MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 16, 2000
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QUIZ 2

USEFUL INFORMATION:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Flat ($\Omega \equiv \rho/\rho_c = 1$): $R(t) \propto t^{2/3}$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

NOTE: Any answer may be expressed in terms of symbols representing the answers to previous parts of the same question, whether or not the previous part was answered correctly. When I say that an answer should be expressed in terms of a specified set of variables, I mean that the variables used in your answer should be drawn from that set; this does not imply that all the variables in the set need be used.

PROBLEM 1: DID YOU DO THE READING? *(30 points)*

(a) *(5 points)* One can do astronomy with detectors besides those that detect electromagnetic radiation. What is the brightest source of detectable *neutrinos* in the sky?

- (i) the radioisotope reactor on the Galileo space probe
- (ii) the Sun
- (iii) the Crab pulsar
- (iv) Sagittarius A (the center of our Galaxy)
- (v) the quasar 3C 273

(b) *(5 points)* To within an order of magnitude, what is the wavelength of the spectral peak of the cosmic microwave background radiation? If you do not remember this, you ought to be able to calculate it. Helpful constants include the speed of light $c = 3 \times 10^8$ m/s, Planck's constant $h = 7 \times 10^{-34}$ J-s, Boltzmann's constant $k = 1 \times 10^{-23}$ J/K, the approximate wavelength of visible light $\lambda_{visible} = 500$ nm (nanometer), and the temperature of the cosmic microwave background radiation $T = 2.7$ K.

(c) *(5 points)* What is the best (i.e. current) interpretation of the Hubble sequence of galaxy types?

- (i) Spirals evolve into ellipticals as their spiral density waves are damped by dust.
- (ii) Ellipticals evolve into spirals as they use up all of their gas.
- (iii) All galaxies form as type S0, and evolve down either the spiral or the elliptical branch depending on their mass.
- (iv) A galaxy's type is determined by its star formation rate.
- (v) A galaxy's type is determined by its mass.

(d) *(5 points)* Roughly how many galaxies does the typical 'rich' galaxy cluster contain? What is the typical mass of such a cluster (in solar masses)?

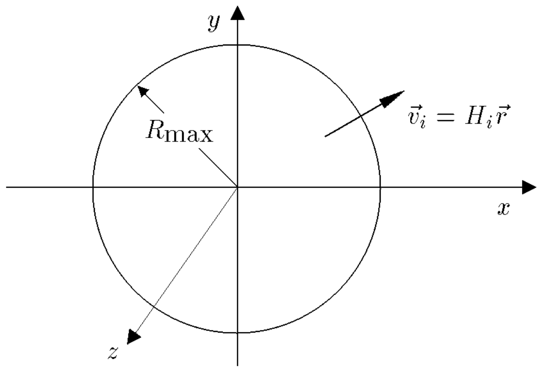
(e) *(5 points)* The distances to tens of thousands of nearby stars, out to a distance of several hundred parsecs, have been measured by which of the following methods:

- (i) radar reflection
- (ii) gravitational lensing
- (iii) parallax achieved by comparing measurements from two telescopes separated by thousands of miles
- (iv) mapping of the 21-cm neutral hydrogen line
- (v) using Cepheid variables
- (vi) parallax achieved by comparing satellite measurements made from different points in the Earth's orbit

(f) *(5 points)* What is the resolution of Olber's paradox?

PROBLEM 2: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(35 points)*

In Lecture Notes 4 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density ρ_i and an initial pattern of velocities corresponding to Hubble expansion: $\vec{v}_i = H_i \vec{r}$:



We denoted the radius at time t of a particle which started at radius r_i by the function $r(r_i, t)$. Assuming Newton's law of gravity, we concluded that each particle would experience an acceleration given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i, t)}\,\hat{r} \ ,$$

where $M(r_i)$ denotes the total mass contained initially in the region $r < r_i$, given by

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i \ .$$

Suppose that the law of gravity is modified to contain a new, repulsive term, producing an acceleration which grows as the nth power of the distance, with a strength that is independent of the mass. That is, suppose $\vec{g}$ is given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i, t)}\,\hat{r} + \gamma r^n(r_i, t)\,\hat{r} \ ,$$

where γ is a constant. The function $r(r_i, t)$ then obeys the differential equation

$$\ddot{r} = -\frac{GM(r_i)}{r^2(r_i, t)} + \gamma r^n(r_i, t) \ .$$

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a) *(6 points)* As done in the lecture notes, we define

$$u(r_i, t) \equiv r(r_i, t)/r_i \ .$$

Write the differential equation obeyed by u.

b) *(6 points)* For what value of the power n is the differential equation found in part (a) independent of r_i?

c) *(8 points)* Write the initial conditions for u which, when combined with the differential equation found in (a), uniquely determine the function u.

d) *(15 points)* If all is going well, then you have learned that for a certain value of n, the function $u(r_i, t)$ will in fact not depend on r_i, so we can define

$$R(t) \equiv u(r_i, t) \ .$$

Show that the differential equation for R can be integrated once to obtain an equation related to the conservation of energy. The desired equation should include terms depending on R and $\dot{R}$, but not $\ddot{R}$ or any higher derivatives.

If all has not gone so well, you may not know what differential equation $R(t)$ obeys. If that is the case, then for a maximum of 12 points you can consider the generic equation

$$\ddot{R} + \frac{A}{R^p} + BR^q = 0 \ ,$$

where A and B are arbitrary constants, and p and q are positive integers, with $p > 1$. Show that this equation can be integrated once, as described in the previous paragraph.

PROBLEM 3: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(35 points)*

The following problem was Problem 4, Problem Set 2, 2000, with one part omitted.

The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 5 as

$$ct = \alpha\,(\sinh\theta - \theta)$$

and

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

where α is a constant with units of length. The following mathematical identities, which you should know, may also prove useful:

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots\ .$$

a) *(5 points)* Find the Hubble "constant" H as a function of α and θ.

b) *(5 points)* Find the mass density ρ as a function of α and θ.

c) *(5 points)* Find the physical value of the horizon distance, $\ell_{p,\mathrm{horizon}}$, as a function of α and θ.

d) *(10 points)* For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then R as a function of t. Give the expression for $R(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

e) *(10 points)* Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1-\Omega$ behaves as a power of t. Find the expression for $1-\Omega$ in this approximation.