MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 3, 2002
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REVIEW PROBLEMS FOR QUIZ 2

QUIZ DATE: Tuesday, April 9, 2002

COVERAGE: Lecture Notes 6 and 7; Problem Sets 3 and 4; Rowan-Robinson, Chapters 4 and 5; Weinberg, Chapters 4 and 5. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems. The starred problems are the ones that I recommend that you review most carefully: Problems 1, 2, 4, 8, 11, 12, and 13. The starred problems do not include any reading questions, but parts of these reading questions may also recur on the upcoming quiz.**

CALCULATORS: Please bring your calculators to this quiz. There may be some numerical problems.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. Except for a few parts which are clearly marked, they are all problems that I would consider fair for the coming quiz. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, and 2000. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. Since we will be having only three quizzes this year, the coverage of each quiz will not necessarily match the quizzes from previous years. The material for the upcoming quiz, however, is an almost perfect match with Quiz 3 of 2000. The only difference is that the reading from Weinberg included in our upcoming quiz was not part of the Quiz 3 material in 2000.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" at the beginning. For the second quiz, this useful information will be the following:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

EVOLUTION OF A FLAT ($\boldsymbol{\Omega \equiv \rho/\rho_c = 1}$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\, \rho(t_i)$$

Closed ($\Omega > 1$): $$ct = \alpha(\theta - \sin\theta) \; ,$$
$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \; ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$): $$ct = \alpha\left(\sinh\theta - \theta\right)$$
$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \; ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2} \; ,$$
$$\kappa \equiv -k \; .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \; ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or: $$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

PHYSICAL CONSTANTS:

$$k = \text{Boltzmann's constant} = 1.381\times10^{-16}\,\text{erg/K} = 8.617\times10^{-5}\,\text{eV/K} \; ,$$

$$\hbar = \frac{h}{2\pi} = 1.055\times10^{-27}\ \text{erg-sec} = 6.582\times10^{-16}\ \text{eV-sec} \; ,$$

$$c = 2.998\times10^{10}\ \text{cm/sec}$$

$$1\ \text{eV} = 1.602\times10^{-12}\ \text{erg} \; .$$

BLACK-BODY RADIATION:

$$u = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = -\frac{1}{3} u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3} \ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions} \ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 \ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45 \hbar^3 c^5}{16 \pi^3 g G} \right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

∗ PROBLEM 1: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(30 points)*

The following problem was Problem 3, Quiz 2, 1998.

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

(a) *(7 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi =$ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(8 points)* Write an expression for the physical horizon distance $\ell_{\rm phys}$ at time t. You should leave your answer in the form of a definite integral.

The form of $R(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.,* dominated by nonrelativistic matter), then $R(t)$ is described by the parametric equations

$$ct = \alpha(\theta - \sin\theta) ,$$

$$R = \alpha(1 - \cos\theta) ,$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{c^2} .$$

These equations are identical to those on the front of the exam, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where θ is the parameter used to describe the evolution.

(d) *(5 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

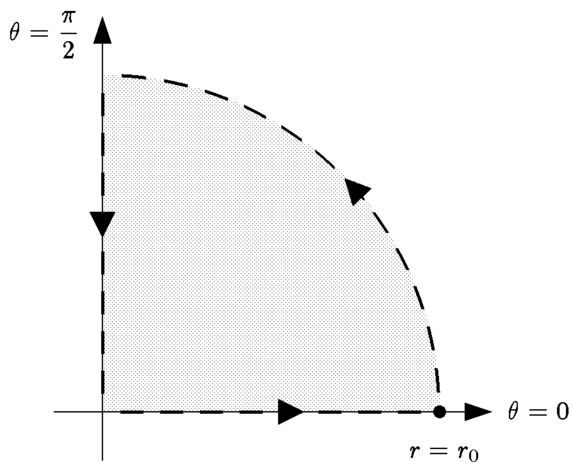
* PROBLEM 2: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC *(25 points)*

The following problem was Problem 3, Quiz 2, 1994:

Suppose a two dimensional space, described in polar coordinates (r, θ), has a metric given by

$$ds^2 = (1 + ar)^2 \, dr^2 + r^2 (1 + br)^2 \, d\theta^2 \ ,$$

where a and b are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the $\theta = 0$ line to $r = r_0$, then moving at fixed r to $\theta = \pi/2$, and then moving back to the origin at fixed θ. The path is shown below:



a) *(10 points)* Find the total length of this path.

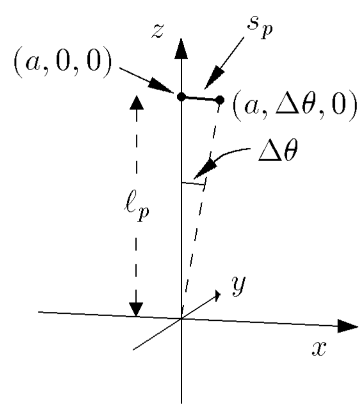
b) *(15 points)* Find the area enclosed by this path.

PROBLEM 3: GEOMETRY IN A CLOSED UNIVERSE *(25 points)*

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with $k = 1$. The questions below all pertain to some fixed time t, so the scale factor can be written simply as R, dropping its explicit t-dependence.

A small rod has one end at the point ($r = a$, $\theta = 0$, $\phi = 0$) and the other end at the point ($r = a$, $\theta = \Delta\theta$, $\phi = 0$). Assume that $\Delta\theta \ll 1$.



(a) Find the physical distance ℓ_p from the origin ($r = 0$) to the first end $(a, 0, 0)$ of the rod. You may find one of the following integrals useful:

$$\int \frac{dr}{\sqrt{1-r^2}} = \sin^{-1} r$$

$$\int \frac{dr}{1-r^2} = \frac{1}{2} \ln \left(\frac{1+r}{1-r} \right) .$$

(b) Find the physical length s_p of the rod. Express your answer in terms of the scale factor R, and the coordinates a and $\Delta\theta$.

(c) Note that $\Delta\theta$ is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance ℓ_p, the physical length s_p, and the scale factor R.

∗ PROBLEM 4: THE GENERAL SPHERICALLY SYMMETRIC METRIC *(20 points)*

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three dimensional space which is spherically symmetric about a particular point, coordinates can be found so that the metric has the form

$$ds^2 = dr^2 + \rho^2(r) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right]$$

for some function $\rho(r)$. The coordinates θ and ϕ have their usual ranges: θ varies between 0 and π, and ϕ varies from 0 to 2π, where $\phi = 0$ and $\phi = 2\pi$ are identified. Given this metric, consider the sphere whose outer boundary is defined by $r = r_0$.

(a) Find the physical radius a of the sphere. (By "radius", I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)

(b) Find the physical area of the surface of the sphere.

(c) Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.

(d) Suppose a new radial coordinate σ is introduced, where σ is related to r by

$$\sigma = r^2 \ .$$

Express the metric in terms of this new variable.

PROBLEM 5: VOLUMES IN A ROBERTSON-WALKER UNIVERSE *(20 points)*

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} \ .$$

Calculate the volume $V(r_{\rm max})$ of the sphere described by

$$r \leq r_{\rm max} \ .$$

You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

PROBLEM 6: THE SCHWARZSCHILD METRIC *(25 points)*

The follow problem was Problem 4, Quiz 3, 1992:

The space outside a spherically symmetric mass M is described by the Schwarzschild metric, given at the front of the exam. Two observers, designated A and B, are located along the same radial line, with values of the coordinate r given by r_A and r_B, respectively, with $r_A < r_B$. You should assume that both observers lie outside the Schwarzschild horizon.

a) *(5 points)* Write down the expression for the Schwarzschild horizon radius R_{Sch}, expressed in terms of M and fundamental constants.

b) *(5 points)* What is the proper distance between A and B? It is okay to leave the answer to this part in the form of an integral that you do not evaluate—but be sure to clearly indicate the limits of integration.

c) *(5 points)* Observer A has a clock that emits an evenly spaced sequence of ticks, with proper time separation $\Delta\tau_A$. What will be the coordinate time separation Δt_A between these ticks?

d) *(5 points)* At each tick of A's clock, a light pulse is transmitted. Observer B receives these pulses, and measures the time separation on his own clock. What is the time interval $\Delta\tau_B$ measured by B.

e) *(5 points)* Suppose that the object creating the gravitational field is a static black hole, so the Schwarzschild metric is valid for all r. Now suppose that one considers the case in which observer A lies on the Schwarzschild horizon, so $r_A \equiv R_{\text{Sch}}$. Is the proper distance between A and B finite for this case? Does the time interval of the pulses received by B, $\Delta\tau_B$, diverge in this case?

PROBLEM 7: GEODESICS *(20 points)*

The following problem was Problem 4, Quiz 2, 1986:

Ordinary Euclidean two-dimensional space can be described in polar coordinates by the metric

$$ds^2 = dr^2 + r^2\, d\theta^2 \ .$$

(a) Suppose that $r(\lambda)$ and $\theta(\lambda)$ describe a geodesic in this space, where the parameter λ is the arc length measured along the curve. Use the general formula on the front of the exam to obtain explicit differential equations which $r(\lambda)$ and $\theta(\lambda)$ must obey.

(b) Now introduce the usual Cartesian coordinates, defined by

$$x = r\cos\theta \ ,$$

$$y = r\sin\theta \ .$$

Use your answer to (a) to show that the line $y = 1$ is a geodesic curve.

∗ PROBLEM 8: METRIC OF A STATIC GRAVITATIONAL FIELD
(30 points)

The following problem was Problem 2, Quiz 3, 1990:

In this problem we will consider the metric

$$ds^2_{\rm ST} = -\left[c^2 + 2\phi(\vec{x})\right]\, dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \quad ,$$

which describes a static gravitational field. Here i runs from 1 to 3, with the identifications $x^1 \equiv x$, $x^2 \equiv y$, and $x^3 \equiv z$. The function $\phi(\vec{x})$ depends only on the spatial variables $\vec{x} \equiv (x^1, x^2, x^3)$, and not on the time coordinate t.

(a) Suppose that a radio transmitter, located at $\vec{x}_e$, emits a series of evenly spaced pulses. The pulses are separated by a proper time interval ΔT_e, as measured by a clock at the same location. What is the coordinate time interval Δt_e between the emission of the pulses? (I.e., Δt_e is the difference between the time coordinate t at the emission of one pulse and the time coordinate t at the emission of the next pulse.)

(b) The pulses are received by an observer at $\vec{x}_r$, who measures the time of arrival of each pulse. What is the ***coordinate*** time interval Δt_r between the reception of successive pulses?

(c) The observer uses his own clocks to measure the proper time interval ΔT_r between the reception of successive pulses. Find this time interval, and also the redshift z, defined by

$$1 + z = \frac{\Delta T_r}{\Delta T_e} \quad .$$

First compute an exact expression for z, and then expand the answer to lowest order in $\phi(\vec{x})$ to obtain a weak-field approximation. (This weak-field approximation is in fact highly accurate in all terrestrial and solar system applications.)

(d) A freely falling particle travels on a spacetime geodesic $x^\mu(\tau)$, where τ is the proper time. (I.e., τ is the time that would be measured by a clock moving with the particle.) The trajectory is described by the geodesic equation

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \quad ,$$

where the Greek indices $(\mu, \nu, \lambda, \sigma$, etc.) run from 0 to 3, and are summed over when repeated. Calculate an explicit expression for

$$\frac{d^2x^i}{d\tau^2} \; ,$$

valid for $i = 1, 2$, or 3. (It is acceptable to leave quantities such as $dt/d\tau$ or $dx^i/d\tau$ in the answer.)

PROBLEM 9: GEODESICS ON THE SURFACE OF A SPHERE

In this problem we will test the geodesic equation by computing the geodesic curves on the surface of a sphere. We will describe the sphere as in Lecture Notes 6, with metric given by

$$ds^2 = a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \quad .$$

(a) Clearly one geodesic on the sphere is the equator, which can be parametrized by $\theta = \pi/2$ and $\phi = \psi$, where ψ is a parameter which runs from 0 to 2π. Show that if the equator is rotated by an angle α about the x-axis, then the equations become:

$$\cos\theta = \sin\psi \sin\alpha$$

$$\tan\phi = \tan\psi \cos\alpha \quad .$$

(b) Using the generic form of the geodesic equation on the front of the exam, derive the differential equation which describes geodesics in this space.

(c) Show that the expressions in (a) satisfy the differential equation for the geodesic. Hint: The algebra on this can be messy, but I found things were reasonably simple if I wrote the derivatives in the following way:

$$\frac{d\theta}{d\psi} = -\frac{\cos\psi \sin\alpha}{\sqrt{1 - \sin^2\psi \sin^2\alpha}} \quad , \qquad \frac{d\phi}{d\psi} = \frac{\cos\alpha}{1 - \sin^2\psi \sin^2\alpha} \quad .$$

PROBLEM 10: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

Today the temperature of the cosmic microwave background radiation is 2.7°K. Calculate the number density of photons in this radiation. What is the number density of thermal neutrinos left over from the big bang?

∗ PROBLEM 11: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

∗ PROBLEM 12: A NEW SPECIES OF LEPTON

The following problem was Problem 2, Quiz 3, 1992, worth 25 points.

Suppose the calculations describing the early universe were modified by including an additional, hypothetical lepton, called an 8.286ion. The 8.286ion has roughly the same properties as an electron, except that its mass is given by $mc^2 = 0.750$ MeV.

Parts (a)-(c) of this question require numerical answers, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included. (For example, if you were asked how many meters a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$.)

a) *(5 points)* What would be the number density of 8.286ions, in particles per cubic meter, when the temperature T was given by $kT = 3$ MeV?

b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the value of the mass density at $t = .01$ sec? You may assume that 0.75 MeV $\ll kT \ll$ 100 MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, e^+-e^- pairs, and 8.286ion-anti8286ion pairs. Express your answer in the units of gm-cm^{-3}.

c) *(5 points)* Under the same assumptions as in (b), what would be the value of kT, in MeV, at $t = .01$ sec?

d) *(5 points)* When nucleosynthesis calculations are modified to include the effect of the 8.286ion, is the production of helium increased or decreased? Explain your answer in a few sentences. *[This part is not appropriate for Quiz 2 of this year (2002), as we have not yet studied nucleosynthesis.]*

e) *(5 points)* Suppose the neutrinos decouple while $kT \gg 0.75$ MeV. If the 8.286ions are included, what does one predict for the value of T_ν/T_γ today? (Here T_ν denotes the temperature of the neutrinos, and T_γ denotes the temperature of the cosmic background radiation photons.)

∗ PROBLEM 13: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(20 points)*

The following problem was Problem 3, Quiz 3, 1998.

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(10 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $R(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{R^5(t)} \ .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(5 points)* Find the behavior of the scale factor $R(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $R(t)$ up to a constant factor.

(c) *(5 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{6}\rho c^2 \ .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{R^n(t)} \ .$$

Find the power n.

PROBLEM 14: DID YOU DO THE READING?

The following problem was Problem 1 on Quiz 3, 2000, where it was worth 25 points. It is based on Chapters 4 and 5 of Rowan-Robinson's book, 3rd edition.

(a) *(5 points)* What does Birkhoff's theorem state?

(i) A uniform medium outside a spherical cavity has no gravitational effect inside the cavity.

(ii) A redshifted version of a blackbody spectrum remains a blackbody spectrum, but at a lower temperature.

(iii) The universe is homogeneous and isotropic.

(iv) The effect of a uniform gravitational field is indistinguishable from the effect of a uniform acceleration.

(v) A Hubble flow is the only global motion allowed in a completely homogeneous and isotropic universe.

(b) *(5 points)* Two special-case cosmological models are the "Milne model" and the "Einstein de-Sitter model". Pick one of them and briefly describe its distinguishing characteristics from other common models (be sure in your answer to specify which one you are describing).

(c) *(5 points)* The observation that only about one-quarter of the primordial gas in the universe is helium means that the Big Bang appears to have produced about seven times as many protons as neutrons. What can help explain this asymmetry?

(i) The very energetic conditions of the early universe forced the GUT proton decay process to run in reverse.

(ii) The early population of muons preferentially decayed into protons, boosting their density.

(iii) The neutron is heavier than the proton, causing the weak reaction rates to shift as the temperature dropped.

(iv) The rest of the neutrons formed into neutron stars, and thus aren't observed in the primordial gas at all.

(v) The same asymmetry which gave us more particles than antiparticles also produced more down quarks than up quarks.

(d) *(5 points)* What causes the dipole anisotropy in the cosmic microwave background radiation?

(e) *(5 points)* Place the following events in order in the standard Big Bang picture, from earliest to latest. A valid answer would read, for instance: v, iv, iii, ii, i.

(i) Primordial nucleosynthesis

(ii) Decoupling of electron neutrinos

(iii) Quark confinement

(iv) Recombination

(v) Muon annihiliation

PROBLEM 15: GEODESICS IN A CLOSED UNIVERSE

The following problem was Problem 3, Quiz 3, 2000, where it was worth 40 points plus 5 points extra credit.

Consider the case of closed Robertson-Walker universe. Taking $k = 1$, the spacetime metric can be written in the form

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} .$$

We will assume that this metric is given, and that $R(t)$ has been specified. While galaxies are approximately stationary in the comoving coordinate system described by this metric, we can still consider an object that moves in this system. In particular, in this problem we will consider an object that is moving in the radial direction (r-direction), under the influence of no forces other than gravity. Hence the object will travel on a geodesic.

(a) *(7 points)* Express $d\tau/dt$ in terms of dr/dt.

(b) *(3 points)* Express $dt/d\tau$ in terms of dr/dt.

(c) *(10 points)* If the object travels on a trajectory given by the function $r_p(t)$ between some time t_1 and some later time t_2, write an integral which gives the total amount of time that a clock attached to the object would record for this journey.

(d) *(10 points)* During a time interval dt, the object will move a coordinate distance

$$dr = \frac{dr}{dt} dt .$$

Let $d\ell$ denote the physical distance that the object moves during this time. By "physical distance," I mean the distance that would be measured by a comoving observer (an observer stationary with respect to the coordinate system) who is located at the same point. The quantity $d\ell/dt$ can be regarded as the physical speed $v_{\rm phys}$ of the object, since it is the speed that would be measured by a comoving observer. Write an expression for $v_{\rm phys}$ as a function of dr/dt and r.

(e) *(10 points)* Using the formulas at the front of the exam, derive the geodesic equation of motion for the coordinate r of the object. Specifically, you should derive an equation of the form

$$\frac{d}{d\tau}\left[A \frac{dr}{d\tau} \right] = B \left(\frac{dt}{d\tau} \right)^2 + C \left(\frac{dr}{d\tau} \right)^2 + D \left(\frac{d\theta}{d\tau} \right)^2 + E \left(\frac{d\phi}{d\tau} \right)^2 ,$$

where A, B, C, D, and E are functions of the coordinates, some of which might be zero.

(f) *(5 points EXTRA CREDIT)* On Problem 4 of Problem Set 3 we learned that in a flat Robertson-Walker metric, the relativistically defined momentum of a particle,

$$p = \frac{m v_{\rm phys}}{\sqrt{1 - \frac{v_{\rm phys}^2}{c^2}}} \ ,$$

falls off as $1/R(t)$. Use the geodesic equation derived in part (e) to show that the same is true in a closed universe.

SOLUTIONS

PROBLEM 1: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\, dt^2 + R^2(t) d\psi^2 \ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{R^2(t)} \ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm \frac{c}{R(t)} \ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\text{hor}} = \int_0^t \frac{c}{R(t')} dt' \ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\text{hor}}$, which according to the metric is given by

$$\ell_{\text{phys}}(t) = \int_{\psi=0}^{\psi=\psi_{\text{hor}}} ds = \int_0^{\psi_{\text{hor}}} R(t)\, d\psi = \boxed{R(t) \int_0^t \frac{c}{R(t')} dt' \ .}$$

(c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{R(t)} \ .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) \ .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1-\cos\theta)}{R(t)} \; .$$

Then using $R = \alpha(1-\cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1 \; .}$$

(d) This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ must also change by 2π. From $R = \alpha(1-\cos\theta)$, one can see that R returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1 \; .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 6. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

$$x^2 + y^2 + z^2 + w^2 = a^2 \; ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0, 0, 0, 1)$ can be called the north pole. Each point (x, y, z, w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x, y, z, w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0, 0, 0, -1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch.

Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\rm Crunch}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\rm Crunch} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\rm Crunch}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\rm Crunch} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost 1/2, and it would approach 1/2 as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 2: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC

a) Along the first segment $d\theta = 0$, so $ds^2 = (1 + ar)^2\, dr^2$, or $ds = (1 + ar)\, dr$. Integrating, the length of the first segment is found to be

$$S_1 = \int_0^{r_0} (1 + ar)\, dr = r_0 + \frac{1}{2} a r_0^2 \ .$$

Along the second segment $dr = 0$, so $ds = r(1 + br)\, d\theta$, where $r = r_0$. So the length of the second segment is

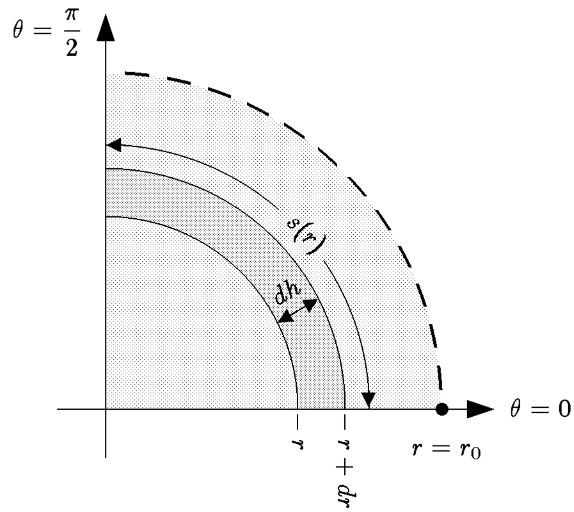
$$S_2 = \int_0^{\pi/2} r_0(1 + br_0)\, d\theta = \frac{\pi}{2} r_0(1 + br_0) \ .$$

Finally, the third segment is identical to the first, so $S_3 = S_1$. The total length is then

$$S = 2S_1 + S_2 = 2\left(r_0 + \frac{1}{2} a r_0^2\right) + \frac{\pi}{2} r_0(1 + br_0)$$

$$= \boxed{\left(2 + \frac{\pi}{2}\right) r_0 + \frac{1}{2}(2a + \pi b) r_0^2 \ .}$$

b) To find the area, it is best to divide the region into concentric strips as shown:



Note that the strip has a coordinate width of dr, but the distance across the width of the strip is determined by the metric to be

$$dh = (1 + ar)\, dr \ .$$

The length of the strip is calculated the same way as S_2 in part (a):

$$s(r) = \frac{\pi}{2} r(1 + br) \ .$$

The area is then

$$dA = s(r)\, dh \ ,$$

so

$$\begin{aligned} A &= \int_0^{r_0} s(r)\, dh \\ &= \int_0^{r_0} \frac{\pi}{2} r(1 + br)(1 + ar)\, dr \\ &= \frac{\pi}{2} \int_0^{r_0} [r + (a + b)r^2 + abr^3]\, dr \\ &= \boxed{\frac{\pi}{2} \left[\frac{1}{2} r_0^2 + \frac{1}{3}(a + b) r_0^3 + \frac{1}{4} abr_0^4 \right]} \end{aligned}$$

PROBLEM 3: GEOMETRY IN A CLOSED UNIVERSE

(a) As one moves along a line from the origin to $(a, 0, 0)$, there is no variation in θ or ϕ. So $d\theta = d\phi = 0$, and

$$ds = \frac{R\,dr}{\sqrt{1-r^2}} \ .$$

So

$$\boxed{\ell_p = \int_0^a \frac{R\,dr}{\sqrt{1-r^2}} = R \sin^{-1} a \ .}$$

(b) In this case it is only θ that varies, so $dr = d\phi = 0$. So

$$ds = Rr\,d\theta \ ,$$

so

$$\boxed{s_p = Ra\,\Delta\theta \ .}$$

(c) From part (a), one has

$$a = \sin(\ell_p/R) \ .$$

Inserting this expression into the answer to (b), and then solving for $\Delta\theta$, one has

$$\boxed{\Delta\theta = \frac{s_p}{R\sin(\ell_p/R)} \ .}$$

Note that as $R \to \infty$, this approaches the Euclidean result, $\Delta\theta = s_p/\ell_p$.

PROBLEM 4: THE GENERAL SPHERICALLY SYMMETRIC METRIC

(a) The metric is given by

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .$$

The radius a is defined as the physical length of a radial line which extends from the center to the boundary of the sphere. The length of a path is just the integral of ds, so

$$a = \int_{\substack{\text{radial path from}\\ \text{origin to } r_0}} ds \ .$$

The radial path is at a constant value of θ and ϕ, so $d\theta = d\phi = 0$, and then $ds = dr$. So

$$a = \int_0^{r_0} dr = \boxed{r_0 \ .}$$

(b) On the surface $r = r_0$, so $dr \equiv 0$. Then

$$ds^2 = \rho^2(r_0)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .$$

To find the area element, consider first a path obtained by varying only θ. Then $ds = \rho(r_0)\, d\theta$. Similarly, a path obtained by varying only ϕ has length $ds = \rho(r_0)\sin\theta\, d\phi$. Furthermore, these two paths are perpendicular to each other, a fact that is incorporated into the metric by the absence of a $dr\, d\theta$ term. Thus, the area of a small rectangle constructed from these two paths is given by the product of their lengths, so

$$dA = \rho^2(r_0)\sin\theta\, d\theta\, d\phi \ .$$

The area is then obtained by integrating over the range of the coordinate variables:

$$\begin{aligned} A &= \rho^2(r_0)\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \\ &= \rho^2(r_0)(2\pi)\left(-\cos\theta\Big|_0^{\pi}\right) \\ \Longrightarrow & \quad \boxed{A = 4\pi\rho^2(r_0) \ .} \end{aligned}$$

As a check, notice that if $\rho(r) = r$, then the metric becomes the metric of Euclidean space, in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere, $4\pi r^2$.

(c) As in Problem 2 of Problem Set 3 (2000), we can imagine breaking up the volume into spherical shells of infinitesimal thickness, with a given shell extending from r to $r + dr$. By the previous calculation, the area of such a shell is $A(r) = 4\pi\rho^2(r)$. (In the previous part we considered only the case $r = r_0$, but the same argument applies for any value of r.) The thickness of the shell is just the path length ds of a radial path corresponding to the coordinate interval dr. For radial paths the metric reduces to $ds^2 = dr^2$, so the thickness of the shell is $ds = dr$. The volume of the shell is then

$$dV = 4\pi\rho^2(r)\, dr \ .$$

The total volume is then obtained by integration:

$$\boxed{V = 4\pi \int_0^{r_0} \rho^2(r)\, dr \ .}$$

Checking the answer for the Euclidean case, $\rho(r) = r$, one sees that it gives $V = (4\pi/3)r_0^3$, as expected.

(d) If r is replaced by a new coordinate $\sigma \equiv r^2$, then the infinitesimal variations of the two coordinates are related by

$$\frac{d\sigma}{dr} = 2r = 2\sqrt{\sigma}\ ,$$

so

$$dr^2 = \frac{d\sigma^2}{4\sigma}\ .$$

The function $\rho(r)$ can then be written as $\rho(\sqrt{\sigma}\,)$, so

$$\boxed{ds^2 = \frac{d\sigma^2}{4\sigma} + \rho^2(\sqrt{\sigma}\,)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\ .}$$

PROBLEM 5: VOLUMES IN A ROBERTSON-WALKER UNIVERSE

The product of differential length elements corresponding to infinitesimal changes in the coordinates r, θ and ϕ equals the differential volume element dV. Therefore

$$dV = R(t)\frac{dr}{\sqrt{1-kr^2}} \times R(t) r d\theta \times R(t) r \sin\theta d\phi$$

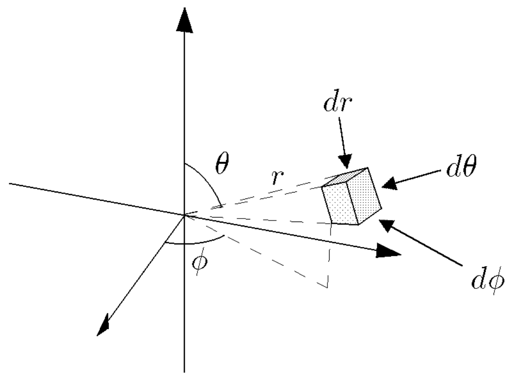
The total volume is then

$$V = \int dV = R^3(t) \int_0^{r_{\max}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\sqrt{1-kr^2}}$$

We can do the angular integrations immediately:

$$\boxed{V = 4\pi R^3(t) \int_0^{r_{max}} \frac{r^2 dr}{\sqrt{1-kr^2}}\ .}$$

[Pedagogical Note: If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:



The cell includes the volume lying between r and $r + dr$, between θ and $\theta + d\theta$, and between ϕ and $\phi + d\phi$. In the limit as dr, $d\theta$, and $d\phi$ all approach zero, the cell approaches a rectangular solid with sides of length:

$$\begin{aligned} ds_1 &= R(t)\frac{dr}{\sqrt{1-kr^2}} \\ ds_2 &= R(t)r\,d\theta \\ ds_3 &= R(t)r\sin\theta\,d\theta\ . \end{aligned}$$

Here each ds is calculated by using the metric to find ds^2, in each case allowing only one of the quantities dr, $d\theta$, or $d\phi$ to be nonzero. The infinitesimal volume element is then $dV = ds_1 ds_2 ds_3$, resulting in the answer above. The derivation relies on the orthogonality of the dr, $d\theta$, and $d\phi$ directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as $dr\,d\theta$.]

[Extension: The integral can in fact be carried out, using the substitution

$$\sqrt{k}\,r = \sin\psi \quad \text{(if } k > 0\text{)}$$

$$\sqrt{-k}\,r = \sinh\psi \quad \text{(if } k > 0\text{)}.$$

The answer is

$$V = \begin{cases} 2\pi R^3(t)\left[\dfrac{\sin^{-1}\left(\sqrt{k}\,r_{\rm max}\right)}{k^{3/2}} - \dfrac{\sqrt{1-kr_{\rm max}^2}}{k}\right] & \text{(if } k > 0\text{)} \\ 2\pi R^3(t)\left[\dfrac{\sqrt{1-kr_{\rm max}^2}}{(-k)} - \dfrac{\sinh^{-1}\left(\sqrt{-k}\,r_{\rm max}\right)}{(-k)^{3/2}}\right] & \text{(if } k < 0\text{)} \quad .] \end{cases}$$

PROBLEM 6: THE SCHWARZSCHILD METRIC

a) The Schwarzschild horizon is the value of r for which the metric becomes singular. Since the metric contains the factor

$$\left(1 - \frac{2GM}{rc^2}\right) ,$$

it becomes singular at

$$\boxed{R_{\text{Sch}} = \frac{2GM}{c^2} .}$$

b) The separation between A and B is purely in the radial direction, so the proper length of a segment along the path joining them is given by

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 ,$$

so

$$ds = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} .$$

The proper distance from A to B is obtained by adding the proper lengths of all the segments along the path, so

$$\boxed{s_{AB} = \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} .}$$

EXTENSION: The integration can be carried out explicitly. First use the expression for the Schwarzschild radius to rewrite the expression for s_{AB} as

$$s_{AB} = \int_{r_A}^{r_B} \frac{\sqrt{r}\, dr}{\sqrt{r - R_{\text{Sch}}}} .$$

Then introduce the hyperbolic trigonometric substitution

$$r = R_{\text{Sch}} \cosh^2 u .$$

One then has

$$\sqrt{r - R_{\text{Sch}}} = \sqrt{R_{\text{Sch}}}\, \sinh u$$

$$dr = 2R_{\rm Sch}\cosh u \sinh u\, du \ ,$$

and the indefinite integral becomes

$$\begin{aligned}\int \frac{\sqrt{r}\,dr}{\sqrt{r-R_{\rm Sch}}} &= 2R_{\rm Sch}\int \cosh^2 u\, du \\ &= R_{\rm Sch}\int (1+\cosh 2u)du \\ &= R_{\rm Sch}\left(u + \frac{1}{2}\sinh 2u\right) \\ &= R_{\rm Sch}(u + \sinh u\, \cosh u) \\ &= R_{\rm Sch}\sinh^{-1}\left(\sqrt{\frac{r}{R_{\rm Sch}} - 1}\right) + \sqrt{r(r-R_{\rm Sch})}\ .\end{aligned}$$

Thus,

$$\begin{aligned}s_{AB} = R_{\rm Sch}&\left[\sinh^{-1}\left(\sqrt{\frac{r_B}{R_{\rm Sch}} - 1}\right) - \sinh^{-1}\left(\sqrt{\frac{r_A}{R_{\rm Sch}} - 1}\right)\right] \\ &+ \sqrt{r_B(r_B - R_{\rm Sch})} - \sqrt{r_A(r_A-R_{\rm Sch})}\ .\end{aligned}$$

c) A tick of the clock and the following tick are two events that differ only in their time coordinates. Thus, the metric reduces to

$$-c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2\ ,$$

so

$$d\tau = \sqrt{1 - \frac{2GM}{rc^2}}\, dt\ .$$

The reading on the observer's clock corresponds to the proper time interval $d\tau$, so the corresponding interval of the coordinate t is given by

$$\boxed{\Delta t_A = \frac{\Delta\tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}}\ .}$$

d) Since the Schwarzschild metric does not change with time, each pulse leaving A will take the same length of time to reach B. Thus, the pulses emitted by A will arrive at B with a time coordinate spacing

$$\Delta t_B = \Delta t_A = \frac{\Delta\tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}}\ .$$

The clock at B, however, will read the proper time and not the coordinate time. Thus,

$$\Delta\tau_B = \sqrt{1 - \frac{2GM}{r_B c^2}}\,\Delta t_B$$

$$= \boxed{\sqrt{\frac{1 - \frac{2GM}{r_B c^2}}{1 - \frac{2GM}{r_A c^2}}}\,\Delta\tau_A\ .}$$

e) From parts (a) and (b), the proper distance between A and B can be rewritten as

$$s_{AB} = \int_{R_{\rm Sch}}^{r_B} \frac{\sqrt{r}\,dr}{\sqrt{r - R_{\rm Sch}}}\ .$$

The potentially divergent part of the integral comes from the range of integration in the immediate vicinity of $r = R_{\rm Sch}$, say $R_{\rm Sch} < r < R_{\rm Sch} + \epsilon$. For this range the quantity $\sqrt{r}$ in the numerator can be approximated by $\sqrt{R_{\rm Sch}}$, so the contribution has the form

$$\sqrt{R_{\rm Sch}} \int_{R_{\rm Sch}}^{R_{\rm Sch}+\epsilon} \frac{dr}{\sqrt{r - R_{\rm Sch}}}\ .$$

Changing the integration variable to $u \equiv r - R_{\rm Sch}$, the contribution can be easily evaluated:

$$\sqrt{R_{\rm Sch}} \int_{R_{\rm Sch}}^{R_{\rm Sch}+\epsilon} \frac{dr}{\sqrt{r - R_{\rm Sch}}} = \sqrt{R_{\rm Sch}} \int_0^{\epsilon} \frac{du}{\sqrt{u}} = 2\sqrt{R_{\rm Sch}\epsilon} < \infty\ .$$

So, although the integrand is infinite at $r = R_{\rm Sch}$, the integral is still finite.

$$\boxed{\text{The proper distance between } A \text{ and } B \text{ does not diverge.}}$$

Looking at the answer to part (d), however, one can see that when $r_A = R_{\rm Sch}$,

$$\boxed{\text{The time interval } \Delta\tau_B \text{ diverges.}}$$

PROBLEM 7: GEODESICS

The geodesic equation for a curve $x^i(\lambda)$, where the parameter λ is the arc length along the curve, can be written as

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\frac{dx^\ell}{d\lambda} \ .$$

Here the indices j, k, and ℓ are summed from 1 to the dimension of the space, so there is one equation for each value of i.

(a) The metric is given by

$$ds^2 = g_{ij}dx^i dx^j = dr^2 + r^2\,d\theta^2 \ ,$$

so

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2 \ , \qquad g_{r\theta} = g_{\theta r} = 0 \ .$$

First taking $i = r$, the nonvanishing terms in the geodesic equation become

$$\frac{d}{d\lambda}\left\{g_{rr}\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r g_{\theta\theta}\right)\frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda} \ ,$$

which can be written explicitly as

$$\frac{d}{d\lambda}\left\{\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r r^2\right)\left(\frac{d\theta}{d\lambda}\right)^2 \ ,$$

or

$$\boxed{\frac{d^2 r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 \ .}$$

For $i = \theta$, one has the simplification that g_{ij} is independent of θ for all (i, j). So

$$\boxed{\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0 \ .}$$

(b) The first step is to parameterize the curve, which means to imagine moving along the curve, and expressing the coordinates as a function of the distance traveled. (I am calling the locus $y = 1$ a curve rather than a line, since the techniques that are used here are usually applied to curves. Since a line is a special case of a curve, there is nothing wrong with treating the line as a curve.) In Cartesian coordinates, the curve $y = 1$ can be parameterized as

$$x(\lambda) = \lambda \ , \qquad y(\lambda) = 1 \ .$$

(The parameterization is not unique, because one can choose $\lambda = 0$ to represent any point along the curve.) Converting to the desired polar coordinates,

$$r(\lambda) = \sqrt{x^2(\lambda) + y^2(\lambda)} = \sqrt{\lambda^2 + 1} \ ,$$

$$\theta(\lambda) = \tan^{-1} \frac{y(\lambda)}{x(\lambda)} = \tan^{-1}(1/\lambda) \ .$$

Calculating the needed derivatives,*

$$\begin{aligned} \frac{dr}{d\lambda} &= \frac{\lambda}{\sqrt{\lambda^2+1}} \\ \frac{d^2 r}{d\lambda^2} &= \frac{1}{\sqrt{\lambda^2+1}} - \frac{\lambda^2}{(\lambda^2+1)^{3/2}} = \frac{1}{(\lambda^2+1)^{3/2}} = \frac{1}{r^3} \\ \frac{d\theta}{d\lambda} &= -\frac{1}{1+\left(\frac{1}{\lambda}\right)^2} \frac{1}{\lambda^2} = -\frac{1}{r^2} \ . \end{aligned}$$

Then, substituting into the geodesic equation for $i = r$,

$$\frac{d^2 r}{d\lambda^2} = r \left(\frac{d\theta}{d\lambda} \right)^2 \Longleftrightarrow \frac{1}{r^3} = r \left(-\frac{1}{r^2} \right)^2 \ ,$$

which checks. Substituting into the geodesic equation for $i = \theta$,

$$\frac{d}{d\lambda} \left\{ r^2 \frac{d\theta}{d\lambda} \right\} = 0 \Longleftrightarrow \frac{d}{d\lambda} \left\{ r^2 \left(-\frac{1}{r^2} \right) \right\} = 0 \ ,$$

which also checks.

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* If you do not remember how to differentiate $\phi = \tan^{-1}(z)$, then you should know how to derive it. Write $z = \tan\phi = \sin\phi / \cos\phi$, so

$$dz = \left(\frac{\cos\phi}{\cos\phi} + \frac{\sin^2\phi}{\cos^2\phi} \right) d\phi = (1 + \tan^2\phi) d\phi \ .$$

Then

$$\frac{d\phi}{dz} = \frac{1}{1+\tan^2\phi} = \frac{1}{1+z^2} \ .$$

PROBLEM 8: METRIC OF A STATIC GRAVITATIONAL FIELD

(a) ds^2_{ST} is the invariant separation between the event at (x^i, t) and the event at $(x^i + dx^i, t + dt)$. Here x^i and t are arbitrary coordinates that are connected to measurements only through the metric. ds^2_{ST} is defined to equal

$$-c^2 dT^2 + d\vec{r}^{\,2} \ ,$$

where $d\vec{r}$ and dT denote the space and time separation as it would be measured by a freely falling observer. Taking the transmitter as the freely falling observer* and taking the emission of two successive pulses as the two events, one has

$$ds^2_{\mathrm{ST}} = -c^2 (\Delta T_e)^2 \ .$$

To connect with the metric, note that the successive emissions have a separation in the time coordinate of Δt_e, and a separation of space coordinates $dx^i = 0$. So

$$ds^2_{ST} = -\left[c^2 + 2\phi(\vec{x}_e)\right](\Delta t_e)^2 \ ,$$

and then

$$-c^2(\Delta T_e)^2 = -[c^2 + 2\phi(\vec{x}_e)](\Delta t_e)^2 \quad \Longrightarrow$$

$$\boxed{\Delta t_e = \frac{\Delta T_e}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}} \ .}$$

(b) Since the metric is independent of t, each pulse follows a trajectory identical to the previous pulse, but delayed in t. Thus each pulse requires the same time interval Δt to travel from emitter to receiver, so the pulses arrive with the same t-separation as they have at emission:

$$\boxed{\Delta t_r = \Delta t_e \ .}$$

(c) This is similar to part (a), but in this case we consider the two events corresponding to the reception of two successive pulses. ds^2_{ST} is related to the physical measurement ΔT_r by

$$ds^2_{\mathrm{ST}} = -c^2(\Delta T_r)^2 \ .$$

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* The transmitter is not really a freely falling observer, but is presumably held at rest in this coordinate system. Thus gravity is acting on the clock, and could in principle affect its speed. It is standard, however, to assume that such effects are negligible. That is, one assumes that the clock is ideal, meaning that it ticks at the same rate as a freely falling clock that is instantaneously moving with the same velocity.

It is connected to the coordinate separation Δt_r through the metric, where again we use the fact that the two events have zero separation in their space coordinates— i.e., $dx^i = 0$. So

$$ds^2_{ST} = -[c^2 + 2\phi(\vec{x}_r)](\Delta t_r)^2 \ .$$

Then

$$-c^2(\Delta T_r)^2 = -[c^2 + 2\phi(\vec{x}_r)](\Delta t_e)^2 \quad \Longrightarrow$$

$$\Delta T_r = \sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}\Delta t_e \ .$$

We can cast this into a more useful form for the problem by using the solution for Δt_e found in part (c). This gives

$$\Delta T_r = \left[\frac{\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}} \right] \Delta T_e \ .$$

Substitute this result for ΔT_r directly into the definition for Z to obtain the exact expression for the redshift,

$$\boxed{1 + Z = \frac{\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}} \ .}$$

Remember that $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for small x. For weak fields, that is, for small values of $\phi(\vec{x})$, we can expand our result to lowest order in $\phi(\vec{x})$. Expanding the numerator we have

$$\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}} \approx 1 + \frac{\phi(\vec{x}_r)}{c^2} \ .$$

Similarly we find for

$$\frac{1}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}} \approx 1 - \frac{\phi(\vec{x}_e)}{c^2} \ .$$

Putting these approximations into our exact expression for $1 + Z$ we obtain

$$1 + Z \approx \left(1 + \frac{\phi(\vec{x}_r)}{c^2}\right)\left(1 - \frac{\phi(\vec{x}_e)}{c^2}\right) \approx 1 + \frac{\phi(\vec{x}_r)}{c^2} - \frac{\phi(\vec{x}_e)}{c^2} \ ,$$

where we dropped terms in $\phi(\vec{x}_e)\phi(\vec{x}_r)$. Finally,

$$\boxed{Z \approx \frac{\phi(\vec{x}_r) - \phi(\vec{x}_e)}{c^2} \quad .}$$

(d) For the metric at hand we know $g_{00} = -[c^2 + 2\phi(\vec{x})]$, $\;g_{k0} = 0$ and $g_{ik} = g_{ki} = \delta_{ik}$. It is useful to notice that only g_{00} depends on $\vec{x}$ and thus $\partial_i g_{km} = 0$. The geodesic equation corresponding to $\mu = i$, where i runs from 1 to 3, is

$$\frac{d}{d\tau}\left(g_{ik}\frac{dx^k}{d\tau}\right) = \frac{1}{2}(\partial_i g_{\lambda\sigma})\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \qquad \Longrightarrow$$

$$\delta_{ik}\frac{d^2x^k}{d\tau^2} = \frac{1}{2}(\partial_i g_{00})\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} \; .$$

Using $x^0 \equiv t$, $\delta_{ik}y^k = y^i$ and

$$\partial_i g_{00} = -\partial_i(c^2 + 2\phi(\vec{x})) = -\frac{2}{c^2}\partial_i\phi(\vec{x})$$

we find

$$\boxed{\frac{d^2x^i}{d^2\tau} = -\partial_i\phi(\vec{x})\left(\frac{dt}{d\tau}\right)^2 \; .}$$

[Pedagogical Note: You might prefer to use the notation $x^0 \equiv ct$, which is also a very common choice. In that case the metric is rewritten as

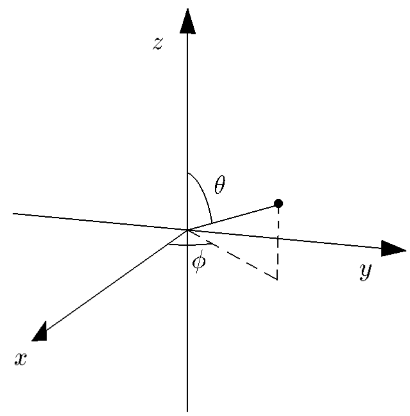
$$ds^2_{\text{ST}} = -\left[1 + \frac{2\phi(\vec{x})}{c^2}\right]\left(dx^0\right)^2 + \sum_{i=1}^{3}\left(dx^i\right)^2 \quad ,$$

so one takes $g_{00} = -\left[1 + (2\phi(\vec{x})/c^2)\right]$. In the end one finds the same answer as the boxed equation above.

Note also that when ϕ is small and velocities are nonrelativistic, then $dt/d\tau \approx 1$. Thus one has $d^2x^i/d^2t \approx -\partial_i\phi(\vec{x})$, so $\phi(\vec{x})$ can be identified with the Newtonian gravitational potential. In the context of general relativity, Newtonian gravity is a distortion of the metric in the time-direction.]

PROBLEM 9: GEODESICS ON THE SURFACE OF A SPHERE

(a) Rotations are easy to understand in Cartesian coordinates. The relationship between the polar and Cartesian coordinates is given by

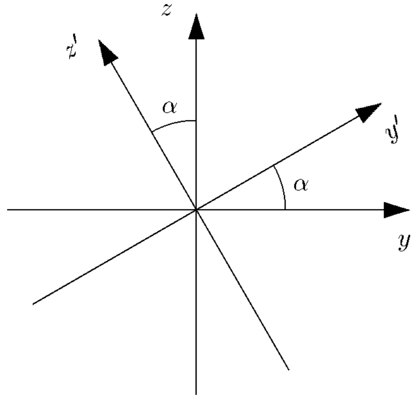


$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta\ .$$

The equator is then described by $\theta = \pi/2$, and $\phi = \psi$, where ψ is a parameter running from 0 to 2π. Thus, the equator is described by the curve $x^i(\psi)$, where

$$x^1 = x = r\cos\psi$$
$$x^2 = y = r\sin\psi$$
$$x^3 = z = 0\ .$$

Now introduce a primed coordinate system that is related to the original system by a rotation in the y-z plane by an angle α:



$$x = x'$$
$$y = y'\cos\alpha - z'\sin\alpha$$
$$z = z'\cos\alpha + y'\sin\alpha\ .$$

The rotated equator, which we seek to describe, is just the standard equator in the primed coordinates:

$$x' = r\cos\psi \ , \qquad y' = r\sin\psi \ , \qquad z' = 0 \ .$$

Using the relation between the two coordinate systems given above,

$$\begin{aligned} x &= r\cos\psi \\ y &= r\sin\psi\cos\alpha \\ z &= r\sin\psi\sin\alpha \ . \end{aligned}$$

Using again the relations between polar and Cartesian coordinates,

$$\boxed{\begin{aligned} \cos\theta &= \frac{z}{r} = \sin\psi\sin\alpha \\ \tan\phi &= \frac{y}{x} = \tan\psi\cos\alpha \ . \end{aligned}}$$

(b) A segment of the equator corresponding to an interval $d\psi$ has length $a\,d\psi$, so the parameter ψ is proportional to the arc length. Expressed in terms of the metric, this relationship becomes

$$ds^2 = g_{ij}\frac{dx^i}{d\psi}\frac{dx^j}{d\psi}d\psi^2 = a^2 d\psi^2 \ .$$

Thus the quantity

$$A \equiv g_{ij}\frac{dx^i}{d\psi}\frac{dx^j}{d\psi}$$

is equal to a^2, so the geodesic equation (6.36) reduces to the simpler form of Eq. (6.38). (Note that we are following the notation of Lecture Notes 6, except that the variable used to parametrize the path is called ψ, rather than λ or s. Although A is not equal to 1 as we assumed in Lecture Notes 6, it is easily seen that Eq. (6.38) follows from (6.36) provided only that $A = \textit{constant}$.) Thus,

$$\frac{d}{d\psi}\left\{g_{ij}\frac{dx^j}{d\psi}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\psi}\,\frac{dx^\ell}{d\psi} \ .$$

For this problem the metric has only two nonzero components:

$$g_{\theta\theta} = a^2 \ , \qquad g_{\phi\phi} = a^2\sin^2\theta \ .$$

Taking $i = \theta$ in the geodesic equation,

$$\frac{d}{d\psi}\left\{ g_{\theta\theta}\frac{d\theta}{d\psi}\right\} = \frac{1}{2}\partial_\theta g_{\phi\phi}\frac{d\phi}{d\psi}\frac{d\phi}{d\psi} \quad \Longrightarrow$$

$$\boxed{\frac{d^2\theta}{d\psi^2} = \sin\theta\cos\theta\left(\frac{d\phi}{d\psi}\right)^2 \ .}$$

Taking $i = \phi$,

$$\frac{d}{d\psi}\left\{ a^2\sin^2\theta\frac{d\phi}{d\psi}\right\} = 0 \quad \Longrightarrow$$

$$\boxed{\frac{d}{d\psi}\left\{ \sin^2\theta\frac{d\phi}{d\psi}\right\} = 0 \ .}$$

(c) This part is mainly algebra. Taking the derivative of

$$\cos\theta = \sin\psi\sin\alpha$$

implies

$$-\sin\theta\, d\theta = \cos\psi\sin\alpha\, d\psi \ .$$

Then, using the trigonometric identity $\sin\theta = \sqrt{1-\cos^2\theta}$, one finds

$$\sin\theta = \sqrt{1-\sin^2\psi\sin^2\alpha} \ ,$$

so

$$\frac{d\theta}{d\psi} = -\frac{\cos\psi\sin\alpha}{\sqrt{1-\sin^2\psi\sin^2\alpha}} \ .$$

Similarly

$$\tan\phi = \tan\psi\cos\alpha \quad \Longrightarrow \quad \sec^2\phi\, d\phi = \sec^2\psi\, d\psi\cos\alpha \ .$$

Then

$$\begin{aligned}\sec^2\phi = \tan^2\phi + 1 &= \tan^2\psi\cos^2\alpha + 1\\ &= \frac{1}{\cos^2\psi}[\sin^2\psi\cos^2\alpha + \cos^2\psi]\\ &= \sec^2\psi[\sin^2\psi(1-\sin^2\alpha)+\cos^2\psi]\\ &= \sec^2\psi[1-\sin^2\psi\sin^2\alpha] \ ,\end{aligned}$$

So

$$\frac{d\phi}{d\psi} = \frac{\cos\alpha}{1 - \sin^2\psi \sin^2\alpha} \ .$$

To verify the geodesic equations of part (b), it is easiest to check the second one first:

$$\begin{aligned}\sin^2\theta \frac{d\phi}{d\psi} &= (1 - \sin^2\psi \sin^2\alpha) \frac{\cos\alpha}{1 - \sin^2\psi \sin^2\alpha} \\ &= \cos\alpha \ ,\end{aligned}$$

so clearly

$$\frac{d}{d\psi}\left\{\sin^2\theta \frac{d\phi}{d\psi}\right\} = \frac{d}{d\psi}(\cos\alpha) = 0 \ .$$

To verify the first geodesic equation from part (b), first calculate the left-hand side, $d^2\theta/d\psi^2$, using our result for $d\theta/d\psi$:

$$\frac{d^2\theta}{d\psi^2} = \frac{d}{d\psi}\left(\frac{d\theta}{d\psi}\right) = \frac{d}{d\psi}\left\{-\frac{\cos\psi \sin\alpha}{\sqrt{1 - \sin^2\psi \sin^2\alpha}}\right\} \ .$$

After some straightforward algebra, one finds

$$\frac{d^2\theta}{d\psi^2} = \frac{\sin\psi \sin\alpha \cos^2\alpha}{\left[1 - \sin^2\psi \sin^2\alpha\right]^{3/2}} \ .$$

The right-hand side of the first geodesic equation can be evaluated using the expression found above for $d\phi/d\psi$, giving

$$\begin{aligned}\sin\theta\cos\theta\left(\frac{d\phi}{d\psi}\right)^2 &= \sqrt{1 - \sin^2\psi \sin^2\alpha}\ \sin\psi \sin\alpha \frac{\cos^2\alpha}{\left[1 - \sin^2\psi \sin^2\alpha\right]^2} \\ &= \frac{\sin\psi \sin\alpha \cos^2\alpha}{\left[1 - \sin^2\psi \sin^2\alpha\right]^{3/2}} \ .\end{aligned}$$

So the left- and right-hand sides are equal.

PROBLEM 10: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

In general, the number density of a particle in the black-body radiation is given by

$$n = g^* \frac{\xi(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$$

For photons, one has $g^* = 2$. Then

$$\left.\begin{aligned} k &= 1.381 \times 10^{-16}\text{erg}/^\circ\text{K} \\ T &= 2.7\,^\circ\text{K} \\ \hbar &= 1.055 \times 10^{-27}\text{erg-sec} \\ c &= 2.998 \times 10^{10}\text{cm/sec} \end{aligned}\right\} \implies \left(\frac{kT}{\hbar c}\right)^3 = 1.638 \times 10^3\text{cm}^{-3} \ .$$

Then using $\xi(3) \simeq 1.202$, one finds

$$\boxed{n_\gamma = 399/\text{cm}^3 \ .}$$

For the neutrinos,

$$g_\nu^* = 2 \times \frac{3}{4} = \frac{3}{2} \quad \text{per species.}$$

The factor of 2 is to account for ν and $\bar{\nu}$, and the factor of $3/4$ arises from the Pauli exclusion principle. So for three species of neutrinos one has

$$g_\nu^* = \frac{9}{2} \ .$$

Using the result

$$T_\nu^3 = \frac{4}{11} T_\gamma^3$$

from Problem 8 of Problem Set 3 (2000), one finds

$$\begin{aligned} n_\nu &= \left(\frac{g_\nu^*}{g_\gamma^*}\right)\left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma \\ &= \left(\frac{9}{4}\right)\left(\frac{4}{11}\right) 399\text{cm}^{-3} \\ &\implies \boxed{n_\nu = 326/\text{cm}^3 \text{ (for all three species combined).}} \end{aligned}$$

PROBLEM 11: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)}\,kT\ .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\,kT\ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{2\pi^4}{45\zeta(3)}\,k\ .}$$

Numerically, this gives $3.602\,k$, where k is the Boltzman constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is $7/8$ times the number of spin states, and g^* is $3/4$ times the number of spin states. So the average energy per particle is

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8}\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{180\zeta(3)}\,kT\ .}$$

Numerically, $E = 3.1514\,kT$.

> Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \frac{\dfrac{7}{8}\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{7\pi^4}{135\zeta(3)}k\ .}$$

Numerically, this gives $S = 4.202\,k$.

PROBLEM 12: A NEW SPECIES OF LEPTON

a) The number density is given by the formula at the start of the exam,

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ .$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$\begin{aligned} n &= 3\frac{\zeta(3)}{\pi^2}\times\left(\frac{3\ \cancel{\text{MeV}}}{6.582\times10^{-16}\ \cancel{\text{eV-sec}}\times2.998\times10^{10}\ \cancel{\text{cm-sec}}^{-1}}\right)^3 \\ &\quad\times\left(\frac{10^6\ \cancel{\text{eV}}}{1\ \cancel{\text{MeV}}}\right)^3\times\left(\frac{10^2\ \cancel{\text{cm}}}{1\ \text{m}}\right)^3 \\ &= 3\frac{\zeta(3)}{\pi^2}\times\left(\frac{3\times10^6\times10^2}{6.582\times10^{-16}\times2.998\times10^{10}}\right)^3\ \text{m}^{-3}\ . \end{aligned}$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2}\times\left(\frac{3\times10^6\times10^2}{6.582\times10^{-16}\times2.998\times10^{10}}\right)^3\ .}$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho \quad .$$

During the radiation-dominated era $R(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot{R}}{R} = \frac{1}{2t} \quad .$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3} G\rho \quad .$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \quad .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}$ cm^3 sec^{-2} g^{-1}, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732 \times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9$ g/cm^3.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \quad .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion – anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\rm tot} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2}\frac{1}{g_{\rm tot}}\hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/R$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\rm rest} T^3(t) R^3(t)$$

where $g_{\text{rest}} = g_{\text{tot}} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 R^3(t) = g_{\text{rest}} T^3(t) R^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\text{rest}}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 13: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION

(a) This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by $dU = -p\,dV$. Using the fact that the energy density u is equal to ρc^2, the energy conservation relation can be written

$$\frac{dU}{dt} = -p\frac{dV}{dt} \quad \Longrightarrow \quad \frac{d}{dt}\left(\rho c^2 R^3\right) = -p\frac{d}{dt}\left(R^3\right) \quad . \tag{1}$$

Setting

$$\rho = \frac{\alpha}{R^5} \tag{2}$$

for some constant α, the conservation of energy formula becomes

$$\frac{d}{dt}\left(\frac{\alpha c^2}{R^2}\right) = -p\frac{d}{dt}\left(R^3\right) \quad ,$$

which implies

$$-2\frac{\alpha c^2}{R^3}\frac{dR}{dt} = -3pR^2\frac{dR}{dt} \quad .$$

Thus

$$p = \frac{2}{3}\frac{\alpha c^2}{R^5} = \boxed{\frac{2}{3}\rho c^2 \quad .}$$

For those students who could not reconstruct Eq. (1) or some equivalent equation from memory, the conservation of energy equation could be derived from the formulas for cosmological evolution on the front of the exam:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \tag{3}$$

$$\ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R\ . \tag{4}$$

By rewriting Eq. (3) as

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 - kc^2\ ,$$

the time derivative becomes

$$2\dot{R}\,\ddot{R} = \frac{8\pi}{3}G\dot{\rho}R^2 + \frac{16\pi}{3}G\rho R\dot{R}\ .$$

This equation can be solved for $\dot{\rho}$ to give

$$\dot{\rho} = \frac{3}{4\pi G}\frac{\dot{R}\,\ddot{R}}{R^2} - 2\frac{\dot{R}}{R}\rho\ .$$

Using Eq. (4) to replace $\ddot{R}$, one finds

$$\dot{\rho} = -\frac{\dot{R}}{R}\left(\rho + \frac{3p}{c^2}\right) - 2\frac{\dot{R}}{R}\rho = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right)\ . \tag{5}$$

It is easy to show that Eq. (5) is equivalent to Eq. (1), but it is not necessary to do so. The question can be answered directly from Eq. (5), by substituting Eq. (2) and manipulating.

(b) For a flat universe, Eq. (3) reduces to

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho\ .$$

Using Eq. (2), this implies that

$$\dot{R} = \frac{\beta}{R^{3/2}}\ ,$$

for some constant β. Rewriting this as

$$R^{3/2}\,dR = \beta\,dt\ ,$$

we can integrate the equation to give

$$\frac{2}{5} R^{5/2} = \beta t + \text{const} \ ,$$

where the constant of integration has no effect other than to shift the origin of the time variable t. Using the standard big bang convention that $R = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$\boxed{R \propto t^{2/5} \ .} \tag{6}$$

The arbitrary constant of proportionality in Eq. (6) is consistent with the wording of the problem, which states that "You should be able to determine the function $R(t)$ up to a constant factor." Note that we could have expressed the constant of proportionality in terms of the constant α in Eq. (2), but there would not really be any point in doing that. The constant α was not a given variable. If the comoving coordinates are measured in "notches," then R is measured in meters per notch, and the constant of proportionality in Eq. (6) can be changed by changing the arbitrary definition of the notch.

(c) Combining Eq. (1) with $p = \frac{1}{6}\rho c^2$, one has

$$\frac{d}{dt}\left(\rho c^2 R^3\right) = -\frac{1}{6}\rho c^2 \frac{d}{dt}\left(R^3\right) \ ,$$

or equivalently

$$\frac{d}{dt}\left(\rho R^3\right) + \frac{1}{6}\rho \frac{d}{dt}\left(R^3\right) = 0 \ . \tag{7}$$

There are various ways to proceed from here. Since the problem told us that

$$\rho = \frac{\text{const}}{R^n} \ ,$$

the most straightforward approach would be to use this expression to replace ρ in Eq. (7), and then solve the equation for n. A cleverer approach would be to multiply Eq. (7) by $R^{1/2}$, and then rewrite it as

$$\frac{d}{dt}\left(\rho R^{7/2}\right) = 0 \ ,$$

from which one can see immediately that

$$\boxed{\rho(t) \propto \frac{1}{R^{7/2}(t)} \ ,}$$

and therefore

$$\boxed{n = 7/2 \ .}$$

PROBLEM 14: DID YOU DO THE READING?

The solutions to this problem were written by Edward Keyes.

(a) Birkhoff's theorem

Birkhoff's theorem states that "the gravitational effect of a uniform medium external to a spherical cavity is zero." This is a theorem from general relativity, and necessary to know in order to extrapolate our Newtonian cosmology results to the whole universe: it might have been the case that the global curvature of space would have interfered with our Newtonian results. The other choices in the question were generally true statements from other areas of cosmology.

(b) Special-case cosmological models

The Einstein-de Sitter model is not, as some answered, Einstein's original, static universe with a cosmological constant. Instead, this model describes a flat ($k = 0$) universe with a critical density of ordinary matter ($\rho = \rho_c$). As we showed earlier in the class, this means that its scale factor grows as $R(t) \propto t^{2/3}$.

The Milne model describes an empty universe: it is open ($k = -1$) and has no matter or radiation in it ($\rho = 0$). Its scale factor grows linearly with time, since there's no matter to slow down the Hubble expansion. (One normally includes "test" particles in the description of the Milne universe, so that we can talk about their motion. But the mass of these test particles is taken to be arbitrarily small, so we completely ignore any gravitational field that they might produce.)

As an interesting aside, we might ask why the Milne model has $k = -1$. Since there is no matter, there shouldn't be any general relativity effects, and so we would ordinarily expect that the metric should be the normal, flat, Minkowski special relativity metric. Why is this space hyperbolic instead?

The answer is an illustration of the subtleties that can arise in changing coordinate systems. In fact, the metric of the Milne universe can be viewed as ***either*** a flat, Minkowski metric, ***or*** as the negatively curved metric of an open universe, depending on what coordinate system one uses. If one uses coordinates for time and space as they would be measured by a single inertial observer, then one finds a Minkowski metric; in this way of describing the model, it is clear that special relativity is sufficient, and general relativity plays no role. In this coordinate system all the test particles start at the origin at time $t = 0$, and they move outward from the origin at speeds ranging from zero, up to (but not including) the speed of light.

On the other hand, we can describe the same universe in a way that treats all the test particles on an equal footing. In this description we define time not as it would be measured by a single observer, but instead we define the time at each location as the time that would be measured by observers riding with the test particles at that location. This definition is what we have been calling "cosmic time" in our description of cosmology. One can also introduce a comoving spatial coordinate system that expands with the motion of the particles. With a particular definition of these spatial coordinates, one can show that the metric is precisely that of an open Robertson-Walker universe with $R(t) = t$.

The derivation is left as an exercise for the curious student. You should find that the normal special-relativistic time dilation and Lorentz contraction formulas, when applied to the velocities of a Hubble expansion to construct the comoving coordinate system, introduce the negative curvature to the metric.

(c) Neutron-proton ratio

In the early universe, neutrons and protons first formed when the temperature dropped far enough to keep them from being torn apart into their constituent quarks. This happened around a microsecond after the Big Bang, and at this time there were roughly equal numbers of neutrons and protons.

In fact, neutrons could be converted into protons and vice-versa in several weak reactions with electrons, positrons, and neutrinos:

$$n + \nu_e \longleftrightarrow p + e^- \quad , \quad n + e^+ \longleftrightarrow p + \bar{\nu}_e \quad , \quad n \longleftrightarrow p + e^- + \bar{\nu}_e$$

Since these are weak-force reactions, though, their rates are strongly dependent on the temperature. Once T drops below 10^{10} K, the neutrinos stop interacting with matter, and these reactions freeze, except for the forward direction of the third reaction, which describes free neutron decay (this process has a half-life of 15 minutes, so it doesn't affect things very much).

Before the freeze-out, which essentially fixes the neutron/proton ratio, the reaction rates shift as the temperature changes. The neutron is 1.3 MeV heavier than the proton, while the mass/energy of an electron is only 0.5 MeV. This means that the conversion of a neutron to a proton and electron is energetically favorable, while the reverse process costs energy. As the temperature drops so that kT is of the order of 1 MeV, these energy differences become significant compared to the available free thermal energy, and the reaction rates shift so that thermal equilibrium favors protons over neutron by an increasing margin.

When the weak reactions freeze out, this unequal ratio of neutrons and protons is preserved. Since essentially all of the neutrons end up in helium after nucleosynthesis, this also fixes the ratio of hydrogen to helium formed by the Big Bang.

(d) The dipole anisotropy

When we look at the temperature of the cosmic microwave background radiation, to first order it appears uniform across the sky. When we look closer, though, we see that it is hotter in one direction and smoothly shades into cooler in the opposite direction, at a level of about one part in 1000. This is the dipole anisotropy.

The explanation is quite simple: the Earth is not at rest with respect to the cosmic background radiation. The motion of our Sun around the center of the Galaxy, and the motion of our Galaxy towards the Virgo Cluster, etc., all give us a net velocity of around 600 km/sec, which causes us to see blueshifted CMB photons in one direction, and redshifted ones in the opposite direction. As we learned earlier in the class, a redshifted blackbody spectrum just shifts its temperature, so we see the effects of this motion as a smooth temperature variation across the sky.

(e) Events in the early universe

The correct order is:

(iii) Quark confinement, at $t \sim 10^{-6}$ sec.

(v) Muon annihilation, at $t \sim 10^{-4}$ sec.

(ii) Decoupling of electron neutrinos, at $t \sim 1$ sec.

(i) Primordial nucleosynthesis, at $t \sim 10^2$ sec.

(iv) Recombination, at $t \sim 10^5$ years.

A surprising number of students did not realize that recombination is the final stage of the early universe. After this event takes place, the universe is transparent to photons and the temperature has dropped to just a few thousand K. Nothing interesting happens after this until the processes of structure formation begins.

PROBLEM 15: GEODESICS IN A CLOSED UNIVERSE

(a) *(7 points)* For purely radial motion, $d\theta = d\phi = 0$, so the line element reduces do

$$-c^2\, d\boldsymbol{\tau}^2 = -c^2\, dt^2 + R^2(t) \left\{ \frac{dr^2}{1-r^2} \right\} \, .$$

Dividing by dt^2,

$$-c^2 \left(\frac{d\boldsymbol{\tau}}{dt}\right)^2 = -c^2 + \frac{R^2(t)}{1-r^2} \left(\frac{dr}{dt}\right)^2 \, .$$

Rearranging,

$$\boxed{\frac{d\boldsymbol{\tau}}{dt} = \sqrt{1 - \frac{R^2(t)}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2}\ .}$$

(b) *(3 points)*

$$\frac{dt}{d\boldsymbol{\tau}} = \frac{1}{\dfrac{d\boldsymbol{\tau}}{dt}} = \boxed{\frac{1}{\sqrt{1 - \dfrac{R^2(t)}{c^2(1-r^2)}\left(\dfrac{dr}{dt}\right)^2}}\ .}$$

(c) *(10 points)* During any interval of clock time dt, the proper time that would be measured by a clock moving with the object is given by $d\boldsymbol{\tau}$, as given by the metric. Using the answer from part (a),

$$d\boldsymbol{\tau} = \frac{d\boldsymbol{\tau}}{dt}\,dt = \sqrt{1 - \frac{R^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\,dt\ .$$

Integrating to find the total proper time,

$$\boxed{\boldsymbol{\tau} = \int_{t_1}^{t_2} \sqrt{1 - \frac{R^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\,dt\ .}$$

(d) *(10 points)* The physical distance $d\ell$ that the object moves during a given time interval is related to the coordinate distance dr by the spatial part of the metric:

$$d\ell^2 = ds^2 = R^2(t)\left\{\frac{dr^2}{1-r^2}\right\} \quad\Longrightarrow\quad d\ell = \frac{R(t)}{\sqrt{1-r^2}}\,dr\ .$$

Thus

$$\boxed{v_{\rm phys} = \frac{d\ell}{dt} = \frac{R(t)}{\sqrt{1-r^2}}\,\frac{dr}{dt}\ .}$$

Discussion: A common mistake was to include $-c^2\,dt^2$ in the expression for $d\ell^2$. To understand why this is not correct, we should think about how an observer would measure $d\ell$, the distance to be used in calculating the velocity of a passing object. The observer would place a meter stick along the path of the object, and she would mark off the position of the object at the beginning and end of a time interval $dt_{\rm meas}$. Then she would read the distance by subtracting the two readings on the meter stick. This subtraction is equal to the physical distance between the two marks, measured at the ***same*** time t. Thus, when we compute the distance between the two marks, we set $dt = 0$. To compute the speed she would then divide the distance by $dt_{\rm meas}$, which is nonzero.

(e) *(10 points)* We start with the standard formula for a geodesic, as written on the front of the exam:

$$\frac{d}{d\boldsymbol{\tau}}\left\{g_{\mu\nu}\frac{dx^\nu}{d\boldsymbol{\tau}}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\boldsymbol{\tau}}\frac{dx^\sigma}{d\boldsymbol{\tau}} \ .$$

This formula is true for each possible value of μ, while the Einstein summation convention implies that the indices ν, λ, and σ are summed. We are trying to derive the equation for r, so we set $\mu = r$. Since the metric is diagonal, the only contribution on the left-hand side will be $\nu = r$. On the right-hand side, the diagonal nature of the metric implies that nonzero contributions arise only when $\lambda = \sigma$. The term will vanish unless $dx^\lambda/d\boldsymbol{\tau}$ is nonzero, so λ must be either r or t (i.e., there is no motion in the θ or ϕ directions). However, the right-hand side is proportional to

$$\frac{\partial g_{\lambda\sigma}}{\partial r} \ .$$

Since $g_{tt} = -c^2$, the derivative with respect to r will vanish. Thus, the only nonzero contribution on the right-hand side arises from $\lambda = \sigma = r$. Using

$$g_{rr} = \frac{R^2(t)}{1-r^2} \ ,$$

the geodesic equation becomes

$$\frac{d}{d\boldsymbol{\tau}}\left\{g_{rr}\frac{dr}{d\boldsymbol{\tau}}\right\} = \frac{1}{2}\left(\partial_r g_{rr}\right)\frac{dr}{d\boldsymbol{\tau}}\frac{dr}{d\boldsymbol{\tau}} \ ,$$

or

$$\frac{d}{d\boldsymbol{\tau}}\left\{\frac{R^2}{1-r^2}\frac{dr}{d\boldsymbol{\tau}}\right\} = \frac{1}{2}\left[\partial_r\left(\frac{R^2}{1-r^2}\right)\right]\frac{dr}{d\boldsymbol{\tau}}\frac{dr}{d\boldsymbol{\tau}} \ ,$$

or finally

$$\boxed{\frac{d}{d\boldsymbol{\tau}}\left\{\frac{R^2}{1-r^2}\frac{dr}{d\boldsymbol{\tau}}\right\} = R^2\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\boldsymbol{\tau}}\right)^2 \ .}$$

This matches the form shown in the question, with

$$A = \frac{R^2}{1-r^2} \ , \ \text{and } C = R^2 \frac{r}{(1-r^2)^2} \ ,$$

with $B = D = E = 0$.

(f) *(5 points EXTRA CREDIT)* The algebra here can get messy, but it is not too bad if one does the calculation in an efficient way. One good way to start is to simplify the expression for p. Using the answer from (d),

$$p = \frac{m v_{\rm phys}}{\sqrt{1 - \frac{v_{\rm phys}^2}{c^2}}} = \frac{m \frac{R(t)}{\sqrt{1-r^2}} \frac{dr}{dt}}{\sqrt{1 - \frac{R^2}{c^2(1-r^2)} \left(\frac{dr}{dt}\right)^2}} \ .$$

Using the answer from (b), this simplifies to

$$p = m \frac{R(t)}{\sqrt{1-r^2}} \frac{dr}{dt} \frac{dt}{d\boldsymbol{\tau}} = m \frac{R(t)}{\sqrt{1-r^2}} \frac{dr}{d\boldsymbol{\tau}} \ .$$

Multiply the geodesic equation by m, and then use the above result to rewrite it as

$$\frac{d}{d\boldsymbol{\tau}} \left\{ \frac{Rp}{\sqrt{1-r^2}} \right\} = m R^2 \frac{r}{(1-r^2)^2} \left(\frac{dr}{d\boldsymbol{\tau}} \right)^2 \ .$$

Expanding the left-hand side,

$$\begin{aligned} LHS = \frac{d}{d\boldsymbol{\tau}} \left\{ \frac{Rp}{\sqrt{1-r^2}} \right\} &= \frac{1}{\sqrt{1-r^2}} \frac{d}{d\boldsymbol{\tau}} \left\{ Rp \right\} + Rp \frac{r}{(1-r^2)^{3/2}} \frac{dr}{d\boldsymbol{\tau}} \\ &= \frac{1}{\sqrt{1-r^2}} \frac{d}{d\boldsymbol{\tau}} \left\{ Rp \right\} + m R^2 \frac{r}{(1-r^2)^2} \left(\frac{dr}{d\boldsymbol{\tau}} \right)^2 \ . \end{aligned}$$

Inserting this expression back into left-hand side of the original equation, one sees that the second term cancels the expression on the right-hand side, leaving

$$\frac{1}{\sqrt{1-r^2}} \frac{d}{d\boldsymbol{\tau}} \left\{ Rp \right\} = 0 \ .$$

Multiplying by $\sqrt{1-r^2}$, one has the desired result:

$$\frac{d}{d\boldsymbol{\tau}} \left\{ Rp \right\} = 0 \quad \Longrightarrow \quad \boxed{p \propto \frac{1}{R(t)} \ .}$$