MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — May 5, 2000
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REVIEW PROBLEMS FOR QUIZ 4

QUIZ DATE: Thursday, May 11, 2000

COVERAGE: Lecture Notes 8–13; Problem Sets 4 and 5; Michael Rowan-Robinson, *Cosmology* (Third Edition), Chapter 6, Sections 8.2 and 8.3, and the Epilogue; Steven Weinberg, *The First Three Minutes*, Chapters 4, 5, 7, and 8. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from this set of Review Problems.**

CALCULATORS: You may need a calculator for this quiz, so please bring one.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They are all problems that I would consider fair for the coming quiz. Since Rowan-Robinson's book has not previously been used as a text in 8.286, there are again no review problems based on this reading. You should expect, however, that the quiz will include some questions based on this reading assignment.

INFORMATION TO BE GIVEN ON QUIZ:

The following material will be included on the quiz, so you need not memorize it. You should, however, make sure that you understand what these formulas mean, and how they can be applied.

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R$$

EVOLUTION OF A FLAT ($\boldsymbol{\Omega \equiv \rho/\rho_c = 1}$) UNIVERSE:

$R(t) \propto t^{2/3}$ (matter-dominated)

$R(t) \propto t^{1/2}$ (radiation-dominated)

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{ g_{ij} \frac{dx^j}{ds} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell}\right) \frac{dx^k}{ds} \frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma}\right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\rm vac} = -\rho_{\rm vac} c^2 \qquad \rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\, \text{erg/K} = 8.617 \times 10^{-5}\, \text{eV/K} \ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27} \text{ erg-sec} = 6.582 \times 10^{-16} \text{ eV-sec} \ ,$$

$$c = 2.998 \times 10^{10} \text{ cm/sec}$$

$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg} \ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\;, \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\;, \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\;.$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3c^5}{16\pi^3 gG}\right)^{1/4}\frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860\text{ MeV}}{\sqrt{t\text{ (in sec)}}}$$

PARTICLE PROPERTIES:

While working on this exam you may refer to any of the tables in Lecture Notes 11. Please bring your copy of Lecture Notes 11 with you to the exam.

PROBLEM 1: SHORT ANSWERS:

(a) According to the unified electroweak theory, at the fundamental level an electron is identical to what other type of elementary particle? According to grand unified theories, it is identical to what two other types of particles?

(b) The Higgs fields that are introduced into the electroweak theory or grand unified theories to spontaneously break the internal symmetries are always (A) scalar fields, corresponding to spin 0 particles, (B) spinor fields, corresponding to spin $\frac{1}{2}$ particles, (C) vector fields, corresponding to spin 1 particles, or (D) tensor fields, corresponding to spin 2 particles?

(c) Hadrons are not thought to be elementary particles but instead are composed of more fundamental particles. What is the general name for these more fundamental constituents of the hadrons? Name a kind of lepton. Do leptons feel the strong interactions? Do hadrons? [Hint: "hadrons" refers to strongly interacting particles, the baryons and mesons.]

(d) Grand unified theories (GUTs) predict the existence of magnetic monopoles, point-like topological defects. Is the mass of the monopole expected to be roughly (a) 10^{16} GeV (b) 1 GeV (c) 1/2 MeV or (d) zero? What is the most serious cosmological problem associated with the existence of GUT monopoles?

(e) The cosmological evolution of the universe is described by the two Einstein equations given on the cover sheet of the exam. When the energy density of the universe is dominated by matter or by radiation, then the acceleration of the scale factor, described by one of the Einstein equations, is negative. Consequently the expansion of the universe is slowed. During an inflationary era, is the acceleration of the scale factor positive or negative? Why?

(f) The V-A theory of the weak interactions, developed in 1958 by Feynman and Gell-Mann and independently by Marshak and Sudarshan, is listed on Table 3 of Lecture Notes 10 as a "flawed theory". In what way is this theory flawed. (Please answer in no more than 2-3 sentences.)

(g) The word "supersymmetry" refers to a symmetry that relates the behavior of one certain class of particles with the behavior of another class. What are these two classes?

PROBLEM 2: TIME SCALES IN COSMOLOGY

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place.

The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-35} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the beginning of the processes involved in big bang nucleosynthesis;

(b) the end of the processes involved in big bang nucleosynthesis;

(c) the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{14}$ GeV;

(d) "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

PROBLEM 3: GRAND UNIFIED THEORIES AND MAGNETIC MONOPOLE PRODUCTION

When grand unified theories are combined with standard (i.e., non-inflationary) cosmology, one is led to the conclusion that far too many magnetic monopoles are produced. This conclusion is based on an estimate of n_M/n_γ, the ratio of the number density of magnetic monopoles to the number density of photons. The estimated value of n_M/n_γ is proportional to a power of the critical temperature T_c of the grand unified theory phase transition. State the power, and explain why.

SOLUTIONS

(a) In the electroweak theory, the electron is fundamentally the same as the neutrino. In grand unified theories, the electron is fundamentally the same as both the neutrino and the quark. (Many students mentioned the symmetry between the electron, the muon, and the tau, which is a part of the symmetry between generations of fermions. This symmetry remains a mystery, not explained by either the electroweak theory or grand unified theories.)

(b) Choice (A): the Higgs fields are scalars. This is essential for the consistency of the theory, since the Higgs fields have a nonzero value in the vacuum. If they were not scalars, then this nonzero value would be measured differently by observers who were rotated with respect to each other, so the rotational invariance of the vacuum would be violated. (The electron, by the way, is a spin-$\frac{1}{2}$ particle, the photon is an example of a spin 1 particle, and the graviton has spin 2.)

(c) Hadrons are thought to be built out of quarks. Electrons, neutrinos, muons are all examples of leptons. Leptons do not feel the strong interactions. Hadrons do feel the strong interactions.

(d) GUT monopoles are expected to have a mass of (a) 10^{16} GeV. With the abundance predicted by GUTs, the incredibly massive monopoles would quickly come to dominate the energy density of the universe. The large energy density would speed up the cosmological evolution so that the universe was a mere few tens of thousands of years old today. Such a young and quickly evolving universe is in serious conflict with observations. Also, there are no confirmed observations of monopoles.

(e) During an inflationary era the acceleration of the scale factor is positive. The pressure associated with the false vacuum which drives inflation is negative. The contribution to the acceleration from the negative pressure dominates over that from the positive energy density in the equation

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2} \right) R \quad .$$

Consequently, the negative pressure powers a repulsive gravitational force.

(f) It was flawed in that it was not "renormalizable". That is, calculations based on quantum mechanical perturbation theory lead to infinities in the sums over intermediate states, which is similar to the situation with QED (quantum electrodynamics). Unlike QED, however, the infinities here cannot be absorbed into a redefinition of the fundamental constants of the theory.

(g) Fermions and bosons.

PROBLEM 2: TIME SCALES IN COSMOLOGY

(a) 1 sec. *[This is the time at which the weak interactions "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) 10^{-35} sec. *[We learned in Lecture Notes 7 that kT was about 1 MeV at $t = 1$ sec. Since 1 GeV = 1000 MeV, the value of kT that we want is 10^{17} times higher. In the radiation-dominated era $T \propto R^{-1} \propto t^{-1/2}$, so we get 10^{-34} sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) 10^{-5} sec. *[As in (c), we can use $t \propto T^{-2}$, with $kT \approx 1$ MeV at $t = 1$ sec.]*

PROBLEM 3: GRAND UNIFIED THEORIES AND MAGNETIC MONOPOLE PRODUCTION

Then number density of monopoles produced in the grand unified theory phase transition is roughly one per horizon volume. The horizon distance is proportional to t, which in turn is proportional to $1/T_c^2$. Thus, the number density of magnetic monopoles is proportional to $1/t^3 \propto T_c^6$. The number density of photons at temperature T_c is proportional to T_c^3, So the ratio

$$\boxed{n_M/n_\gamma \propto T_c^3 \ .}$$