MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 4, 2000
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QUIZ 2 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(35 points)*

Solution to Problem 1 written by Edward Keyes.

(a) Neutrinos in astrophysics

Fusion reactions in the Sun produce a very plentiful supply of neutrinos. In fact, about 3% of the Sun's energy is radiated as neutrinos, giving a flux here on Earth of about 60 billion neutrinos per square centimeter per second. Naturally, we don't notice this ordinarily because neutrinos interact only weakly with matter. Interestingly enough, although we believe we understand the fusion processes in the Sun pretty well, detectors tend to only see about a third to a half of the expected neutrino flux, a discrepancy known as the **solar neutrino problem**. The favored explanation is to say that some of the neutrinos, which are produced in the Sun as electron neutrinos, can "oscillate" en route into mu neutrinos, tau neutrinos, or perhaps some unknown type of neutrino, which normal detectors aren't designed to catch. Recent experimental results seem to confirm this picture.

The other choices given probably all give off some amount of neutrinos, but the only other detected astrophysical source of neutrinos was Supernova 1987a. Eleven neutrinos from this explosion were detected, and the time-of-flight measurements between the visible flash and the neutrino detections provided a good upper bound to the neutrino mass.

(b) The blackbody spectral shape

There were two straightforward ways to do this problem, depending on how much you remembered from the reading. First, if you remembered that the Sun's surface temperature was around 6000 K, you could use the fact that visible light has a wavelength of 500 nm to say:

$$\lambda_{\rm CMB} = \lambda_{\rm visible} \frac{T_{\rm visible}}{T_{\rm CMB}} = 500 \text{ nm} \frac{6000 \text{ K}}{3 \text{ K}} \simeq 1 \text{ mm}$$

This relies on the fact that for a blackbody, wavelengths are inversely proportional to temperature, as we learned earlier in the class. Note that the Sun's spectrum naturally peaks in the visible range, because our eyes have simply evolved to be sensitive to the region of the spectrum where the Sun radiates the most.

The second way to do the problem was to remember that blackbody radiation is a condition of equilibrium between matter and radiation, so the average energy per photon is about equal to the average kinetic energy of a particle at the same temperature:

$$kT \simeq h\nu = \frac{hc}{\lambda}$$

$$\lambda_{\rm CMB} \simeq \frac{hc}{kT_{\rm CMB}} \simeq 5 \text{ mm}$$

The correct answer is 2 mm, since we've dropped a constant of order unity in our approximation, but we were only looking for the right order of magnitude.

As an aside, note that the formula for the spectral peak which appears on p. 13 of Rowan-Robinson's book, $\nu = 0.354kT/h$, is written incorrectly. The intensity of the radiation per unit frequency is given by Eq. (1.2) on p. 12 of the book,

$$B_\nu(\nu) = \frac{2h\nu^3}{c^2\left(e^{h\nu/kT} - 1\right)} ,$$

and the maximum of this function can be found numerically to be at $\nu = 2.821kT/h$. It is probably not a coincidence that $0.354 = 1/2.821$. Also note that it makes a difference whether you are looking for the peak of intensity per unit frequency or the intensity per unit wavelength. The intensity per unit wavelength is given by

$$B_\lambda(\nu) = B_\nu(\nu)\left|\frac{d\nu}{d\lambda}\right| = \frac{\nu^2}{c}B_\nu(\nu) ,$$

and it peaks at $\nu = 4.965kT/h$.

(c) The Hubble sequence

Hubble introduced a classification system for galactic shapes, dividing them into elliptical and spiral galaxies of certain visible characteristics. However, this "tuning fork" diagram is not an evolutionary sequence. Instead, a galaxy's type is mainly determined by its star formation rate: ellipticals formed most of their stars quickly early in their life, so today they have mainly older, redder stars and very little gas. Spirals, on the other hand, are still actively forming stars in their discs, because their initial star formation rate was small enough so they still have some gas left. Note that all galaxies are essentially the same age, so even spirals and irregulars have a population of very old stars too.

(d) Galaxy clusters

A typical "rich" cluster contains hundreds to thousands of galaxies, and has a mass of 10^{14}-10^{15} solar masses. Answers to this question ranged all over the place, with mass estimates going from the size of a small molecular cloud up to larger than the mass of the visible universe. The scale of galaxy clusters is particularly important because they are the largest gravitationally bound structures that have had time to form so far: measurements of their abundance can put good constraints on the processes of structure formation in the universe and on cosmological parameters.

(e) Distance measurements

The correct answer was (vi), that tens of thousands of nearby stars have had their distances measured by parallax using the orbit of the Earth. The vast majority of these were measured by the Hipparcos satellite, launched in 1989. This was a very important achievement, because among other benefits it allowed astronomers to recalibrate their distance and age estimates for globular clusters, virtually eliminating the alarming problem of having stars older than the age of the universe.

Radar reflection has been used to get distance measurements within the solar system. Gravitational lensing can be used over cosmological distances, turning a time delay of brightness fluctuations seen between two lensed images of the same quasar into a distance estimate. Cepheid variables are useful for getting the distances to nearby galaxies, and recent Hubble observations of Cepheids in the Virgo cluster gave a very important refinement to the astronomical distance ladder.

(f) Olber's paradox

The paradox is as follows: we know that the brightness of a source falls off as the square of its distance, but its angular size also falls off that way, so surface brightness remains constant. In an infinite universe, every line of sight would eventually intersect the surface of a star, and hence the night sky should be as bright as the Sun. Why is the sky dark instead?

The resolution of the paradox is simply the finite age of the universe: we can only see objects out to a certain distance, since the light from other objects hasn't had time to reach us yet.

Another resolution is the expansion of the universe, with its redshift effects. From a certain perspective, Olber's paradox is actually true: the night sky **does** shine uniformly bright, but only in microwaves, due to the cosmic background radiation. Every line of sight intersects a piece of brightly glowing plasma, that used to be about as hot as the Sun a hundred thousand years after the Big Bang, but due to redshift effects, we don't see it in visible light any more.

PROBLEM 2: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(35 points)*

(a) Substituting the equation for $M(r_i)$, given on the quiz, into the differential equation for r, also given on the quiz, one finds:

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} + \gamma r^n \ .$$

Dividing both sides of the equation by r_i, one has

$$\frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{G r_i^2 \rho_i}{r^2} + \gamma \frac{r^n}{r_i} \ .$$

Substituting $u = r/r_i$, this becomes

$$\boxed{\ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} + \gamma u^n r_i^{n-1} \ .}$$

(b) The only dependence on r_i occurs in the last term, which is proportional to r_i^{n-1}. This dependence disappears if $\boxed{n = 1}$, since the zeroth power of any positive number is 1.

(c) This is exactly the same as the case discussed in the lecture notes, since the initial conditions do not depend on the differential equation. At $t = 0$,

$$r(r_i, 0) = r_i \ \text{(definition of } r_i\text{)}$$

$$\dot{r}(r_i, 0) = H_i r_i \ \text{(since } \vec{v}_i = H_i \vec{r}\text{)}.$$

Dividing these equations by r_i one has the intial conditions

$$\boxed{\begin{aligned} u &= 1 \\ \dot{u} &= H_i \ . \end{aligned}}$$

(d) $R(t)$ should obey the differential equation obtained in part (a) for u, for the value of n that was obtained in part (b): $n = 1$. So,

$$\ddot{R} = -\frac{4\pi}{3}\frac{G\rho_i}{R^2} + \gamma R \ .$$

Multiplying the equation by $\dot{R} \equiv dR/dt$, one finds

$$\frac{d^2R}{dt^2}\frac{dR}{dt} = \left\{ -\frac{4\pi}{3}\frac{G\rho_i}{R^2} + \gamma R \right\} \frac{dR}{dt} \ ,$$

which can be rewritten as

$$\frac{d}{dt}\left\{ \frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{R} - \frac{1}{2}\gamma R^2 \right\} = 0 \ .$$

Thus the quantity inside the curly brackets must be constant. Following the lecture notes, I will call this constant E:

$$\boxed{\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{R} - \frac{1}{2}\gamma R^2 = E \ .}$$

Or one can use the standardly defined quantity

$$k = -\frac{2E}{c^2} \ ,$$

in which case the equation can be written

$$-\frac{2}{c^2}\left\{ \frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{R} - \frac{1}{2}\gamma R^2 \right\} = k \ .$$

One can then rewrite the equation in the more standard form

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}\frac{G\rho_i}{R^3} + \gamma - \frac{kc^2}{R^2} \ .}$$

Additional Note: Historically, the constant γ corresponds to the "cosmological constant" which was introduced by Albert Einstein in 1917 in an effort to build a static model of the universe. The cosmological constant Λ is related to γ by

$$\gamma = \frac{1}{3}\Lambda c^2 \ .$$

Since the time-dependent mass density $\rho(t) = \rho_i/R^3$, the differential equation can be rewritten as

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - \frac{kc^2}{R^2} \ .$$

Since the cosmological constant contributes like a constant addition to the mass density, modern physicists interpret it as the mass density of the vacuum. The mass density of the vacuum is then related to Einstein's cosmological constant Λ by

$$\rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G} \ .$$

PROBLEM 3: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(35 points)*

(a) The general formula for H is

$$H = \frac{1}{R}\frac{dR}{dt} \ ,$$

but in this case there is the complication that R is given as a function of θ rather than t. But θ depends on t, so one must apply the chain rule:

$$\frac{d}{dt}R(\theta) = \frac{dR}{d\theta}\frac{d\theta}{dt} \ .$$

The standard formula for the Hubble constant can then be rewritten as

$$H(\theta) = \frac{1}{R}\frac{dR}{d\theta}\frac{d\theta}{dt} \ .$$

The parametric equations for R and t for an open, matter-dominated universe are given by

$$ct = \alpha\left(\sinh\theta - \theta\right)$$
$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ .$$

The hyperbolic trigonometric functions are defined by

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \ ,$$
$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \ ,$$

and they are differentiated as

$$\frac{d}{d\theta}\sinh\theta = \cosh\theta \ ,$$
$$\frac{d}{d\theta}\cosh\theta = \sinh\theta \ .$$

So, differentiating the parametric equations,

$$\frac{dR}{d\theta} = \alpha\sqrt{k}\sinh\theta \ ,$$
$$\frac{dt}{d\theta} = \frac{\alpha}{c}(\cosh\theta - 1) = \frac{1}{d\theta/dt} \ .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{\kappa}\alpha(\cosh\theta - 1)}\right]\left[\alpha\sqrt{\kappa}\sinh\theta\right]\left[\frac{c}{\alpha(\cosh\theta-1)}\right]$$

$$= \boxed{\frac{c\sinh\theta}{\alpha(\cosh\theta-1)^2}\ .}$$

(b) This problem can be attacked by at least three different methods. While you were expected to use only one, we will show all three.

(i) The equation from the front of the exam,

$$\alpha = \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

can be solved for ρ to give

$$\rho = \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{GR^3}\ .$$

Then substitute the parametric equation for $R(\theta)$:

$$\rho = \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{G}\frac{1}{\alpha^3\kappa^{3/2}(\cosh\theta-1)^3}$$

$$= \boxed{\frac{3c^2}{4\pi G\alpha^2(\cosh\theta-1)^3}\ .}$$

(ii) Starting from

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}\ ,$$

one can write

$$\frac{8\pi}{3}G\rho = H^2 + \frac{kc^2}{R^2}\ .$$

Recalling that we described open universes by using $\kappa \equiv -k$, this can be rewritten as

$$\frac{8\pi}{3}G\rho = H^2 - \frac{\kappa c^2}{R^2}\ .$$

Replacing H by the answer in part (a) and R by its parametric equation, one finds

$$\frac{8\pi}{3}G\rho = \frac{c^2\sinh^2\theta}{\alpha^2(\cosh\theta-1)^4} - \frac{\kappa c^2}{\alpha^2\kappa(\cosh\theta-1)^2}$$

$$= \frac{c^2}{\alpha^2(\cosh\theta-1)^4}\left[\sinh^2\theta - (\cosh\theta-1)^2\right]\ .$$

Now make use of the hypertrigonometric identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

to simplify:

$$\sinh^2\theta - (\cosh\theta-1)^2 = \sinh^2\theta - \cosh^2\theta + 2\cosh\theta - 1$$

$$= 2(\cosh\theta - 1)\ ,$$

so

$$\frac{8\pi}{3}G\rho = \frac{2c^2}{\alpha^2(\cosh\theta-1)^3}\ .$$

Dividing both sides of the equation by $(8\pi/3)G$, one duplicates the previous result.

(iii) ρ can also be found from $\ddot{R} = -(4\pi/3)G\rho R$. To evaluate $\ddot{R}$, again use the chain rule. Starting with $\dot{R}$,

$$\dot{R} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \alpha\sqrt{\kappa}\sinh\theta\frac{c}{\alpha(\cosh\theta-1)} = \frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\ .$$

Then

$$\begin{aligned}
\ddot{R} &= \frac{d\dot{R}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left[\frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\right]\frac{c}{\alpha(\cosh\theta - 1)} \\
&= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)}\left[\frac{\cosh\theta}{\cosh\theta - 1} - \frac{\sinh^2\theta}{(\cosh\theta - 1)^2}\right] \\
&= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}\left[\cosh\theta(\cosh\theta - 1) - \sinh^2\theta\right] \\
&= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}(1 - \cosh\theta) = -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} \; .
\end{aligned}$$

So

$$\ddot{R} = -\frac{4\pi}{3}G\rho R \quad \Longrightarrow \quad -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} = -\frac{4\pi}{3}G\rho\alpha\sqrt{\kappa}(\cosh\theta - 1) \; ,$$

and

$$\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3} \; .$$

(c) The basic formula that determines the physical value of the horizon distance is given by Eq. (5.7) of the lecture notes:

$$\ell_{p,\text{horizon}}(t) = R(t)\int_0^t \frac{c}{R(t')}dt' \; .$$

The complication here is that R is given as a function of θ, rather than t. The problem is handled, however, by a simple change of integration variables. One can change the integral over t' to an integral over θ', provided that one replaces

$$dt' \to \frac{dt'}{d\theta'}d\theta' = \frac{\alpha}{c}(\cosh\theta' - 1)d\theta' \; .$$

One must also re-express the limits of integration in terms of θ. So

$$\begin{aligned}
\ell_{p,\text{horizon}}(\theta) &= R(\theta)\int_0^\theta \frac{c}{R(\theta')}\frac{dt'}{d\theta'}d\theta' \\
&= \alpha\sqrt{\kappa}(\cosh\theta - 1)\int_0^\theta \frac{c}{\alpha\sqrt{\kappa}(\cosh\theta' - 1)}\frac{\alpha}{c}(\cosh\theta' - 1)\,d\theta' \; . \\
&= \alpha(\cosh\theta - 1)\int_0^\theta d\theta' = \boxed{\alpha\,\theta(\cosh\theta - 1) \; .}
\end{aligned}$$

(d) The key to this problem is the use of power series expansions. When this problem appeared as a quiz problem in 1992, I was rather surprised to find that many of the students seemed very inexperienced in this technique. It is a very useful method of approximation, so I strongly urge you to learn it if you don't know it already.

In general, any sufficiently smooth function $f(x)$ can be expanded about the point x_0 by the series

$$\begin{aligned}
f(x) = f(x_0) &+ \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 \\
&+ \frac{1}{3!}f'''(x_0)(x - x_0)^3 + \ldots \; ,
\end{aligned}$$

where the prime is used to denote a derivative. In particular, the exponential, sinh, and cosh functions can be expanded about $\theta = 0$ by the formulas

$$\begin{aligned}
e^\theta &= 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots \\
\sinh\theta &= \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^5}{7!} \ldots \\
\cosh\theta &= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots \; .
\end{aligned}$$

For this problem, we expand the parametric equations for $R(\theta)$ and $t(\theta)$, keeping the first nonvanishing term in the power series expansions:

$$t = \frac{\alpha}{c}(\sinh\theta - \theta) = \frac{\alpha}{c}\left(\frac{\theta^3}{3!} + \ldots\right)$$

$$R = \alpha\sqrt{\kappa}(\cosh\theta - 1) = \alpha\sqrt{\kappa}\left(\frac{\theta^2}{2!} + \ldots\right) .$$

The first expression can be solved for θ, giving

$$\theta \approx \left(\frac{6ct}{\alpha}\right)^{1/3} ,$$

which can be substituted into the second expression to give

$$\boxed{R \approx \frac{1}{2}\alpha\sqrt{\kappa}\left(\frac{6ct}{\alpha}\right)^{2/3} .}$$

The power series expansions for the sinh and cosh are valid whenever the terms left out are much smaller than the last term kept, which happens when $\theta \ll 1$. Given the above relation between θ and t, this condition is equivalent to

$$t \ll \frac{\alpha}{6c} .$$

Thus,

$$\boxed{t^* \approx \frac{\alpha}{6c}, \text{ or } t^* \approx \frac{\alpha}{c} .}$$

Since there is no precise meaning to the statement that an approximation is valid, there is no precise value for t^*. Some students placed criteria on the size of the first omitted term in the series, and then derived a more

precise value for t^*. These expressions for t^* were always in the form of a dimensionless constant times α/c. This approach is very good, but it was not required to get full credit for this problem.

(e) *Due to a delay in getting the exams printed, part (e) was dropped from the exam, and all students were automatically given the 10 points allocated to this part. However, students who did this part of the problem could earn up to 5 points of extra credit.*

The critical mass density satisfies the cosmological evolution equations for $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c .$$

Then

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} .$$

Now replace H by the answer to part (a), and ρ by the answer to part (b):

$$\begin{aligned}\Omega &= \frac{8\pi G}{3}\left[\frac{3}{4\pi}\frac{c^2}{G\alpha^2(\cosh\theta-1)^3}\right]\left[\frac{\alpha^2(\cosh\theta-1)^4}{c^2\sinh^2\theta}\right] \\ &= 2\frac{\cosh\theta-1}{\sinh^2\theta} = 2\frac{\cosh\theta-1}{\cosh^2\theta-1} \\ &= 2\frac{\cosh\theta-1}{(\cosh\theta+1)(\cosh\theta-1)} = \frac{2}{\cosh\theta+1} .\end{aligned}$$

So,

$$1 - \Omega = 1 - \frac{2}{\cosh\theta+1} = \frac{\cosh\theta-1}{\cosh\theta+1} .$$

Expanding numerator and denominator in power series,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots}{2 + \frac{\theta^2}{2!} + \ldots} .$$

Keeping only the leading terms,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2}}{2} = \frac{1}{4}\theta^2 \ ,$$

so

$$\boxed{1 - \Omega \approx \frac{1}{4}\left(\frac{6ct}{\alpha}\right)^{2/3} \ .}$$

This result shows that the deviation of Ω from 1 is amplified with time. This fact leads to a conundrum called the "flatness problem", which will be discussed later in the course.