MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 11, 2000
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 2

QUIZ DATE: Thursday, March 16, 2000

COVERAGE: Lecture Notes 4; Lecture Notes 5; Problem Set 2; Michael Rowan-Robinson, *Cosmology* (Third Edition), Chapters 1-3. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignment, or from this set of Review Problems.**

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. Two of the four problems are from quizzes in previous years, and the other two are new. They are all problems that I would consider fair for the coming quiz. Since Rowan-Robinson's book has not previously been used as a text in 8.286, there are no review problems based on this reading. You should expect, however, that the quiz will include a set of questions based on this reading assignment.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1996 and 1998. The relevant problems from those quizzes have been incorporated into Problem Set 2 or these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. However, due to changes in the pacing of the quizzes, the quizzes from 1996 and 1998 do not match well with the coverage of the coming quiz. From the quizzes of 1998, only Problem 2 of Quiz 2 would be appropriate for the coming quiz, and you have already worked this problem in Problem Set 2. From 1996, only Problem 2 of Quiz 1 would be appropriate, but it was also included in this year's Problem Set 2.

INFORMATION TO BE GIVEN ON QUIZ:

The following material will be included on the quiz, so you need not memorize it. You should, however, make sure that you understand what these formulas mean, and how they can be applied.

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Flat ($\Omega \equiv \rho/\rho_c = 1$): $\quad R(t) \propto t^{2/3}$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

PROBLEM 1: THE DECELERATION PARAMETER

The following problem was Problem 2, Quiz 2, 1992, where it counted 10 points out of 100.

Many standard references in cosmology define a quantity called the ***deceleration parameter*** q, which is a direct measure of the slowing down of the cosmic expansion. The parameter is defined by

$$q \equiv -\ddot{R}(t)\frac{R(t)}{\dot{R}^2(t)}\ .$$

Find the relationship between q and Ω for a matter-dominated universe. *[In case you have forgotten, Ω is defined by*

$$\Omega = \rho/\rho_c \ ,$$

where ρ is the mass density and ρ_c is the critical mass density (i.e., that mass density which corresponds to $k = 0$).]

PROBLEM 2: A RADIATION-DOMINATED FLAT UNIVERSE

We have learned that a matter-dominated homogeneous and isotropic universe can be described by a scale factor $R(t)$ obeying the equation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \ .$$

This equation in fact applies to any form of mass density, so we can apply it to a universe in which the mass density is dominated by the energy of photons. Recall that the mass density of nonrelativistic matter falls off as $1/R^3(t)$ as the universe expands; the mass of each particle remains constant, and the density of particles falls off as $1/R^3(t)$ because the volume increases as $R^3(t)$. For the photon-dominated universe, the density of photons falls of as $1/R^3(t)$, but in addition the frequency (and hence the energy) of each photon redshifts in proportion to $1/R(t)$. Since mass and energy are equivalent, the mass density of the gas of photons falls off as $1/R^4(t)$.

For a flat (i.e., $k = 0$) matter-dominated universe we learned that the scale factor $R(t)$ is proportional to $t^{2/3}$. How does $R(t)$ behave for a photon-dominated universe?

PROBLEM 3: EVOLUTION OF AN OPEN UNIVERSE

The following problem was taken from Quiz 2, 1990, where it counted 10 points out of 100.

Consider an open, matter-dominated universe, as described by the evolution equations on the front of the quiz. Find the time t at which $R/\sqrt{\kappa} = 2\alpha$.

PROBLEM 4: ANTICIPATING A BIG CRUNCH

Suppose that we lived in a closed, matter-dominated universe, as described by the equations on the front of the quiz. Suppose further that we measured the mass density parameter Ω to be $\Omega_0 = 2$, and we measured the Hubble "constant" to have some value H_0. How much time would we have before our universe ended in a big crunch, at which time the scale factor $R(t)$ would collapse to 0?

SOLUTIONS

PROBLEM 1: THE DECELERATION PARAMETER

From the front of the exam, we are reminded that

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

and

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \ ,$$

where a dot denotes a derivative with respect to time t. The critical mass density ρ_c is defined to be the mass density that corresponds to a flat ($k = 0$) universe, so from the equation above it follows that

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho_c \ .$$

Substituting into the definition of q, we find

$$\begin{aligned} q = -\ddot{R}(t)\frac{R(t)}{\dot{R}^2(t)} &= -\frac{\ddot{R}}{R}\left(\frac{R}{\dot{R}}\right)^2 \\ &= \left(\frac{4\pi}{3}G\rho\right)\left(\frac{3}{8\pi G\rho_c}\right) = \frac{1}{2}\frac{\rho}{\rho_c} = \boxed{\frac{1}{2}\,\Omega\ .} \end{aligned}$$

PROBLEM 2: A RADIATION-DOMINATED FLAT UNIVERSE

The flatness of the model universe means that $k = 0$, so

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho \ .$$

Since

$$\rho(t) \propto \frac{1}{R^4(t)} \ ,$$

it follows that

$$\frac{dR}{dt} = \frac{\text{const}}{R} \ .$$

Rewriting this as

$$R\,dR = \text{const}\,dt \ ,$$

the indefinite integral becomes

$$\frac{1}{2}R^2 = (\text{const})t + c' \ ,$$

where c' is a constant of integration. Different choices for c' correspond to different choices for the definition of $t = 0$. We will follow the standard convention of choosing $c' = 0$, which sets $t = 0$ to be the time when $R = 0$. Thus the above equation implies that $R^2 \propto t$, and therefore

$$\boxed{R(t) \propto t^{1/2}}$$

for a photon-dominated flat universe.

PROBLEM 3: EVOLUTION OF AN OPEN UNIVERSE

The evolution of an open, matter-dominated universe is described by the following parametric equations:

$$ct = \alpha(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1) \ .$$

Evaluating the second of these equations at $R/\sqrt{\kappa} = 2\alpha$ yields a solution for θ:

$$2\alpha = \alpha(\cosh\theta - 1) \quad \Longrightarrow \quad \cosh\theta = 3 \quad \Longrightarrow \quad \theta = \cosh^{-1}(3) \ .$$

We can use these results in the first equation to solve for t. Noting that

$$\sinh\theta = \sqrt{\cosh^2\theta - 1} = \sqrt{8} = 2\sqrt{2} \ ,$$

we have

$$\boxed{t = \frac{\alpha}{c}\left[2\sqrt{2} - \cosh^{-1}(3)\right] \ .}$$

Numerically, $t \approx 1.06567\,\alpha/c$.

PROBLEM 4: ANTICIPATING A BIG CRUNCH

The critical density is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} \ ,$$

so the mass density is given by

$$\rho = \Omega_0 \rho_c = 2\rho_c = \frac{3H_0^2}{4\pi G} \ . \tag{1}$$

Substituting this relation into

$$H_0^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \ ,$$

we find

$$H_0^2 = 2H_0^2 - \frac{kc^2}{R^2} \ ,$$

from which it follows that

$$\frac{R}{\sqrt{k}} = \frac{c}{H_0} \ . \tag{2}$$

Now use

$$\alpha = \frac{4\pi}{3} \frac{G\rho R^3}{k^{3/2} c^2} \ .$$

Substituting the values we have from Eqs. (1) and (2) for ρ and $R/\sqrt{k}$, we have

$$\alpha = \frac{c}{H_0} \ . \tag{3}$$

To determine the value of the parameter θ, use

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

which when combined with Eqs. (2) and (3) implies that $\cos\theta = 0$. The equation $\cos\theta = 0$ has multiple solutions, but we know that the θ-parameter for a closed matter-dominated universe varies between 0 and π during the expansion phase of the universe. Within this range, $\cos\theta = 0$ implies that $\theta = \pi/2$. Thus, the age of the universe at the time these measurements are made is given by

$$\begin{aligned} t &= \frac{\alpha}{c}(\theta - \sin\theta) \\ &= \frac{1}{H_0}\left(\frac{\pi}{2} - 1\right) \ . \end{aligned}$$

The total lifetime of the closed universe corresponds to $\theta = 2\pi$, or

$$t_{\text{final}} = \frac{2\pi\alpha}{c} = \frac{2\pi}{H_0} \; ,$$

so the time remaining before the big crunch is given by

$$t_{\text{final}} - t = \frac{1}{H_0}\left[2\pi - \left(\frac{\pi}{2} - 1\right)\right] = \boxed{\left(\frac{3\pi}{2} + 1\right)\frac{1}{H_0} \; .}$$