MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — February 28, 2000
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 1

Corrected Version

QUIZ DATE: Tuesday, February 29, 2000

COVERAGE: Lecture Notes 1 (sections on the Doppler shift only); Lecture Notes 3; Problem Set 1; Weinberg, Chapters 1-3. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignment, or from this set of Review Problems.**

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. Except for a few parts which are clearly marked, they are all problems that I would consider fair for the coming quiz. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1996 and 1998. The relevant problems from those quizzes have been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. You should be aware that the 1996 quiz was given slightly later in the term than the first quiz this year, so it covered a bit more material than the coming quiz will cover. In particular, Problem 2 from that quiz would not be appropriate for Quiz 1 of this year.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" at the beginning, but for the first quiz there seems to be only a few equations that are appropriate:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(relativistic)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

These formulas will be included at the start of this year's quiz.

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following problem was Problem 1, Quiz 1, 1994:

The following questions are worth 5 points each:

a) In 1750 the English instrument maker Thomas Wright published *Original Theory or New Hypothesis of the Universe.* In this book Wright described an astronomical object that is known today as the Crab Nebula, the solar system, the Milky Way, or the local supercluster?

b) In 1755 Immanuel Kant published his *Universal Natural History and Theory of the Heavens.* What new hypothesis was put forward in this book?

c) Estimate the diameter and thickness of the disk of the Milky Way galaxy. Any numbers within a factor of 2 of those given in Weinberg's book will be accepted.

d) The mathematical theory of an expanding universe was first published in 1922 by the Russian mathematician Alexandre Friedmann, the Dutch Astronomer Willem de Sitter, the American astronomer Edwin Hubble, or the Belgian cleric Georges Lemaître?

e) After discovering an inexplicable hiss coming from their radio telescope, Arno Penzias and Robert Wilson of Bell Laboratories learned that P.J.E. Peebles, a Princeton theorist, had calculated that the big bang would produce a background of cosmic radiation with a temperature today of 10° K. What MIT radio astronomer informed them of Peebles' work?

PROBLEM 2: AN EXPONENTIALLY EXPANDING UNIVERSE *(20 points)*

The following problem was Problem 2, Quiz 2, 1994, and had also appeared on the 1994 Review Problems. As is the case this year, it was announced that one of the problems on the quiz would come from either the homework or the Review Problems.

Consider a flat (i.e., a $k = 0$, or a Euclidean) universe with scale factor given by

$$R(t) = R_0 e^{\chi t} ,$$

where R_0 and χ are constants.

(a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

(b) *(5 points)* Let (x, y, z, t) be the coordinates of a comoving coordinate system. Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive x-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) *(5 points)* Suppose that we are living on a galaxy along the positive x-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift z. Express the time t_r at which we receive the pulse in terms of z, χ, and any relevant physical constants.

(d) *(5 points)* At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of z, χ, and any relevant physical constants.

PROBLEM 3: "DID YOU DO THE READING?"

(a) The assumptions of homogeneity and isotropy greatly simplify the description of our universe. We find that there are three possibilities for a homogeneous and isotropic universe: an open universe, a flat universe, and a closed universe. What quantity or condition distinguishes between these three cases: the temperature of the microwave background, the value of $\Omega = \rho/\rho_c$, matter vs. radiation domination, or redshift? *[Note for 2000: this question is beyond the material for Quiz 1 of this year.]*

(b) What is the temperature, in Kelvin, of the cosmic microwave background today?

(c) Which of the following supports the hypothesis that the universe is isotropic: the distances to nearby clusters, observations of the cosmic microwave background, clustering of galaxies on large scales, or the age and distribution of globular clusters?

(d) Is the distance to the Andromeda Nebula (roughly) 10 kpc, 5 billion light years, 2 million light years, or 3 light years?

(e) Did Hubble discover the law which bears his name in 1862, 1880, 1906, 1929, or 1948?

(f) When Hubble measured the value of his constant, he found $H^{-1} \approx 100$ million years, 2 billion years, 10 billion years, or 20 billion years?

(g) Cepheid variables are important to cosmology because they can be used to estimate the distances to the nearby galaxies. What property of Cepheid variables makes them useful for this purpose, and how are they used?

(h) Cepheid variable stars can be used as estimators of distance for distances up to about 100 light-years, 10^4 light-years, 10^7 light years, or 10^{10} light-years? *[Note for 2000: this question is beyond the material for Quiz 1 of this year.]*

(i) Name the two men who in 1964 discovered the cosmic background radiation. With what institution were they affiliated?

(j) At the time of the discovery of the cosmic microwave background, an active but independent effort was taking place elsewhere. P.J.E. Peebles had estimated that the universe must contain background radiation with a temperature of at least 10°K, and Robert H. Dicke, P.G. Roll, and D.T. Wilkinson had mounted an experiment to look for it. At what institution were these people working?

PROBLEM 4: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The following problem was Problem 3, Quiz 2, 1988:

Consider a flat universe filled with a new and peculiar form of matter, with a Robertson–Walker scale factor that behaves as

$$R(t) = bt^{1/3} .$$

Here b denotes a constant.

(a) If a light pulse is emitted at time t_e and observed at time t_o, find the physical separation $\ell_p(t_o)$ between the emitter and the observer, at the time of observation.

(b) Again assuming that t_e and t_o are given, find the observed redshift z.

(c) Find the physical distance $\ell_p(t_o)$ which separates the emitter and observer at the time of observation, expressed in terms of c, t_o, and z (i.e., without t_e appearing).

(d) At an arbitrary time t in the interval $t_e < t < t_o$, find the physical distance $\ell_p(t)$ between the light pulse and the observer. Express your answer in terms of c, t, and t_o.

PROBLEM 5: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(30 points)*

The following problem was Problem 4, Quiz 2, 1992:

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{\gamma},$$

where b and γ are constants. *[This universe differs from the matter-dominated universe described in the lecture notes in that ρ is not proportional to $1/R^3(t)$. Such behavior is possible when pressures are large, because a gas expanding under pressure can lose energy (and hence mass) during the expansion.]* For the following questions, any of the answers may depend on γ, whether it is mentioned explicitly or not.

a) *(5 points)* Let t_0 denote the present time, and let t_e denote the time at which the light that we are currently receiving was emitted by a distant object. In terms of these quantities, find the value of the redshift parameter z with which the light is received.

b) *(5 points)* Find the "look-back" time as a function of z and t_0. The look-back time is defined as the length of the interval in cosmic time between the emission and observation of the light.

c) *(6 points)* Express the present value of the physical distance to the object as a function of H_0, z, and γ.

d) *(7 points)* Find the present, physical value of the horizon distance, $\ell_{p,\text{horizon}}$, for this model. *[Note for 2000: this question is beyond the material for Quiz 1 of this year.]*

e) *(7 points)* At the time of emission, the distant object had a power output P (measured, say, in ergs/sec) which was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux J from this object at the earth today? Express your answer in terms of P, H_0, z, and γ. *[Energy flux (which might be measured in erg-cm^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow.]*

PROBLEM 6: DID YOU DO THE READING? *(25 points)*

The following problem was Problem 1, Quiz 1, 1996:

The following questions are worth 5 points each.

a) In 1814-1815, the Munich optician Joseph Frauenhofer allowed light from the sun to pass through a slit and then through a glass prism. The light was spread into a spectrum of colors, showing lines that could be identified with known elements — sodium, iron, magnesium, calcium, and chromium. Were these lines dark, or bright *(2 points)*? Why *(3 points)*?

b) The Andromeda Nebula was shown conclusively to lie outside our own galaxy when astronomers acquired telescopes powerful enough to resolve the individual stars of Andromeda. Was this feat accomplished by Galileo in 1609, by Immanuel Kant in 1755, by Henrietta Swan Leavitt in 1912, by Edwin Hubble in 1923, or by Walter Baade and Allan Sandage in the 1950s?

c) Some of the earliest measurements of the cosmic background radiation were made indirectly, by observing interstellar clouds of a molecule called cyanogen (CN). State whether each of the following statements is true or false *(1 point each)*:

(i) The first measurements of the temperature of the interstellar cyanogen were made over twenty years before the cosmic background radiation was directly observed.

(ii) Cyanogen helps to measure the cosmic background radiation by reflecting it toward the earth, so that it can be received with microwave detectors.

(iii) One reason why the cyanogen observations were important was that they gave the first measurements of the equivalent temperature of the cosmic background radiation at wavelengths shorter than the peak of the black-body spectrum.

(iv) By measuring the spectrum of visible starlight that passes through the cyanogen clouds, astronomers can infer the intensity of the microwave radiation that bathes the clouds.

(v) By observing chemical reactions in the cyanogen clouds, astronomers can infer the temperature of the microwave radiation in which they are bathed.

d) In about 280 B.C., a Greek philosopher proposed that the Earth and the other planets revolve around the sun. What was the name of this person? *[Note for 2000: this question was based on readings from Joseph Silk's The Big Bang, and therefore is not appropriate for Quiz 1 of this year.]*

e) In 1832 Heinrich Wilhelm Olbers presented what we now know as "Olbers' Paradox," although a similar argument had been discussed as early as 1610 by Johannes Kepler. Olbers argued that if the universe were transparent, static, infinitely old, and was populated by a uniform density of stars similar to our sun, then one of the following consequences would result:

(i) The brightness of the night sky would be infinite.

(ii) Any patch of the night sky would look as bright as the surface of the sun.

(iii) The total energy flux from the night sky would be about equal to the total energy flux from the sun.

(iv) Any patch of the night sky would look as bright as the surface of the moon.

Which one of these statements is the correct statement of Olbers' paradox? *[Note for 2000: this question was based on readings from Joseph Silk's The Big Bang, and therefore is not appropriate for Quiz 1 of this year.]*

PROBLEM 7: A FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$

The following problem was Problem 3, Quiz 1, 1996:

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{3/5} ,$$

where b is a constant.

a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

b) *(5 points)* What is the physical horizon distance at time t? *[Note for 2000: the concept of the horizon distance is not introduced until Lecture Notes 4, and is therefore off limits for Quiz 1 of this year.]*

c) *(5 points)* Suppose a light pulse leaves galaxy A at time t_A and arrives at galaxy B at time t_B. What is the coordinate distance between these two galaxies?

d) *(5 points)* What is the physical separation between galaxy A and galaxy B at time t_A? At time t_B?

e) *(5 points)* At what time is the light pulse equidistant from the two galaxies?

f) *(5 points)* What is the speed of B relative to A at the time t_A? (By "speed," I mean the rate of change of the physical distance with respect to cosmic time, $d\ell_p/dt$.)

g) *(5 points)* For observations made at time t, what is the present value of the physical distance as a function of the redshift z (and the time t)? What physical distance corresponds to $z = \infty$? How does this compare with the horizon distance? (Note that this question does not refer to the galaxies A and B discussed in the earlier parts. In particular, you should not assume that the light pulse left its source at time t_A.) *[Note for 2000: the concept of the horizon distance is not introduced until Lecture Notes 4, and is therefore off limits for Quiz 1 of this year.]*

h) *(5 points)* Returning to the discussion of the galaxies A and B which were considered in parts (c)-(f), suppose the radiation from galaxy A is emitted with total power P. What is the power per area received at galaxy B?

i) *(5 points)* When the light pulse is received by galaxy B, a pulse is immediately sent back toward galaxy A. At what time does this second pulse arrive at galaxy A?

PROBLEM 8: DID YOU DO THE READING? *(20 points)*

The following questions were taken from Problem 1, Quiz 1, 1998:

The following questions are worth 5 points each.

a) In 1917, Einstein introduced a model of the universe which was based on his newly developed general relativity, but which contained an extra term in the equations which he called the "cosmological term." (The coefficient of this term is called the "cosmological constant.") What was Einstein's motivation for introducing this term?

b) When the redshift of distant galaxies was first discovered, the earliest observations were analyzed according to a cosmological model invented by the Dutch astronomer W. de Sitter in 1917. At the time of its discovery, was this model thought to be static or expanding? From the modern perspective, is the model thought to be static or expanding?

c) The early universe is believed to have been filled with thermal, or black-body, radiation. For such radiation the number density of photons and the energy density are each proportional to powers of the absolute temperature T. Say

$$\text{Number density} \propto T^{n_1}$$

$$\text{Energy density} \propto T^{n_2}$$

Give the values of the exponents n_1 and n_2.

d) At about 3,000 K the matter in the universe underwent a certain chemical change in its form, a change that was necessary to allow the differentiation of matter into galaxies and stars. What was the nature of this change?

PROBLEM 9: ANOTHER FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$ *(40 points)*

The following was Problem 3, Quiz 1, 1998:

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{3/5} \ ,$$

where b is a constant.

a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

b) *(10 points)* Suppose a message is transmitted by radio signal (traveling at the speed of light c) from galaxy A to galaxy B. The message is sent at cosmic time t_1, when the physical distance between the galaxies is ℓ_0. At what cosmic time t_2 is the message received at galaxy B? (Express your answer in terms of ℓ_0, t_1, and c.)

c) *(5 points)* Upon receipt of the message, the creatures on galaxy B immediately send back an acknowledgement, by radio signal, that the message has been received. At what cosmic time t_3 is the acknowledgment received on galaxy A? (Express your answer in terms of ℓ_0, t_1, t_2, and c.)

d) *(10 points)* The creatures on galaxy B spend some time trying to decode the message, finally deciding that it is an advertisement for Kellogg's Corn Flakes (whatever that is). At a time Δt after the receipt of the message, as measured on their clocks, they send back a response, requesting further explanation. At what cosmic time t_4 is the response received on galaxy A? In answering this part, you should not assume that Δt is necessarily small. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, Δt, and c.)

e) *(5 points)* When the response is received by galaxy A, the radio waves will be redshifted by a factor $1 + z$. Give an expression for z. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, t_4, Δt, and c.)

f) *(5 points; No partial credit)* If the time Δt introduced in part (d) is small, the time difference $t_4 - t_3$ can be expanded to first order in Δt. Calculate $t_4 - t_3$ to first order accuracy in Δt. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, t_4, Δt, and c.) [Hint: while this part can be answered by using brute force to expand the answer in part (d), there is an easier way.]

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

a) Wright's book described the disk structure of the Milky Way.

b) Kant proposed that the faint nebulae seen in the sky are distant galaxies, similar to the Milky Way.

c) The Milky Way galaxy has a diameter of about 80,000 light-years, and a thickness of 6,000 light-years.

d) The mathematical theory of an expanding universe, in the context of general relativity, was invented by Alexandre Friedmann in 1922. (Actually the 1922 paper discussed only closed universes, but Friedmann published a second paper on open universes in 1924.) Willem de Sitter published his model of the universe in 1917. De Sitter's model was initially believed to be static, but it was later discovered that it appeared static only because it was written in peculiar coordinates— in fact it was also an expanding model. While Friedmann's equations described the general case of a homogeneous isotropic expanding universe, de Sitter's model was more specific: it was a model devoid of matter, with the expansion driven by a positive cosmological constant. The intended answer for this question was Friedmann, but full credit was given for either Friedmann or de Sitter.

e) It was Bernard Burke who told Arno Penzias about the prediction of radio noise from the big bang.

PROBLEM 2: AN EXPONENTIALLY EXPANDING UNIVERSE

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by

$$H = \dot{R}/R \ .$$

So

$$H = \frac{\chi R_0 e^{\chi t}}{R_0 e^{\chi t}} = \boxed{\chi \ .}$$

(b) According to Eq. (3.8), the coordinate velocity of light is given by

$$\frac{dx}{dt} = \frac{c}{R(t)} = \frac{c}{R_0} e^{-\chi t} \ .$$

Integrating,

$$\begin{aligned} x(t) &= \frac{c}{R_0}\int_0^t e^{-\chi t'}\,dt' \\ &= \frac{c}{R_0}\left[-\frac{1}{\chi}e^{-\chi t'}\right]_0^t \\ &= \boxed{\frac{c}{\chi R_0}\left[1-e^{-\chi t}\right]\ .} \end{aligned}$$

(c) From Eq. (3.11), or from the front of the quiz, one has

$$1+z = \frac{R(t_r)}{R(t_e)}\ .$$

Here $t_e = 0$, so

$$\begin{aligned} 1+z &= \frac{R_0 e^{\chi t_r}}{R_0} \\ \implies\quad & e^{\chi t_r} = 1+z \\ \implies\quad & \boxed{t_r = \frac{1}{\chi}\ln(1+z)\ .} \end{aligned}$$

(d) The coordinate distance is $x(t_r)$, where $x(t)$ is the function found in part (b), and t_r is the time found in part (c). So

$$e^{\chi t_r} = 1+z\ ,$$

and

$$\begin{aligned} x(t_r) &= \frac{c}{\chi R_0}\left[1-e^{-\chi t_r}\right] \\ &= \frac{c}{\chi R_0}\left[1-\frac{1}{1+z}\right] \\ &= \frac{cZ}{\chi R_0(1+z)\ .} \end{aligned}$$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$\ell_p(t_r) = R(t_r)x(t_r) = \frac{cze^{\chi t_r}}{\chi(1+z)} = \boxed{\frac{cz}{\chi}\ .}$$

PROBLEM 3: "DID YOU DO THE READING?"

(a) The distinguishing quantity is $\Omega \equiv \rho/\rho_c$. The universe is open if $\Omega < 1$, flat if $\Omega = 1$, or closed if $\Omega > 1$.

(b) The temperature of the microwave background today is about 3 Kelvin. (The best determination to date* was made by the COBE satellite, which measured the temperature as 2.728 ± 0.004 Kelvin. The error here is quoted with a 95% confidence limit, which means that the experimenters believe that the probability that the true value lies outside this range is only 5%.)

(c) The cosmic microwave background is observed to be highly isotropic.

(d) The distance to the Andromeda nebula is roughly 2 million light years.

(e) 1929.

(f) 2 billion years. Hubble's value for Hubble's constant was high by modern standards, by a factor of 5 to 10.

(g) The absolute luminosity (*i.e.*, the total light output) of a Cepheid variable star appears to be highly correlated with the period of its pulsations. This correlation can be used to estimate the distance to the Cepheid, by measuring the period and the apparent luminosity. From the period one can estimate the absolute luminosity of the star, and then one uses the apparent luminosity and the $1/r^2$ law for the intensity of a point source to determine the distance r.

(h) 10^7 light-years.

(i) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

(j) Princeton University.

PROBLEM 4: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The key to this problem is to work in comoving coordinates.

[Some students have asked me why one cannot use "physical" coordinates, for which the coordinates really measure the physical distances. In principle one can use any coordinate system on likes, but the comoving coordinates are the simplest. In any other system it is difficult to write down the trajectory of either a particle or a light-beam. In comoving coordinates it is easy to write the trajectory of either a light beam, or a particle which is moving with the expansion of the universe (and

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* Astrophysical Journal, vol. **473**, p. 576 (1996): *The Cosmic Microwave Background Spectrum from the Full COBE FIRAS Data Sets*, D.J. Fixsen, E.S. Cheng, J.M. Gales, J.C. Mather, R.A. Shafer, and E.L. Wright.

hence standing still in the comoving coordinates). Note, by the way, that when one says that a particle is standing still in comoving coordinates, one has not really said very much about it's trajectory. One has said that it is moving with the matter which fills the universe, but one has not said, for example, how the distance between the particle and origin varies with time. The answer to this latter question is then determined by the evolution of the scale factor, $R(t)$.]

(a) The physical separation at t_o is given by the scale factor times the coordinate distance. The coordinate distance is found by integrating the coordinate velocity, so

$$\ell_p(t_o) = R(t_o)\int_{t_e}^{t_o} \frac{c\,dt'}{R(t')} = bt_o^{1/3}\int_{t_e}^{t_o} \frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2}ct_o^{1/3}\left[t_o^{2/3} - t_e^{2/3}\right]$$

$$\boxed{= \frac{3}{2}ct_o\left[1 - (t_e/t_o)^{2/3}\right] .}$$

(b) From the front of the exam,

$$1 + z = \frac{R(t_o)}{R(t_e)} = \left(\frac{t_o}{t_e}\right)^{1/3}$$

$$\Longrightarrow \quad \boxed{z = \left(\frac{t_o}{t_e}\right)^{1/3} - 1 .}$$

(c) By combining the answers to (a) and (b), one has

$$\boxed{\ell_p(t_o) = \frac{3}{2}ct_o\left[1 - \frac{1}{(1+z)^2}\right] .}$$

(d) The physical distance of the light pulse at time t is equal to $R(t)$ times the coordinate distance. The coordinate distance at time t is equal to the starting coordinate distance, $\ell_c(t_e)$, minus the coordinate distance that the light pulse travels between time t_e and time t. Thus,

$$\begin{aligned}\ell_p(t) &= R(t)\left[\ell_c(t_e) - \int_{t_e}^{t} \frac{c\,dt'}{R(t')}\right] \\ &= R(t)\left[\int_{t_e}^{t_o} \frac{c\,dt'}{R(t')} - \int_{t_e}^{t} \frac{c\,dt'}{R(t')}\right] \\ &= R(t)\int_{t}^{t_o} \frac{c\,dt'}{R(t')} \\ &= bt^{1/3}\int_{t}^{t_o} \frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2}ct^{1/3}\left[t_o^{2/3} - t^{2/3}\right] \\ &= \boxed{\frac{3}{2}ct\left[\left(\frac{t_o}{t}\right)^{2/3} - 1\right]\ .}\end{aligned}$$

PROBLEM 5: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION

a) The cosmological redshift is given by the usual form,

$$1 + z = \frac{R(t_0)}{R(t_e)}\ .$$

For light emitted by an object at time t_e, the redshift of the received light is

$$1 + z = \frac{R(t_0)}{R(t_e)} = \left(\frac{t_0}{t_e}\right)^{\gamma}\ .$$

So,

$$\boxed{z = \left(\frac{t_0}{t_e}\right)^{\gamma} - 1\ .}$$

b) The coordinates t_0 and t_e are cosmic time coordinates. The “look-back” time as defined in the exam is then the interval $t_0 - t_e$. We can write this as

$$t_0 - t_e = t_0\left(1 - \frac{t_e}{t_0}\right)\ .$$

We can use the result of part (a) to eliminate t_e/t_0 in favor of z. From (a),

$$\frac{t_e}{t_0} = (1+z)^{-1/\gamma} \quad .$$

Therefore,

$$\boxed{t_0 - t_e = t_0[1 - (1+z)^{-1/\gamma}] \quad .}$$

c) The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0) = R(t_0)\int_{t_e}^{t_0} \frac{c}{R(t)}dt \quad .$$

Calculating this integral gives

$$\ell_p(t_0) = \frac{ct_0^\gamma}{1-\gamma}\left[\frac{1}{t_0^{\gamma-1}} - \frac{1}{t_e^{\gamma-1}}\right] \quad .$$

Pulling $t_0^{\gamma-1}$ out of the parantheses gives

$$\ell_p(t_0) = \frac{ct_0}{1-\gamma}\left[1 - \left(\frac{t_0}{t_e}\right)^{\gamma-1}\right] \quad .$$

This can be rewritten in terms of z and H_0 using the result of part (a) as well as,

$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{\gamma}{t_0} \quad .$$

Finally then,

$$\boxed{\ell_p(t_0) = cH_0^{-1}\frac{\gamma}{1-\gamma}\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right] \quad .}$$

d) The present physical value of the horizon distance is given by a similar integral to that in part (b) with different limits of integration,

$$\ell_{\text{horiz}}(t_0) = R(t_0)\int_0^{t_0} \frac{c}{R(t)}dt \quad .$$

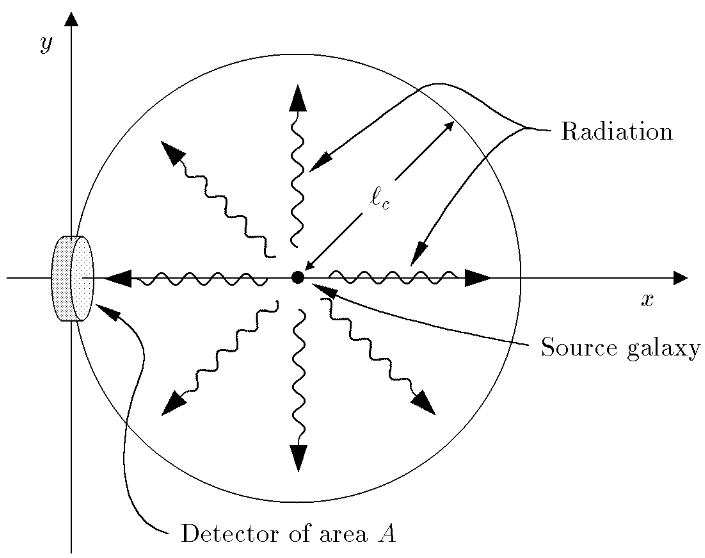
$$= \frac{ct_0}{1-\gamma} \quad .$$

Using H_0 from above, this can be reexpressed as,

$$\boxed{\ell_{\text{horiz}}(t_0) = \frac{c\gamma}{1-\gamma} H_0^{-1} \quad .}$$

e) A nearly identical problem was worked through in Problem 8 of Problem Set 1.

The energy of the observed photons will be redshifted by a factor of $(1+z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of photon emmission, reducing the flux by another factor of $(1+z)$. Consequently, the observed power will be redshifted by two factors of $(1+z)$ to $P/(1+z)^2$.



Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance ℓ_c. Now consider the photons passing through a patch of the sphere with physical area A. In comoving coordinates the present area of the patch is $A/R(t_0)^2$. Since the object radiates uniformly in all directions, the patch will intercept a fraction $(A/R(t_0)^2)/(4\pi\ell_c^2)$ of the photons passing through the sphere. Thus the power hitting the area A is

$$\frac{(A/R(t_0)^2)}{4\pi\ell_c^2}\frac{P}{(1+z)^2} \quad .$$

The radiation energy flux J, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4\pi \ell_p(t_0)^2} \frac{P}{(1+z)^2}$$

where we used $\ell_p(t_0) = R(t_0)\ell_c$. Using the result of part (c) to write J in terms of $P, H_0, z,$ and γ gives,

$$\boxed{J = \frac{H_0^2}{4\pi c^2} \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{P}{(1+z)^2 \left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right]^2} \quad .}$$

PROBLEM 6: DID YOU DO THE READING?

a) The lines were dark, caused by absorption of the radiation in the cooler, outer layers of the sun.

b) Individual stars in the Andromeda Nebula were resolved by Hubble in 1923.

[The other names and dates are not without significance. In 1609 Galileo built his first telescope; during 1609-10 he resolved the individual stars of the Milky Way, and also discovered that the surface of the moon is irregular, that Jupiter has moons of its own, that Saturn has handles (later recognized as rings), that the sun has spots, and that Venus has phases. In 1755 Immanuel Kant published his *Universal Natural History and Theory of the Heavens*, in which he suggested that at least some of the nebulae are galaxies like our own. In 1912 Henrietta Leavitt discovered the relationship between the period and luminosity of Cepheid variable stars. In the 1950s Walter Baade and Allan Sandage recalibrated the extra-galactic distance scale, reducing the accepted value of the Hubble constant by about a factor of 10.]

c)

(i) True. [In 1941, A. McKellar discovered that cyanogen clouds behave as if they are bathed in microwave radiation at a temperature of about 2.3° K, but no connection was made with cosmology.]

(ii) False. [Any radiation reflected by the clouds is far too weak to be detected. It is the bright starlight shining through the cloud that is detectable.]

(iii) True. [Electromagnetic waves at these wavelengths are mostly blocked by the Earth's atmosphere, so they could not be detected directly until high altitude balloons and rockets were introduced into cosmic background radiation research in the 1970s. Precise data was not obtained until the COBE satellite, in 1990.]

(iv) True. [The microwave radiation can boost the CN molecule from its ground state to a low-lying excited state, a state in which the C and N atoms rotate about each other. The population of this low-lying state is therefore determined by the intensity of the microwave radiation. This population is measured by observing the absorption of starlight passing through the clouds, since there are absorption lines in the visible spectrum caused by transitions between the low-lying state and higher energy excited states.]

(v) False. [No chemical reactions are seen.]

d) Aristarchus. [The heliocentric picture was never accepted by other Greek philosophers, however, and was not revived until the publication of *De Revolutionibus Orbium Coelestium* (*On the Revolutions of the Celestial Spheres*) by Copernicus in 1543.]

e) (ii) Any patch of the night sky would look as bright as the surface of the sun. [Explanation: The crux of the argument is that the brightness of an object, measured for example by the power per area (i.e., flux) hitting the retina of your eye, does not change as the object is moved further away. The power falls off with the square of the distance, but so does the area of the image on your retina — so the power per area is independent of distance. Under the assumptions stated, your line of sight will eventually hit a star no matter what direction you are looking. The energy flux on your retina will therefore be the same as in the image of the sun, so the entire sky will appear as bright as the surface of the sun.]

PROBLEM 7: A FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$

a) In general, the Hubble constant is given by $H = \dot{R}/R$, where the overdot denotes a derivative with respect to cosmic time t. In this case

$$H = \frac{1}{bt^{3/5}} \, \frac{3}{5} \, bt^{-2/5} = \boxed{\frac{3}{5t} \ .}$$

b) In general, the (physical) horizon distance is given by

$$\ell_{p,\text{horizon}}(t) = R(t) \int_0^t \frac{c}{R(t')} dt' \ .$$

In this case one has

$$\ell_{p,\text{horizon}}(t) = bt^{3/5} \int_0^t \frac{c}{bt'^{3/5}} dt' = ct^{3/5} \, \frac{5}{2} \left[t^{2/5} - 0^{2/5} \right] = \boxed{\frac{5}{2} \, ct \ .}$$

c) The coordinate speed of light is $c/R(t)$, so the coordinate distance that light travels between t_A and t_B is given by

$$\ell_c = \int_{t_A}^{t_B} \frac{c}{R(t')}\,dt' = \int_{t_A}^{t_B} \frac{c}{bt'^{3/5}}\,dt' = \boxed{\frac{5c}{2b}\left(t_B^{2/5} - t_A^{2/5}\right)\;.}$$

d) The physical separation is just the scale factor times the coordinate separation, so

$$\ell_p(t_A) = R(t_A)\,\ell_c = \boxed{\frac{5}{2}ct_A\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right]\;.}$$

$$\ell_p(t_B) = R(t_B)\,\ell_c = \boxed{\frac{5}{2}ct_B\left[1 - \left(\frac{t_A}{t_B}\right)^{2/5}\right]\;.}$$

e) Let $t_{\rm eq}$ be the time at which the light pulse is equidistant from the two galaxies. At this time it will have traveled a coordinate distance $\ell_c/2$, where ℓ_c is the answer to part (c). Since the coordinate speed is $c/R(t)$, the time $t_{\rm eq}$ can be found from:

$$\int_{t_A}^{t_{\rm eq}} \frac{c}{R(t')}\,dt' = \frac{1}{2}\ell_c$$

$$\frac{5c}{2b}\left(t_{\rm eq}^{2/5} - t_A^{2/5}\right) = \frac{5c}{4b}\left(t_B^{2/5} - t_A^{2/5}\right)$$

Solving for $t_{\rm eq}$,

$$\boxed{t_{\rm eq} = \left[\frac{t_A^{2/5} + t_B^{2/5}}{2}\right]^{5/2}\;.}$$

f) According to Hubble's law, the speed is equal to Hubble's constant times the physical distance. By combining the answers to parts (a) and (d), one has

$$v = H(t_A)\,\ell_p(t_A)$$

$$= \frac{3}{5t_A}\frac{5}{2}ct_A\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right] = \boxed{\frac{3}{2}\,c\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right]\;.}$$

g) The redshift for radiation observed at time t can be written as

$$1 + z = \frac{R(t)}{R(t_e)} ,$$

where t_e is the time that the radiation was emitted. Solving for t_e,

$$t_e = \frac{t}{(1+z)^{5/3}} .$$

As found in part (d), the physical distance that the light travels between t_e and t, as measured at time t, is given by

$$\ell_p(t) = R(t) \int_{t_e}^{t} \frac{c}{R(t')}\, dt' = \frac{5}{2} ct \left[1 - \left(\frac{t_e}{t} \right)^{2/5} \right] .$$

Substituting the expression for t_e, one has

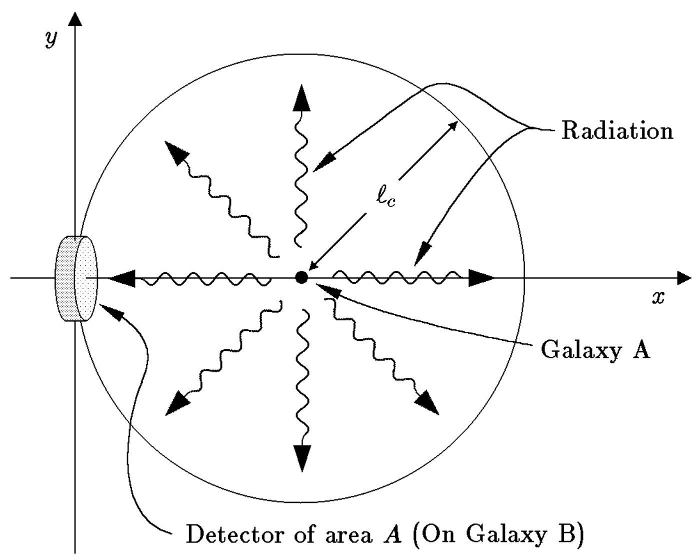
$$\boxed{\ell_p(t) = \frac{5}{2}\, ct \left[1 - \frac{1}{(1+z)^{2/3}} \right] .}$$

As $z \to \infty$, this expression approaches

$$\boxed{\lim_{z\to\infty} \ell_p(t) = \frac{5}{2}\, ct ,}$$

which is exactly equal to the horizon distance. It is a general rule that the horizon distance corresponds to infinite redshift z.

h) Again we will view the problem in comoving coordinates. Put galaxy B at the origin, and galaxy A at a coordinate distance ℓ_c along the x-axis. Draw a sphere of radius ℓ_c, centered galaxy A. Also draw a detector on galaxy B, with physical area A (measured at the present time).



The energy from the quasar will radiate uniformly on the sphere. The detector has a physical area A, so in the comoving coordinate picture its area in square notches would be $A/R(t_B)^2$. The detector therefore occupies a fraction of the sphere given by

$$\frac{[A/R(t_B)^2]}{4\pi\ell_c^2} = \frac{A}{4\pi\ell_p(t_B)^2} ,$$

so this fraction of the emitted photons will strike the detector.

Next consider the rate of arrival of the photons at the sphere. In lecture we figured out that if a periodic wave is emitted at time t_A and observed at time t_B, then the rate of arrival of the wave crests will be slower than the rate of emission by a redshift factor $1 + z = R(t_B)/R(t_A)$. The same argument will apply to the rate of arrival of photons, so the rate of photon arrival at the sphere will be slower than the rate of emission by the factor $1+z$, reducing the energy flux by this factor. In addition, each photon is redshifted in frequency by $1 + z$. Since the energy of each photon is proportional to its frequency, the energy flux is reduced by an additional factor of $1+z$. Thus, the rate at which energy reaches the detector is

$$\text{Power hitting detector} = \frac{A}{4\pi\ell_p(t_B)^2}\frac{P}{(1+z)^2} .$$

The red shift z of the light pulse received at galaxy B is given by

$$1 + z = \frac{R(t_B)}{R(t_A)} = \left(\frac{t_B}{t_A}\right)^{3/5} .$$

Using once more the expression for $\ell_P(t_B)$ from part (d), one has

$$\boxed{J = \frac{\text{Power hitting detector}}{A} = \frac{P(t_A/t_B)^{6/5}}{25\pi\, c^2\, t_B^2 \left[1 - \left(\frac{t_A}{t_B}\right)^{2/5}\right]^2} \ .}$$

The problem is worded so that t_A, and not z, is the given variable that determines how far galaxy A is from galaxy B. In practice, however, it is usually more useful to express the answer in terms of the redshift z of the received radiation. One can do this by using the above expression for $1+z$ to eliminate t_A in favor of z, finding

$$J = \frac{P}{25\pi\, c^2\, t_B^2 (1+z)^{2/3} \left[(1+z)^{2/3} - 1\right]^2} \ .$$

i) Let t'_A be the time at which the light pulse arrives back at galaxy A. The pulse must therefore travel a coordinate distance ℓ_c (the answer to part (c)) between time t_B and t'_A, so

$$\int_{t_B}^{t'_A} \frac{c}{R(t')}\, dt' = \ell_c \ .$$

Using the answer from (c) and integrating the left-hand side,

$$\frac{5c}{2b}\left(t_A'^{2/5} - t_B^{2/5}\right) = \frac{5c}{2b}\left(t_B^{2/5} - t_A^{2/5}\right) \ .$$

Solving for t'_A,

$$\boxed{t'_A = \left(2t_B^{2/5} - t_A^{2/5}\right)^{5/2} \ .}$$

PROBLEM 8: DID YOU DO THE READING?

a) Einstein believed that the universe was static, and the cosmological term was necessary to prevent a static universe from collapsing under the attractive force of normal gravity. [The repulsive effect of a cosmological constant grows linearly with distance, so if the coefficient is small it is important only when the separations are very large. Such a term can be important cosmologically while still being too small to be detected by observations of the solar system or even the galaxy. Recent measurements of distant supernovas ($z \approx 1$), which you may have read about in the newspapers, make it look like maybe there is a cosmological constant after all! Since the cosmological constant is the hot issue in cosmology this season, we will want to look at it more carefully. The best time will be after Lecture Notes 7.]

b) At the time of its discovery, de Sitter's model was thought to be static [although it was known that the model predicted a redshift which, at least for nearby galaxies, was proportional to the distance]. From a modern perspective the model is thought to be expanding.

[It seems strange that physicists in 1917 could not correctly determine if the theory described a universe that was static or expanding, but the mathematical formalism of general relativity can be rather confusing. The basic problem is that when space is not Euclidean there is no simple way to assign coordinates to it. The mathematics of general relativity is designed to be valid for any coordinate system, but the underlying physics can sometimes be obscured by a peculiar choice of coordinates. A change of coordinates can not only distort the apparent geometry of space, but it can also mix up space and time. The de Sitter model was first written down in coordinates that made it look static, so everyone believed it was. Later Arthur Eddington and Hermann Weyl (independently) calculated the trajectories of test particles, discovering that they flew apart.]

c) $n_1 = 3$, and $n_2 = 4$.

d) Above 3,000 K the universe was so hot that the atoms were ionized, dissociated into nuclei and free electrons. At about this temperature, however, the universe was cool enough so that the nuclei and electrons combined to form neutral atoms.

[This process is usually called "recombination," although the prefix "re-" is totally inaccurate, since in the big bang theory these constituents had never been previously combined. As far as I know the word was first used in this context by P.J.E. Peebles, so I once asked him why the prefix was used. He replied that this word is standard terminology in plasma physics, and was carried over into cosmology.]

[Regardless of its name, recombination was crucial for the clumping of matter into galaxies and stars, because the pressure of the photons in the early universe was enormous. When the matter was ionized, the free electrons interacted strongly with the photons, so the pressure of these photons prevented the matter from clumping. After recombination, however, the matter became very transparent to radiation, and the pressure of the radiation became ineffective.]

[Incidentally, at roughly the same time as recombination (with big uncertainties), the mass density of the universe changed from being dominated by radiation (photons and neutrinos) to being dominated by nonrelativistic matter. There is no known underlying connection between these two events, and it seems to be something of a coincidence that they occurred at about the same time. The transition from radiation-domination to matter-domination also helped to promote the clumping of matter, but the effect was much weaker than the effect of recombination— because of the very high velocity of photons and neutrinos, their pressure remained a significant force even after their mass density became much smaller than that of matter.]

PROBLEM 9: ANOTHER FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$

a) According to Eq. (3.7) of the Lecture Notes,

$$H(t) = \frac{1}{R(t)} \frac{dR}{dt} \ .$$

For the special case of $R(t) = bt^{3/5}$, this gives

$$H(t) = \frac{1}{bt^{3/5}} \frac{3}{5} bt^{-2/5} = \boxed{\frac{3}{5t} \ .}$$

b) According to Eq. (3.8) of the Lecture Notes, the coordinate velocity of light (in comoving coordinates) is given by

$$\frac{dx}{dt} = \frac{c}{R(t)} \ .$$

Since galaxies A and B have physical separation ℓ_0 at time t_1, their coordinate separation is given by

$$\ell_c = \frac{\ell_0}{bt_1^{3/5}} \ .$$

The radio signal must cover this coordinate distance in the time interval from t_1 to t_2, which implies that

$$\int_{t_1}^{t_2} \frac{c}{R(t)} \, dt = \frac{\ell_0}{bt_1^{3/5}} \ .$$

Using the expression for $R(t)$ and integrating,

$$\frac{5c}{2b} \left(t_2^{2/5} - t_1^{2/5} \right) = \frac{\ell_0}{bt_1^{3/5}} \ ,$$

which can be solved for t_2 to give

$$\boxed{t_2 = \left(1 + \frac{2\ell_0}{5ct_1}\right)^{5/2} t_1 \ .}$$

c) The method is the same as in part (b). The coordinate distance between the two galaxies is unchanged, but this time the distance must be traversed in the time interval from t_2 to t_3. So,

$$\int_{t_2}^{t_3} \frac{c}{R(t)}\, dt = \frac{\ell_0}{bt_1^{3/5}} \ ,$$

which leads to

$$\frac{5c}{2b}\left(t_3^{2/5} - t_2^{2/5}\right) = \frac{\ell_0}{bt_1^{3/5}} \ .$$

Solving for t_3 gives

$$\boxed{t_3 = \left[\left(\frac{t_2}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{5/2} t_1 \ .}$$

The above answer is perfectly acceptable, but one could also replace t_2 by using the answer to part (b), which gives

$$\boxed{t_3 = \left(1 + \frac{4\ell_0}{5ct_1}\right)^{5/2} t_1 \ .}$$

[Alternatively, one could have begun the problem by considering the full round trip of the radio signal, which travels a coordinate distance $2\ell_c$ during the time interval from t_1 to t_3. The problem then becomes identical to part (b), except that the coordinate distance ℓ_c is replaced by $2\ell_c$, and t_2 is replaced by t_3. One is led immediately to the answer in the form of the previous equation.]

d) Cosmic time is defined by the reading of suitably synchronized clocks which are each at rest with respect to the matter of the universe at the same location. (For this problem we will not need to think about the method of synchronization.) Thus, the cosmic time interval between the receipt of the message and the response is the same as what is measured on the galaxy B clocks, which is Δt. The response is therefore sent at cosmic time $t_2 + \Delta t$. The coordinate distance between the galaxies is still $\ell_0/R(t_1)$, so

$$\int_{t_2+\Delta t}^{t_4} \frac{c}{R(t)}\,dt = \frac{\ell_0}{b t_1^{3/5}} \ .$$

Integration gives

$$\frac{5c}{2b}\left[t_4^{2/5} - (t_2+\Delta t)^{2/5}\right] = \frac{\ell_0}{b t_1^{3/5}} \ ,$$

which can be solved for t_4 to give

$$\boxed{t_4 = \left[\left(\frac{t_2+\Delta t}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{5/2} t_1 \ .}$$

e) From the formula at the front of the exam,

$$1+z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} = \frac{R(t_4)}{R(t_2+\Delta t)} = \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} \ .$$

So,

$$\boxed{z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} = \frac{R(t_4)}{R(t_2+\Delta t)} = \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} - 1 \ .}$$

f) If Δt is small compared to the time that it takes $R(t)$ to change significantly, then the interval between a signal sent at t_3 and a signal sent at $t_3 + \Delta t$ will be received with a redshift identical to that observed between two successive crests of a wave. Thus, the separation between the receipt of the acknowledgement and the receipt of the response will be a factor $(1+z)$ times longer than the time interval between the sending of the two signals, and therefore

$$\begin{aligned} t_4 - t_3 &= (1+z)\Delta t + \mathcal{O}(\Delta t^2) \\ &= \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) \ . \end{aligned}$$

Since the answer contains an explicit factor of Δt, the other factors can be evaluated to zeroth order in Δt:

$$\boxed{t_4 - t_3 = \left(\frac{t_4}{t_2}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2)\ ,}$$

where to first order in Δt the t_4 in the numerator could equally well have been replaced by t_3.

For those who prefer the brute force approach, the answer to part (d) can be Taylor expanded in powers of Δt. To first order one has

$$t_4 = t_3 + \left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} \Delta t + \mathcal{O}(\Delta t^2)\ .$$

Evaluating the necessary derivative gives

$$\frac{\partial t_4}{\partial \Delta t} = \left[\left(\frac{t_2+\Delta t}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{3/2} \left(\frac{t_2+\Delta t}{t_1}\right)^{-3/5}\ ,$$

which when specialized to $\Delta t = 0$ becomes

$$\left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} = \left[\left(\frac{t_2}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{3/2} \left(\frac{t_2}{t_1}\right)^{-3/5}\ .$$

Using the first boxed answer to part (c), this can be simplified to

$$\begin{aligned}
\left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} &= \left(\frac{t_3}{t_1}\right)^{3/5} \left(\frac{t_2}{t_1}\right)^{-3/5} \\
&= \left(\frac{t_3}{t_2}\right)^{3/5}\ .
\end{aligned}$$

Putting this back into the Taylor series gives

$$t_4 - t_3 = \left(\frac{t_3}{t_2}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2)\ ,$$

in agreement with the previous answer.