MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — May 11, 2000
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QUIZ 4

USEFUL INFORMATION:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\,\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\,\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2\ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\rm vac} = -\rho_{\rm vac}c^2 \qquad \rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$\begin{aligned} k &= \text{Boltzmann's constant} = 1.381 \times 10^{-16}\,\text{erg/K} \\ &= 8.617 \times 10^{-5}\,\text{eV/K}\ , \end{aligned}$$

$$\begin{aligned} \hbar &= \frac{h}{2\pi} = 1.055 \times 10^{-27}\ \text{erg-sec} \\ &= 6.582 \times 10^{-16}\ \text{eV-sec}\ , \end{aligned}$$

$$c = 2.998 \times 10^{10}\ \text{cm/sec}$$

$$G = 6.672 \times 10^{-8}\ \text{cm}^3 \cdot \text{gm}^{-1} \cdot \text{sec}^{-2}$$

$$1\ \text{eV} = 1.602 \times 10^{-12}\ \text{erg}\ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G} \right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

PARTICLE PROPERTIES:

While working on this exam you may refer to any of the tables in Lecture Notes 11. Please bring your copy of Lecture Notes 11 with you to the exam.

NOTE: Any answer may be expressed in terms of symbols representing the answers to previous parts of the same question, whether or not the previous part was answered correctly. When I say that an answer should be expressed in terms of a specified set of variables, I mean that the variables used in your answer should be drawn from that set; this does not imply that all the variables in the set need be used.

PROBLEM 1: DID YOU DO THE READING? *(30 points)*

(a) *(5 points)* When orbital velocities of stars in spiral galaxies are measured, we find that they are mostly constant over a large range in radius. What explanation is usually given to understand these flat rotation curves?

(i) The density waves producing the spiral arms perturb the stellar orbits.

(ii) A flat rotation curve is exactly what you'd expect from Kepler's laws applied to the observed mass profile of spiral galaxies.

(iii) The measurements are dominated by bright young stars in the spiral arms, so we're mistaking the wave velocity of the arms for the rotation of the galaxy as a whole.

(iv) Spiral galaxies contain a halo of dark matter in addition to their normal disk mass.

(v) The stellar orbits aren't circular, so we're measuring stars with more and more elliptical orbits at larger radii.

(b) *(5 points)* Briefly describe the distinguishing characteristics of the Eddington-Lemaître cosmological models. (Hint: they are related to Einstein's static closed universe model.)

(c) *(5 points)* What is the Jeans length?

(i) The size at which the sound-crossing time is equal to the age of the universe

(ii) The minimum size of density fluctuations which are unstable to gravitational collapse

(iii) The size of the first peak in the power spectrum of the cosmic microwave background fluctuations

(iv) The size where we expect the effects of quantum gravity to have a significant influence

(v) Approximately equal to the Jeans waist size

(d) *(5 points)* By what factor does the lepton number per comoving volume of the universe change between temperatures of $kT = 10$ MeV and $kT = 0.1$ MeV? You should assume the existence of the normal three species of neutrinos for your answer.

(e) *(5 points)* Measurements of the primordial deuterium abundance would give good constraints on the baryon density of the universe. However, this abundance is hard to measure accurately. Which of the following is NOT a reason why this is hard to do?

(i) The neutron in a deuterium nucleus decays on the time scale of 15 minutes, so almost none of the primordial deuterium produced in the Big Bang is still present.

(ii) The deuterium abundance in the Earth's oceans is biased because, being heavier, less deuterium than hydrogen would have escaped from the Earth's surface.

(iii) The deuterium abundance in the Sun is biased because nuclear reactions tend to destroy it by converting it into helium-3.

(iv) The spectral lines of deuterium are almost identical with those of hydrogen, so deuterium signatures tend to get washed out in spectra of primordial gas clouds.

(v) The deuterium abundance is so small (a few parts per million) that it can be easily changed by astrophysical processes other than primordial nucleosynthesis.

(f) *(5 points)* Give three examples of hadrons.

PROBLEM 2: TIME SCALES IN COSMOLOGY *(20 points)*

The following problem was on the Review Problems for Quiz 4, although a few numbers have been updated for this quiz.

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-37} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) *(4 points)* the beginning of the processes involved in big bang nucleosynthesis;

(b) *(4 points)* the end of the processes involved in big bang nucleosynthesis;

(c) *(4 points)* the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{16}$ GeV;

(d) *(4 points)* "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) *(4 points)* the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

PROBLEM 3: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS *(20 points)*

(a) *(5 points)* When the temperature of the early universe was 5×10^{10} K, what was the ratio of neutrons to protons? You may assume thermal equilibrium, and that the mass difference is given by $(m_n - m_p)c^2 = 1.293$ MeV.

Questions (b), (c), and (d) all refer to calculations that describe a hypothetical world, which differs from the real world in a specified way. In each case you are asked about the calculation of the predicted helium abundance for this hypothetical world. Each of these three parts are to be answered independently; that is, in each part you are to consider a hypothetical world that differs from the real world only by the characteristics stated in that part.

Part (b) was taken from the Review Problems for Quiz 4.

(b) *(5 points)* Suppose the proton-neutron mass difference were larger than the actual value of 1.293 MeV/c^2. Would the predicted helium abundance be larger or smaller than in the standard calculation? Explain your answer in a sentence, or in a few sentences.

(c) *(5 points)* Suppose that the nucleosynthesis calculations were carried out with an electron mass given by $m_e c^2 = 1$ KeV, instead of the physical value of 0.511 MeV. Would this calculation predict a larger or smaller helium abundance than the standard calculation? Explain your answer in a sentence, or in a few sentences.

(d) *(5 points)* Suppose, due to some significant difference in the nuclear reaction rates, that nucleosynthesis occurred suddenly at a temperature of 5×10^{10} K. In that case, what would be the predicted value of Y, the fraction of the baryonic mass density of the universe which is helium?

PROBLEM 4: INFLATION AND THE HORIZON PROBLEM *(30 points)*

(a) *(5 points)* Consider two photons of the cosmic background radiation that are arriving at Earth from opposite directions in the sky. Such photons are observed to have the same energy to within one part in N. What is the numerical value of N? (Answers within a factor of 10 of the intended answer will be accepted for full credit.)

(b) *(5 points)* In the standard cosmological model (i.e., without inflation), the two photons described in the previous part can be traced back to points in spacetime that were separated from each other by a number of horizon lengths. By approximately how many horizon lengths were these points separated? (If you remember this number, you can quote it from memory, and any number within a factor of 10 of the intended answer will be considered correct. If you don't remember it, you can calculate it from first principles, using the fact that the photons were released when the temperature of the universe was about 4000 K, and that the present temperature is 2.7 K.)

(c) *(5 points)* In inflationary cosmology, the "horizon problem" described in the previous part disappears. In a few sentences, explain why.

(d) *(15 points)* Assuming that a grand unified theory phase transition occurred at a temperature of $kT = M_{\rm GUT} \simeq 10^{16}$ GeV, calculate the ratio of the diameter at that time of the region that will evolve to become the observed universe to the horizon length at that time. Work in the approximation that the universe is described by a flat Robertson-Walker metric.