MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — May 7, 2002
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QUIZ 4 SOLUTIONS, 2000

PROBLEM 1: DID YOU DO THE READING? *(30 points)*

(a) The correct answer, of course, is (iv). The other items are supposed to be plausible-sounding but flawed explanations. See Rowan-Robinson section 6.4, pages 102-3.

(b) The key fact is that they have a critical value for the cosmological constant, leading to an asymptotic period of near-static evolution. Technically one asymptotes to the Einstein static model at infinite negative time and then expands (this has no Big Bang), and the other asymptotes to the Einstein case at infinite positive time after a Big Bang and initial period of expansion. See Rowan-Robinson section 8.2, pages 133-4.

(c) The correct answer is, of course, (ii). Answer (i) is intended to be evil and tricky, but the others are merely wrong, and (v) is just a joke to compensate for confusing the students with (i). See Rowan-Robinson section 6.1, pages 97-9.

(d) This is a total trick question. Lepton number is, of course, conserved, so the factor is just 1. See Weinberg chapter 4, pages 91-4.

(e) The correct answer is (i). The others are all real reasons why it's hard to measure, although Weinberg's book emphasizes reason (v) a bit more than modern astrophysicists do: astrophysicists have been looking for other ways that deuterium might be produced, but no significant mechanism has been found. See Weinberg chapter 5, pages 114-7.

(f) The most obvious answers would be proton, neutron, and pi meson. However, any of the particles listed as baryons or mesons in Lecture Notes 11 would be correct. See Weinberg chapter 7, pages 136-8.

PROBLEM 2: TIME SCALES IN COSMOLOGY *(20 points)*

(a) 1 sec. *[This is the time at which the weak interactions begin to "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) 10^{-37} sec. *[We learned in Lecture Notes 7 that kT was about 1 MeV at $t = 1$ sec. Since 1 GeV = 1000 MeV, the value of kT that we want is 10^{19} times higher. In the radiation-dominated era $T \propto R^{-1} \propto t^{-1/2}$, so we get 10^{-38} sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) 10^{-5} sec. *[As in (c), we can use $t \propto T^{-2}$, with $kT \approx 1$ MeV at $t = 1$ sec.]*

PROBLEM 3: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS *(20 points)*

(a) In thermal equilibrium, the ratio of neutrons to protons is given by a Boltzmann factor,

$$\boxed{\frac{n_n}{n_p} = e^{-\Delta m\, c^2/kT}\ ,}$$

where $\Delta m = (m_n - m_p)$. For $\Delta m\, c^2 = 1.293 \times 10^6$ eV, $k = 8.617 \times 10^{-5}$ eV/K, and $T = 5 \times 10^{10}$ K, this gives

$$\frac{n_n}{n_p} = \exp\left\{-1.293 \times 10^6/(8.617 \times 10^{-5} \times 5 \times 10^{10})\right\} = \boxed{0.741\ .}$$

Caveat (for stat mech experts): The above formula would be a precise consequence of statistical mechanics if the neutron and proton were two possible energy levels of the same system. In this case one would describe the system using the *canonical ensemble*, which implies that the probability of the system existing in any specific state i is proportional to $\exp(-E_i/kT)$, where E_i is the energy of the state. However, the neutron and proton are not really different energy levels of the same system, because the conversion between neutrons and protons involves other particles as well; a sample conversion reaction would be

$$n + \nu_e \longleftrightarrow p + e^- \ ,$$

where ν_e is the electron neutrino, and e^- is the electron. This means that if the universe contained a very large density of electron neutrinos, then n-ν_e collisions would occur more frequently, and the reaction would be driven in the forward direction. Thus, a large density of electron neutrinos would lead to a lower ratio of neutrons to protons than the Boltzmann factor given above. Similarly, if the universe contained a large density of electrons, then the reaction would be driven in the reverse direction, and the ratio of neutrons to protons would be higher than the Boltzmann factor. A complete statistical mechanical treatment of this situation would use the *grand canonical ensemble*, which describes systems in which the number of particles of a given type can change by chemical reactions. In this formalism the density of each type of particle is related to a quantity called the *chemical potential* μ, where in general the relationship is given by

$$n = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}}{\exp\left[(E-\mu)/(kT)\right]\pm 1} E\,\mathrm{d}E$$

where the $+$ sign holds for Fermi particles, the $-$ sign holds for Bose particles, and the factor g has the same meaning as in Lecture Notes 7. The ratio of neutrons to protons is then given by

$$\frac{n_n}{n_p} = e^{-(\Delta m\, c^2 + \mu_\nu - \mu_e)/kT} \ ,$$

where μ_ν and μ_e represent the chemical potentials for electron neutrinos and electrons, respectively. In the early universe, however, the standard theories imply that the chemical potentials for electrons and neutrinos were both negligible.

(b) A larger Δm would mean that the Boltzmann factor described in the previous answer would be smaller, so that there would be fewer neutrons at any given temperature. Fewer neutrons implies less helium, since essentially all the neutrons that exist when the temperature falls enough for deuterium to become stable become bound into helium.

(c) There are at least four effects that occur when the electron mass/energy is taken as 1 KeV instead of 0.511 MeV. When I graded the problem, I gave full credit to any student who accurately described any one of the following:

(i) For the real mass/energy of 0.511 MeV the electron-positron pairs freeze out before nucleosynthesis, but a mass/energy of 1 KeV would mean that electron-positron pairs would behave as massless particles throughout the nucleosynthesis process. Just like adding an extra species of neutrino, this additional massless particle would mean that the expansion rate would be larger, since for a flat universe,

$$H^2 = \frac{8\pi}{3}G\rho \ ,$$

and

$$\rho = \frac{u}{c^2} = g\frac{\pi^2}{30}\frac{(kT)^4}{\hbar^3 c^5} \ .$$

Faster expansion means that the weak interactions "freeze out" earlier, since the freeze-out point is the time at which the interactions can no longer maintain equilibrium as the universe expands. An earlier freeze-out means a higher temperature of freeze-out and hence more neutrons at the time of freeze-out. In addition, the faster expansion rate means faster cooling, which means less time before the temperature of nucleosynthesis is reached, and therefore less time for neutrons to decay.

Thus, faster expansion means more neutrons. Since essentially all the neutrons present when the deuterium bottleneck breaks are collected into helium, this implies more helium.

(ii) The most important reactions that keep protons and neutrons in thermal equilibrium all involve electrons and positrons:

$$n + e^+ \longleftrightarrow p + \bar{\nu}_e$$

$$n + \nu_e \longleftrightarrow p + e^- \ .$$

If the electron-positron mass/energy were smaller, then the rates of all of these reactions would be enhanced. The reactions in which an e^+ or e^- appears in the initial state will be enhanced by the presence of more e^+'s and e^-'s, and the reactions in which they appear in the final state will be enhanced because a lighter final state is easier to produce. The enhanced rate for these reactions will keep neutrons and protons in thermal equilibrium longer, and hence to lower temperatures, and this would decrease the final abundance of neutrons. Thus this effect will go in the opposite direction as effect (i), leading to the production of less helium.

(iii) If the electron mass is decreased, then the neutron decay

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

becomes more exothermic, so it will happen more quickly. Thus more neutrons can decay, leading to less helium.

(iv) As mentioned in (i), lowering the mass/energy of electron-positron pairs to 1 KeV would mean that their freeze-out would not occur until after nucleosynthesis is over. In the real case, however, with $m_e c^2 = 0.511$ MeV, the electron-positron pairs start to freeze out at $t \approx 10$ sec. The energy released by this freeze-out heats the photons, protons, and neutrons, and this extra heat delays the time when the universe cools enough to break the deuterium bottleneck so that helium production can proceed. The delay allows more time for the neutrons to decay, resulting in less helium. Since the freeze-out that occurs for $m_e c^2 = 0.511$ MeV results in less helium, the absence of this freeze-out if $m_e c^2 = 1$ KeV would result in more helium.

Since the effects point in different directions, there is no easy way to know what the net effect will be. I (AHG) tried carrying out a full numerical integration, using the equations from P.J.E. Peebles, "Primordial helium abundance and the primordial fireball II," *Astrophysical Journal* **146**, 542-552 (1966). I found that the net effect of changing $m_e c^2$ to 1 KeV was to produce less helium. Apparently effects (ii) and (iii) above are the most significant. Of course I did not expect students to figure this out during the exam!

(d) Part (a) asked for the ratio of neutrons to protons, so its answer is

$$A = \frac{n_{\rm neutron}}{n_{\rm proton}} \ .$$

The fraction of the baryonic mass in neutrons is then

$$\frac{n_{\rm neutron}}{n_B} = \frac{n_{\rm neutron}}{n_{\rm neutron} + n_{\rm proton}} = \frac{\frac{n_{\rm neutron}}{n_{\rm proton}}}{\frac{n_{\rm neutron}}{n_{\rm proton}} + 1} = \frac{A}{1+A} \ .$$

The fraction of the baryonic mass in helium is twice this number, since after nucleosynthesis essentially all neutrons are in helium, and the mass of each helium nucleus is twice the mass of the neutrons within it. Thus

$$Y = \frac{2A}{1+A} \ .$$

This gives $\boxed{Y = 0.851.}$

PROBLEM 4: INFLATION AND THE HORIZON PROBLEM *(30 points)*

(a) As described in Lecture Notes 10, the photons of the cosmic background radiation have a uniform temperature (and hence energy) to an accuracy of $\boxed{\text{about one part in } 10^5.}$ (In addition, the motion of the earth through the cosmic background radiation produces an anisotropy of about one part in 10^3, but this is not really an anisotropy of the radiation itself.)

(b) Again, as described in Lecture Notes 10, the number is $\boxed{\text{about 90.}}$

(c) In an inflationary model, at very early times the region from which the observed universe evolves can be much smaller than in conventional cosmology. The uniformity can be established before inflation, when the region was incredibly small. Then inflation can expand this tiny region to become large enough to easily encompass everything that we see.

(d) To estimate the time $t_{\rm GUT}$ at which kT was equal to $M_{\rm GUT}$, we use the formula from the front of the exam for a flat radiation-dominated universe:

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 gG}\right)^{1/4} \frac{1}{\sqrt{t}} .$$

For a typical grand unified theory, we will use $g_{\rm GUT} \approx 100$ (as stated in Lecture Notes 12). So

$$t_{\rm GUT} \approx \frac{1}{M_{\rm GUT}^2}\sqrt{\frac{45\hbar^3 c^5}{16\pi^3 g_{\rm GUT} G}} .$$

Numerically,

$$\begin{aligned} t_{\rm GUT} &\approx \\ &\frac{1}{(10^{25}\ \text{eV})^2}\sqrt{\frac{45\cdot(6.582\times10^{-16}\ \text{eV}\cdot\text{s})^3\cdot(2.998\times10^{10}\ \text{cm}\cdot\text{s}^{-1})^5}{16\pi^3\cdot 100\cdot 5.573\times10^{-8}\ \text{cm}^3\cdot\text{g}^{-1}\cdot\text{s}^{-2}}} \\ &\times\sqrt{\frac{1\ \text{eV}}{1.602\times10^{-12}\text{erg}}\,\frac{1\ \text{erg}}{\text{g}\cdot\text{cm}^2\cdot\text{s}^{-2}}} \\ &= 2.42\times10^{-39}\ \text{s} . \end{aligned}$$

For a radiation-dominated universe the physical horizon length $\ell_{\rm hor}$ is $2ct$, so

$$\begin{aligned} \ell_{\rm hor}(t_{\rm GUT}) &= 2ct_{\rm GUT} \\ &\approx 2\cdot 2.998\times10^{10}\ \text{cm}\cdot\text{s}^{-1}\cdot 2.42\times10^{-39}\ \text{s} \\ &= 1.45\times10^{-28}\ \text{cm} . \end{aligned}$$

We are asked to compare the horizon length $\ell_{\rm hor}(t_{\rm GUT})$ with the diameter at that time of the region that will evolve to become the presently observed universe. The radius of the presently observed universe is about equal to the horizon distance calculated for a matter-dominated universe,

$$\ell_{\rm hor}(t_0) \approx 3ct_0 .$$

At $t_{\rm GUT}$, the diameter of this region was

$$d(t_{\rm GUT}) \approx 6ct_0\,\frac{R(t_{\rm GUT})}{R(t_0)} .$$

To determine this ration of scale factors, one can use the fact that entropy is roughly conserved between $t_{\rm GUT}$ and the present. (It is this statement that would be radically changed in an inflationary model, in which a huge amount of entropy is produced by the reheating at the end of the period of inflation.) Since entropy density s is proportional to gT^3, and conservation of entropy implies that $R^3 s = const$, one has

$$R^3(t_{\rm GUT})g_{\rm GUT}T_{\rm GUT}^3 \approx R^3(t_0)g_0T_0^3 .$$

Here g_0 is the present value of g, which we will take to be about equal to 10. Then

$$d(t_{\rm GUT}) \approx 6ct_0\left(\frac{g_0}{g_{\rm GUT}}\right)^{1/3}\frac{T_0}{T_{\rm GUT}} = 6ct_0\left(\frac{g_0}{g_{\rm GUT}}\right)^{1/3}\frac{kT_0}{M_{\rm GUT}} .$$

Numerically,

$$
\begin{aligned}
d(t_{\rm GUT}) &\approx 6 \cdot \left(2.998 \times 10^{10} \text{ cm} \cdot \text{s}^{-1}\right) \cdot \left(1.5 \times 10^{10} \text{ yr}\right) \\
&\qquad \times \left(3.156 \times 10^{7} \text{ s} \cdot \text{yr}^{-1}\right) \cdot \left(\frac{10}{100}\right)^{1/3} \\
&\qquad \times \frac{\left(8.617 \times 10^{-5} \text{ eV} \cdot \text{K}^{-1}\right) \cdot 2.7 \text{ K}}{10^{25} \text{ eV}} \\
&\approx 0.920 \text{ cm} \ .
\end{aligned}
$$

Finally, the ratio is given by

$$
\frac{d(t_{\rm GUT})}{\ell_{\rm hor}(t_{\rm GUT})} \approx \frac{0.920 \text{ cm}}{1.45 \times 10^{-28} \text{ cm}} \approx 6.34 \times 10^{27} \sim \boxed{10^{28}} \ .
$$

That is, at the time when kT was equal to $T_{\rm GUT}$, the diameter of the region that will evolve to become the presently observed universe was about 10^{28} times larger than the horizon length at that time.