent calculation to find out if they got it right.

(a) *(15 points)* Write an equation that determines this redshift z. The equation may involve one or more integrals which are not evaluated, and the equation itself does not have to be solved to determine z.

(b) *(5 points)* The integrals that should appear in your answer to part (a) can be evaluated numerically, and the whole equation you found in part (a) can be solved numerically to find z. Do this, and see how your z compares with 3.65.

PROBLEM 4: MASS DENSITY OF VACUUM FLUCTUATIONS *(25 points)*

The energy density of vacuum fluctuations has been discussed qualitatively in lecture. In this problem we will calculate in detail the energy density associated with quantum fluctuations of the electromagnetic field. To keep the problem finite, we will not consider all of space at once, but instead we will consider the electromagnetic field inside a cube of side L, defined by coordinates

$$0 \leq x \leq L \; ,$$

$$0 \leq y \leq L \; ,$$

$$0 \leq z \leq L \; .$$

Our goal, however, will be to compute the energy density in the limit as the size of the box is taken to infinity.

(a) *(10 points)* The electromagnetic waves inside the box can be decomposed into a Fourier sum of sinusoidal normal modes. Suppose we consider only modes that extend up to a maximum wave number $k_{\rm max}$, or equivalently modes that extend down to a minimum wavelength $\lambda_{\rm min}$, where

$$k_{\rm max} = \frac{2\pi}{\lambda_{\rm min}} \ .$$

How many such modes are there? I do not expect an exact answer, but your approximations should become arbitrarily accurate when $\lambda_{\rm min} \ll L$. (These mode counting techniques are probably familiar to many of you, but in case they are not I have attached an extended hint after part (c).)

(b) *(10 points)* When the electromagnetic field is described quantum mechanically, each normal mode behaves exactly as a harmonic oscillator: if the angular frequency of the mode is ω, then the quantized energy levels have energies given by

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega \ ,$$

where $\hbar$ is Planck's original constant divided by 2π, and n is an integer. The integer n is called the "occupation number," and is interpreted as the number of photons in the specified mode. The minimum energy is not zero, but instead is $\frac{1}{2}\hbar\omega$, which is the energy of the quantum fluctuations of the electromagnetic field. Assuming that the mode sum is cut off at $\lambda_{\rm min}$ equal to the Planck length ($\lambda_{\rm Planck} \equiv \sqrt{\hbar G/c^3}$, as defined in Lecture Notes 7), what is the total mass density of these quantum fluctuations?

(c) *(5 points)* How does the mass density of the quantum fluctuations of the electromagnetic field compare with the critical density of our universe?

Extended Hint:

The electromagnetic fields inside a closed box can be expanded as the sum of modes, each of which has a sinusoidally varying time dependence, but the precise form of these modes depends on the nature of the boundary conditions on the walls of the box. Physically reasonable boundary conditions, such as total reflection, are in fact difficult to use in calculations. However, it is known that in the limit of an infinite-sized box, the nature of the boundary conditions will not make any difference. We are therefore free to choose the simplest boundary conditions that we can imagine, and for this purpose we will choose periodic boundary conditions. That is, we will assume that the fields and their normal derivatives on each wall are fixed to precisely match the fields and their normal derivatives on the opposite wall.

To begin, we consider a wave in one dimension, moving at the speed of light. Such waves are most easily described in terms of complex exponentials. If $A(x,t)$ represents the amplitude of the wave, then a sinusoidal wave moving in the positive x-direction can be written as

$$A(x,t) = \text{Re}\left[Be^{ik(x-ct)}\right] ,$$

where B is a complex constant and k is a real constant. Defining $\omega = c|k|$, waves in either direction can be written as

$$A(x,t) = \text{Re}\left[Be^{i(kx-\omega t)}\right] ,$$

where the sign of k determines the direction. To be periodic with period L, the parameter k must satisfy

$$kL = 2\pi n ,$$

where n is an integer. The spacing between modes is therefore $\Delta k = 2\pi/L$. The density of modes dN/dk (i.e., the number of modes per interval of k) is then one divided by the spacing, or $1/\Delta k$, so

$$\frac{dN}{dk} = \frac{L}{2\pi} \quad \text{(one dimension)} .$$

In three dimensions, a sinusoidal wave can be written as

$$A(\vec{x},t) = \text{Re}\left[Be^{i(\vec{k}\cdot\vec{x}-\omega t)}\right] ,$$

where $\omega = c|\vec{k}|$, and

$$k_x L = 2\pi n_x \ , \quad k_y L = 2\pi n_y \ , \quad k_z L = 2\pi n_z \ ,$$

where n_x, n_y, and n_z are integers. Thus, in three-dimensional $\vec{k}$-space the allowed values of $\vec{k}$ lie on a cubical lattice, with spacing $2\pi/L$. In counting the modes, one should also remember that for photons there is an extra factor of 2 associated with the fact that electromagnetic waves have two possible polarizations for each allowed value of $\vec{k}$.

PROBLEM 5: PLOTTING THE SUPERNOVA DATA *(EXTRA CREDIT, 20 pts)*

The original data on the Hubble diagram based on Type Ia supernovae are found in two papers. One paper is authored by the High Z Supernova Search Team,* and the other

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* http://arXiv.org/abs/astro-ph/9805201, later published as Riess *et al.*, *Astronomical Journal* **116**, 1009 (1998).

is by the Supernova Cosmology Project.† More recent data from the High Z team, which includes many more data points, can be found in Riess *et al.*, http://arXiv.org/abs/astro-ph/0402512.¶ (By the way, the lead author Adam Riess was an MIT undergraduate physics major, graduating in 1992.)

You are asked to plot the data from either the 2nd or 3rd of these papers, and to include on the graph the theoretical predictions for several cosmological models.

The plot will be similar to the plots contained in these papers, and to the plot on p. 121 of Ryden's book, showing a graph of (corrected) magnitude m vs. redshift z. Your graph should include the error bars. If you plot the Perlmutter *et al.* data, you will be plotting "effective magnitude" m vs. redshift z. The magnitude is related to the flux J of the observed radiation by $m = -\frac{5}{2}\log_{10}(J) + \text{const}$. The value of the constant in this expression will not be needed. The word "corrected" refers both to corrections related to the spectral sensitivity of the detectors and to the brightness of the supernova explosions themselves. That is, the supernova at various distances are observed with different redshifts, and hence one must apply corrections if the detectors used to measure the radiation do not have the same sensitivity at all wavelengths. In addition, to improve the uniformity of the supernova as standard candles, the astronomers apply a correction based on the duration of the light output. Note that our ignorance of the absolute brightness of the supernova, of the precise value of the Hubble constant, and of the constant that appears in the definition of magnitude all combine to give an unknown but constant contribution to the predicted magnitudes. The consequence is that you will be able to move your predicted curves up or down (i.e., translate them by a fixed distance along the m axis). You should choose the vertical positioning of your curve to optimize your fit, either by eyeball or by some more systematic method.

If you choose to plot the data from the 3rd paper, Riess *et al.* 2004, then you should see the note at the end of this problem.

For your convenience, the magnitudes and redshifts for the Supernova Cosmology Project paper, from Tables 1 and 2, are summarized in a text file on the 8.286 web page. The data from Table 5 of the Riess *et al.* 2004 paper, mentioned above, is also posted on the 8.286 web page.

For the cosmological models to plot, you should include:

(i) A matter-dominated universe with $\Omega_m = 1$.

(ii) An open universe, with $\Omega_{m,0} = 0.3$.

(iii) A universe with $\Omega_{m,0} = 0.3$ and a cosmological constant, with $\Omega_{\text{vac},0} = 0.7$. (If you prefer to avoid the flat case, you can use $\Omega_{\text{vac},0} = 0.6$. Or, if you want to compare

† http://arXiv.org/abs/astro-ph/9812133, later published as Perlmutter *et al.*, *Astrophysical Journal* **517**:565–586 (1999).

¶ Published as *Astrophysical Journal* **607**:665-687 (2004).

directly with Figure 4 of the Riess *et al.* (2004) paper, you should use $\Omega_{m,0} = 0.29$, $\Omega_{\text{vac},0} = 0.71$.)

You may include any other models if they interest you. You can draw the plot with either a linear or a logarithmic scale in z. I would recommend extending your theoretical plot to $z = 3$, if you do it logarithmically, or $z = 2$ if you do it linearly, even though the data does not go out that far. That way you can see what possible knowledge can be gained by data at higher redshift.

NOTE FOR THOSE PLOTTING DATA FROM RIESS ET AL. 2004:

Unlike the Perlmutter *et al.* data, the Riess *et al.* data is expressed in terms of the distance modulus, which is a direct measure of the luminosity distance. The distance modulus is defined both in the Riess *et al.* paper and in Ryden's book (p. 118) as

$$\mu = 5 \log_{10} \left(\frac{d_L}{1 \text{ Mpc}} \right) + 25 \ ,$$

where Ryden uses the notation $m - M$ for the distance modulus, and d_L is the luminosity distance. The luminosity distance, in turn, is really a measure of the observed brightness of the object. It is defined as the distance that the object would have to be located to result in the observed brightness, if we were living in a static Euclidean universe. More explicitly, if we lived in a static Euclidean universe and an object radiated power P in a spherically symmetric pattern, then the energy flux J at a distance d would be

$$J = \frac{P}{4\pi d^2} \ .$$

That is, the power would be distributed uniformly over the surface of a sphere at radius d. The luminosity distance is therefore defined as

$$d_L = \sqrt{\frac{P}{4\pi J}} \ .$$

Thus, a specified value of the distance modulus μ implies a definite value of the ratio J/P.

In plotting a theoretical curve, you will need to choose a value for H_0. Riess *et al.* do not specify what value they used, but I found that their curve is most closely reproduced if I choose $H_0 = 66$ km-sec^{-1}-Mpc^{-1}. This seems a little on the low side, since the value is usually estimated as 70–72 km-sec^{-1}-Mpc^{-1}, but Riess *et al.* emphasize that they were not concerned with this value. They were concerned with the relative values of the distance moduli, and hence the shape of the graph of the distance modulus vs. z. In their own words, from Appendix A, "The zeropoint, distance scale, absolute magnitude of the fiducial SN Ia or Hubble constant derived from Table 5 are all closely related (or even equivalent) quantities which were arbitrarily set for the sample presented here. Their correct value is not relevant for the analyses presented which only make use of differences between SN Ia magnitudes. Thus the analysis are independent of the aforementioned normalization parameters."

PROBLEM 6: THE HORIZON PROBLEM *(20 points)*

The success of the big bang predictions for the abundances of the light elements suggests that the universe was already in thermal equilibrium at one second after the big bang. At this time, the region that later evolves to become the observed universe was, in the context of the conventional (non-inflationary) cosmological model, many horizon distances across. Try to estimate how many. You may assume that the universe is flat, that it was radiation-dominated for $t \lesssim 50{,}000$ yr, and for this crude estimate you can also assume that it has been matter-dominated for all $t \gtrsim 50{,}000$ yr, and that $a(t)T(t)$ was about equal to its present value for the whole period from 1 second to the present.

PROBLEM 7: THE FLATNESS PROBLEM *(20 points)*

Although we now know that $\Omega_0 = 1$ to an accuracy of about half a percent, let us pretend that the value of Ω today is 0.1. It nonetheless follows that at 10^{-37} second after the big bang (about the time of the grand unified theory phase transition), Ω must have been extraordinarily close to one. Writing $\Omega = 1 - \delta$, estimate the value of δ at $t = 10^{-37}$ sec (using the standard cosmological model). You may again use any of the approximations mentioned in Problem 6, except of course the assumption that the universe is flat.

Total points for Problem Set 9: 130, plus an optional 20 points of extra credit.