MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 13, 2000
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QUIZ 3

USEFUL INFORMATION:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{k^{3/2} c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{\kappa^{3/2} c^2} \ ,$$

$$\kappa \equiv -k \ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\,d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

PHYSICAL CONSTANTS:

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\,\text{erg/K}$$

$$= 8.617 \times 10^{-5}\,\text{eV/K} \ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27} \text{ erg-sec}$$
$$= 6.582 \times 10^{-16} \text{ eV-sec} ,$$
$$c = 2.998 \times 10^{10} \text{ cm/sec}$$
$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg} .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3} , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions} , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3c^5}{16\pi^3 gG}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

NOTE: Any answer may be expressed in terms of symbols representing the answers to previous parts of the same question, whether or not the previous part was answered correctly. When I say that an answer should be expressed in terms of a specified set of variables, I mean that the variables used in your answer should be drawn from that set; this does not imply that all the variables in the set need be used.

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(5 points)* What does Birkhoff's theorem state?

(i) A uniform medium outside a spherical cavity has no gravitational effect inside the cavity.

(ii) A redshifted version of a blackbody spectrum remains a blackbody spectrum, but at a lower temperature.

(iii) The universe is homogeneous and isotropic.

(iv) The effect of a uniform gravitational field is indistinguishable from the effect of a uniform acceleration.

(v) A Hubble flow is the only global motion allowed in a completely homogeneous and isotropic universe.

(b) *(5 points)* Two special-case cosmological models are the "Milne model" and the "Einstein de-Sitter model". Pick one of them and briefly describe its distinguishing characteristics from other common models (be sure in your answer to specify which one you are describing).

(c) *(5 points)* The observation that only about one-quarter of the primordial gas in the universe is helium means that the Big Bang appears to have produced about seven times as many protons as neutrons. What can help explain this asymmetry?

(i) The very energetic conditions of the early universe forced the GUT proton decay process to run in reverse.

(ii) The early population of muons preferentially decayed into protons, boosting their density.

(iii) The neutron is heavier than the proton, causing the weak reaction rates to shift as the temperature dropped.

(iv) The rest of the neutrons formed into neutron stars, and thus aren't observed in the primordial gas at all.

(v) The same asymmetry which gave us more particles than antiparticles also produced more down quarks than up quarks.

(d) *(5 points)* What causes the dipole anisotropy in the cosmic microwave background radiation?

(e) *(5 points)* Place the following events in order in the standard Big Bang picture, from earliest to latest. A valid answer would read, for instance: v, iv, iii, ii, i.

(i) Primordial nucleosynthesis

(ii) Decoupling of electron neutrinos

(iii) Quark confinement

(iv) Recombination

(v) Muon annihiliation

PROBLEM 2: FREEZE-OUT OF MUONS *(35 points)*

The following problem was Problem 13 from the Review Problems for Quiz 3:

A particle called the muon seems to be essentially identical to the electron, except that it is heavier— the mass/energy of a muon is 106 MeV, compared to 0.511 MeV for the electron. The muon (μ^-) has the same charge as an electron, denoted by $-e$. There is also an antimuon (μ^+), analogous to the positron, with charge $+e$. The muon and antimuon have the same spin as the electron. There is no known particle with a mass between that of an electron and that of a muon.

(a) *(10 points)* The black-body radiation formula, as shown at the front of this quiz, is written in terms of a normalization constant g. What is the value of g for the muons, taking μ^+ and μ^- together?

(b) *(10 points)* When kT is just above 106 MeV as the universe cools, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the contribution to g from each of these particles?

(c) *(15 points)* As kT falls below 106 MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting R denote the Robertson-Walker scale factor, by what factor does the quantity RT increase when the muons disappear?

PROBLEM 3: GEODESICS IN A CLOSED UNIVERSE (*40 points + 5 points extra credit*)

Consider the case of closed Robertson-Walker universe. Taking $k = 1$, the spacetime metric can be written in the form

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} .$$

We will assume that this metric is given, and that $R(t)$ has been specified. While galaxies are approximately stationary in the comoving coordinate system described by this metric, we can still consider an object that moves in this system. In particular, in this problem we will consider an object that is moving in the radial direction (r-direction), under the influence of no forces other than gravity. Hence the object will travel on a geodesic.

(a) *(7 points)* Express $d\tau/dt$ in terms of dr/dt.

(b) *(3 points)* Express $dt/d\tau$ in terms of dr/dt.

(c) *(10 points)* If the object travels on a trajectory given by the function $r_p(t)$ between some time t_1 and some later time t_2, write an integral which gives the total amount of time that a clock attached to the object would record for this journey.

(d) *(10 points)* During a time interval dt, the object will move a coordinate distance

$$dr = \frac{dr}{dt} dt .$$

Let $d\ell$ denote the physical distance that the object moves during this time. By "physical distance," I mean the distance that would be measured by a comoving observer (an observer stationary with respect to the coordinate system) who is located at the same point. The quantity $d\ell/dt$ can be regarded as the physical speed v_{phys} of the object, since it is the speed that would be measured by a comoving observer. Write an expression for v_{phys} as a function of dr/dt and r.

(e) *(10 points)* Using the formulas at the front of the exam, derive the geodesic equation of motion for the coordinate r of the object. Specifically, you should derive an equation of the form

$$\frac{d}{d\tau}\left[A \frac{dr}{d\tau} \right] = B \left(\frac{dt}{d\tau} \right)^2 + C \left(\frac{dr}{d\tau} \right)^2 + D \left(\frac{d\theta}{d\tau} \right)^2 + E \left(\frac{d\phi}{d\tau} \right)^2 ,$$

where A, B, C, D, and E are functions of the coordinates, some of which might be zero.

— *CONTINUED* —

(f) *(5 points EXTRA CREDIT)* On Problem 4 of Problem Set 3 we learned that in a flat Robertson-Walker metric, the relativistically defined momentum of a particle,

$$p = \frac{m v_{\rm phys}}{\sqrt{1 - \frac{v_{\rm phys}^2}{c^2}}} \ ,$$

falls off as $1/R(t)$. Use the geodesic equation derived in part (e) to show that the same is true in a closed universe.