MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — May 4, 2002
Prof. Alan Guth

# REVIEW PROBLEMS FOR QUIZ 3

**QUIZ DATE:** Thursday, May 9, 2002

**COVERAGE:** Lecture Notes 8-13; Problem Sets 5 and 6; Rowan-Robinson, Chapters 6, 8.2 and 8.3, and Epilogue; Weinberg, Chapters 6 through the end of the book. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from this set of Review Problems.** Since big bang nucleosynthesis (Lecture Notes 9) was brushed over rather quickly in class, any questions on this subject will be very similar to the questions on the homework and in this set of Review Problems. Similarly, Lecture Notes 11 (Introduction to Particle Physics) were not discussed in full, so you will be responsible only for Table 3 and the accompanying class discussion.

**CALCULATORS:** Please bring your calculators to this quiz. There may be some numerical problems.

**PURPOSE:** These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. They are all problems that I would consider fair for the coming quiz. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, and 2000. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. Since we are having only three quizzes this year, the coverage of each quiz will not necessarily match the quizzes from previous years. The material for the upcoming quiz, however, is an almost perfect match with Quiz 4 of 2000. The only difference is that the reading from Weinberg included in our upcoming quiz was a little different from 2000: this year it includes Chapter 6, but it excludes Chapters 4 and 5.

## INFORMATION TO BE GIVEN ON QUIZ:

The following material will be included on the quiz, so you need not memorize it. You should, however, make sure that you understand what these formulas mean, and how they can be applied.

**DOPPLER SHIFT:**

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

**COSMOLOGICAL REDSHIFT:**

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

**COSMOLOGICAL EVOLUTION:**

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

**EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:**

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

**EVOLUTION OF A MATTER-DOMINATED UNIVERSE:**

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):
$$ct = \alpha(\theta - \sin\theta) \ ,$$
$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):
$$ct = \alpha\left(\sinh\theta - \theta\right)$$
$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2} \ ,$$
$$\kappa \equiv -k \ .$$

**ROBERTSON-WALKER METRIC:**

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

**SCHWARZSCHILD METRIC:**

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \ ,$$

**GEODESIC EQUATION:**

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or:
$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

**COSMOLOGICAL CONSTANT:**

$$p_{\text{vac}} = -\rho_{\text{vac}}c^2 \qquad \rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G}$$

where $\Lambda$ is the cosmological constant.

**PHYSICAL CONSTANTS:**

$$G = 6.673 \times 10^{-8}\ \mathrm{cm}^3 \cdot \mathrm{g}^{-1} \cdot \mathrm{s}^{-2}$$

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381 \times 10^{-16}\,\mathrm{erg/K} \\ &= 8.617 \times 10^{-5}\,\mathrm{eV/K}\ , \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055 \times 10^{-27}\ \text{erg-sec} \\ &= 6.582 \times 10^{-16}\ \text{eV-sec}\ , \end{aligned}$$

$$c = 2.998 \times 10^{10}\ \mathrm{cm/sec}$$

$$1\ \mathrm{yr} = 3.156 \times 10^7\ \mathrm{s}$$

$$1\ \mathrm{eV} = 1.602 \times 10^{-12}\ \mathrm{erg}\ .$$

**BLACK-BODY RADIATION:**

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

**EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:**

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

**PARTICLE PROPERTIES:**

While working on this exam you may refer to any of the tables in Lecture Notes 11. Please bring your copy of Lecture Notes 11 with you to the exam.

## PROBLEM 1: SHORT ANSWERS:

(a) According to the unified electroweak theory, at the fundamental level an electron is identical to what other type of elementary particle? According to grand unified theories, it is identical to what two other types of particles?

(b) The Higgs fields that are introduced into the electroweak theory or grand unified theories to spontaneously break the internal symmetries are always (A) scalar fields, corresponding to spin 0 particles, (B) spinor fields, corresponding to spin $\frac{1}{2}$ particles, (C) vector fields, corresponding to spin 1 particles, or (D) tensor fields, corresponding to spin 2 particles?

(c) Hadrons are not thought to be elementary particles but instead are composed of more fundamental particles. What is the general name for these more fundamental constituents of the hadrons? Name a kind of lepton. Do leptons feel the strong interactions? Do hadrons? [Hint: "hadrons" refers to strongly interacting particles, the baryons and mesons.]

(d) Grand unified theories (GUTs) predict the existence of magnetic monopoles, point-like topological defects. Is the mass of the monopole expected to be roughly (a) $10^{16}$ GeV (b) 1 GeV (c) 1/2 MeV or (d) zero? What is the most serious cosmological problem associated with the existence of GUT monopoles?

(e) The cosmological evolution of the universe is described by the two Einstein equations given on the cover sheet of the exam. When the energy density of the universe is dominated by matter or by radiation, then the acceleration of the scale factor, described by one of the Einstein equations, is negative. Consequently the expansion of the universe is slowed. During an inflationary era, is the acceleration of the scale factor positive or negative? Why?

(f) The V-A theory of the weak interactions, developed in 1958 by Feynman and Gell-Mann and independently by Marshak and Sudarshan, is listed on Table 3 of Lecture Notes 10 as a "flawed theory". In what way is this theory flawed. (Please answer in no more than 2-3 sentences.)

(g) The word "supersymmetry" refers to a symmetry that relates the behavior of one certain class of particles with the behavior of another class. What are these two classes?

## PROBLEM 2: TIME SCALES IN COSMOLOGY

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

$10^{-43}$ sec.
$10^{-37}$ sec.
$10^{-12}$ sec.
$10^{-5}$ sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the beginning of the processes involved in big bang nucleosynthesis;

(b) the end of the processes involved in big bang nucleosynthesis;

(c) the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{16}$ GeV;

(d) "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

**PROBLEM 3: DID YOU DO THE READING?** *(30 points)*

(a) *(5 points)* When orbital velocities of stars in spiral galaxies are measured, we find that they are mostly constant over a large range in radius. What explanation is usually given to understand these flat rotation curves?

(i) The density waves producing the spiral arms perturb the stellar orbits.

(ii) A flat rotation curve is exactly what you'd expect from Kepler's laws applied to the observed mass profile of spiral galaxies.

(iii) The measurements are dominated by bright young stars in the spiral arms, so we're mistaking the wave velocity of the arms for the rotation of the galaxy as a whole.

(iv) Spiral galaxies contain a halo of dark matter in addition to their normal disk mass.

(v) The stellar orbits aren't circular, so we're measuring stars with more and more elliptical orbits at larger radii.

(b) *(5 points)* Briefly describe the distinguishing characteristics of the Eddington-Lemaître cosmological models. (Hint: they are related to Einstein's static closed universe model.)

(c) *(5 points)* What is the Jeans length?

(i) The size at which the sound-crossing time is equal to the age of the universe

(ii) The minimum size of density fluctuations which are unstable to gravitational collapse

(iii) The size of the first peak in the power spectrum of the cosmic microwave background fluctuations

(iv) The size where we expect the effects of quantum gravity to have a significant influence

(v) Approximately equal to the Jeans waist size

(d) *(5 points)* By what factor does the lepton number per comoving volume of the universe change between temperatures of $kT = 10$ MeV and $kT = 0.1$ MeV? You should assume the existence of the normal three species of neutrinos for your answer. *[Note: this question is based on Chapter 4 of Weinberg, and so it would not be appropriate for Quiz 3 of 2002.]*

(e) *(5 points)* Measurements of the primordial deuterium abundance would give good constraints on the baryon density of the universe. However, this abundance is hard to measure accurately. Which of the following is NOT a reason why this is hard to do? *[Note: this question is based on Chapter 4 of Weinberg, and so it would not be appropriate for Quiz 3 of 2002.]*

(i) The neutron in a deuterium nucleus decays on the time scale of 15 minutes, so almost none of the primordial deuterium produced in the Big Bang is still present.

(ii) The deuterium abundance in the Earth's oceans is biased because, being heavier, less deuterium than hydrogen would have escaped from the Earth's surface.

(iii) The deuterium abundance in the Sun is biased because nuclear reactions tend to destroy it by converting it into helium-3.

(iv) The spectral lines of deuterium are almost identical with those of hydrogen, so deuterium signatures tend to get washed out in spectra of primordial gas clouds.

(v) The deuterium abundance is so small (a few parts per million) that it can be easily changed by astrophysical processes other than primordial nucleosynthesis.

(f) *(5 points)* Give three examples of hadrons.

**PROBLEM 4: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS** *(20 points)*

*The following problem was on Quiz 4, 2000, except that part (c) has been modified:*

(a) *(5 points)* When the temperature of the early universe was $5 \times 10^{10}$ K, what was the ratio of neutrons to protons? You may assume thermal equilibrium, and that the mass difference is given by $(m_n - m_p)c^2 = 1.293$ MeV.

Questions (b), (c), and (d) all refer to calculations that describe a hypothetical world, which differs from the real world in a specified way. In each case you are asked about the calculation of the predicted helium abundance for this hypothetical world. Each of these three parts are to be answered independently; that is, in each part you are to consider a hypothetical world that differs from the real world only by the characteristics stated in that part.

(b) *(5 points)* Suppose the proton-neutron mass difference were larger than the actual value of 1.293 MeV/c$^2$. Would the predicted helium abundance be larger or smaller than in the standard calculation? Explain your answer in a sentence, or in a few sentences.

(c) *(5 points)* Suppose that the nucleosynthesis calculations were carried out with an electron mass given by $m_e c^2 = 1$ KeV, instead of the physical value of 0.511 MeV. This change would affect the production of helium in several ways. Describe one way in which the helium production process would be affected, and explain in a few sentences whether this change would increase or decrease the predicted helium abundance.

(d) *(5 points)* Suppose, due to some significant difference in the nuclear reaction rates, that nucleosynthesis occurred suddenly at a temperature of $5 \times 10^{10}$ K. In that case, what would be the predicted value of $Y$, the fraction of the baryonic mass density of the universe which is helium?

**PROBLEM 5: INFLATION AND THE HORIZON PROBLEM** *(30 points)*

*The following problem was on Quiz 4, 2000:*

(a) *(5 points)* Consider two photons of the cosmic background radiation that are arriving at Earth from opposite directions in the sky. Such photons are observed to have the same energy to within one part in $N$. What is the numerical value of $N$? (Answers within a factor of 10 of the intended answer will be accepted for full credit.)

(b) *(5 points)* In the standard cosmological model (i.e., without inflation), the two photons described in the previous part can be traced back to points in spacetime that were separated from each other by a number of horizon lengths. By approximately how many horizon lengths were these points separated? (If you remember this number, you can quote it from memory, and any number within a factor of 10 of the intended answer will be considered correct. If you don't remember it, you can calculate it from first principles, using the fact that the photons were released when the temperature of the universe was about 4000 K, and that the present temperature is 2.7 K.)

(c) *(5 points)* In inflationary cosmology, the "horizon problem" described in the previous part disappears. In a few sentences, explain why.

(d) *(15 points)* Assuming that a grand unified theory phase transition occurred at a temperature of $kT = E_{\mathrm{GUT}} \simeq 10^{16}$ GeV, calculate (in conventional cosmology, without inflation) the ratio of the diameter at that time of the region that will evolve to become the observed universe to the horizon length at that time. Work in the approximation that the universe is described by a flat Robertson-Walker metric.

# SOLUTIONS

## PROBLEM 1: SHORT ANSWERS:

(a) In the electroweak theory, the electron is fundamentally the same as the neutrino. In grand unified theories, the electron is fundamentally the same as both the neutrino and the quark. (Many students mentioned the symmetry between the electron, the muon, and the tau, which is a part of the symmetry between generations of fermions. This symmetry remains a mystery, not explained by either the electroweak theory or grand unified theories.)

(b) Choice (A): the Higgs fields are scalars. This is essential for the consistency of the theory, since the Higgs fields have a nonzero value in the vacuum. If they were not scalars, then this nonzero value would be measured differently by observers who were rotated with respect to each other, so the rotational invariance of the vacuum would be violated. (The electron, by the way, is a spin-$\frac{1}{2}$ particle, the photon is an example of a spin 1 particle, and the graviton has spin 2.)

(c) Hadrons are thought to be built out of quarks. Electrons, neutrinos, muons are all examples of leptons. Leptons do not feel the strong interactions. Hadrons do feel the strong interactions.

(d) GUT monopoles are expected to have a mass of (a) $10^{16}$ GeV. With the abundance predicted by GUTs, the incredibly massive monopoles would quickly come to dominate the energy density of the universe. The large energy density would speed up the cosmological evolution so that the universe was a mere few tens of thousands of years old today. Such a young and quickly evolving universe is in serious conflict with observations. Also, there are no confirmed observations of monopoles.

(e) During an inflationary era the acceleration of the scale factor is positive. The pressure associated with the false vacuum which drives inflation is negative. The contribution to the acceleration from the negative pressure dominates over that from the positive energy density in the equation

$$\ddot{R} = -\frac{4\pi}{3} G \left( \rho + \frac{3p}{c^2} \right) R \quad .$$

Consequently, the negative pressure powers a repulsive gravitational force.

(f) It was flawed in that it was not "renormalizable". That is, calculations based on quantum mechanical perturbation theory lead to infinities in the sums over intermediate states, which is similar to the situation with QED (quantum electrodynamics). Unlike QED, however, the infinities here cannot be absorbed into a redefinition of the fundamental constants of the theory.

(g) Fermions and bosons.

**PROBLEM 2: TIME SCALES IN COSMOLOGY**

(a) 1 sec. *[This is the time at which the weak interactions "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) $10^{-35}$ sec. *[We learned in Lecture Notes 7 that* $kT$ *was about 1 MeV at* $t = 1$ *sec. Since 1 GeV = 1000 MeV, the value of* $kT$ *that we want is* $10^{17}$ *times higher. In the radiation-dominated era* $T \propto R^{-1} \propto t^{-1/2}$*, so we get* $10^{-34}$ *sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) $10^{-5}$ sec. *[As in (c), we can use* $t \propto T^{-2}$*, with* $kT \approx 1$ *MeV at* $t = 1$ *sec.]*

**PROBLEM 3: DID YOU DO THE READING?** *(30 points)*

(a) The correct answer, of course, is (iv). The other items are supposed to be plausible-sounding but flawed explanations. See Rowan-Robinson section 6.4, pages 102-3.

(b) The key fact is that they have a critical value for the cosmological constant, leading to an asymptotic period of near-static evolution. Technically one asymptotes to the Einstein static model at infinite negative time and then expands (this has no Big Bang), and the other asymptotes to the Einstein case at infinite positive time after a Big Bang and initial period of expansion. See Rowan-Robinson section 8.2, pages 133-4.

(c) The correct answer is, of course, (ii). Answer (i) is intended to be evil and tricky, but the others are merely wrong, and (v) is just a joke to compensate for confusing the students with (i). See Rowan-Robinson section 6.1, pages 97-9.

(d) This is a total trick question. Lepton number is, of course, conserved, so the factor is just 1. See Weinberg chapter 4, pages 91-4.

(e) The correct answer is (i). The others are all real reasons why it's hard to measure, although Weinberg's book emphasizes reason (v) a bit more than modern astrophysicists do: astrophysicists have been looking for other ways that deuterium might be produced, but no significant mechanism has been found. See Weinberg chapter 5, pages 114-7.

(f) The most obvious answers would be proton, neutron, and pi meson. However, any of the particles listed as baryons or mesons in Lecture Notes 11 would be correct. See Weinberg chapter 7, pages 136-8.

**PROBLEM 4: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS**

(a) In thermal equilibrium, the ratio of neutrons to protons is given by a Boltzmann factor,

$$\boxed{\frac{n_n}{n_p} = e^{-\Delta m\, c^2/kT} \ ,}$$

where $\Delta m = (m_n - m_p)$. For $\Delta m\, c^2 = 1.293 \times 10^6$ eV, $k = 8.617 \times 10^{-5}$ eV/K, and $T = 5 \times 10^{10}$ K, this gives

$$\frac{n_n}{n_p} = \exp\left\{-1.293 \times 10^6/(8.617 \times 10^{-5} \times 5 \times 10^{10})\right\} = \boxed{0.741\ .}$$

*Caveat (for stat mech experts):* The above formula would be a precise consequence of statistical mechanics if the neutron and proton were two possible energy levels of the same system. In this case one would describe the system using the *canonical ensemble*, which implies that the probability of the system existing in any specific state $i$ is proportional to $\exp(-E_i/kT)$, where $E_i$ is the energy of the state. However, the neutron and proton are not really different energy levels of the same system, because the conversion between neutrons and protons involves other particles as well; a sample conversion reaction would be

$$n + \nu_e \longleftrightarrow p + e^- \ ,$$

where $\nu_e$ is the electron neutrino, and $e^-$ is the electron. This means that if the universe contained a very large density of electron neutrinos, then $n$-$\nu_e$ collisions would occur more frequently, and the reaction would be driven in the forward direction. Thus, a large density of electron neutrinos would lead to a lower ratio of neutrons to protons than the Boltzmann factor given above. Similarly, if the universe contained a large density of electrons, then the reaction would be driven in the reverse direction, and the ratio of neutrons to protons would be higher than the Boltzmann factor. A complete statistical mechanical treatment of this situation would use the *grand canonical ensemble*, which describes systems in which the number of particles of a given type can change by chemical reactions. In this formalism the density of each type of particle is related to a quantity called the *chemical potential* $\mu$, where in general the relationship is given by

$$n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp\left[(E-\mu)/(kT)\right] \pm 1} E\, \mathrm{d}E$$

where the + sign holds for Fermi particles, the − sign holds for Bose particles, and the factor $g$ has the same meaning as in Lecture Notes 7. The ratio of neutrons to protons is then given by

$$\frac{n_n}{n_p} = e^{-(\Delta m\, c^2 + \mu_\nu - \mu_e)/kT} ,$$

where $\mu_\nu$ and $\mu_e$ represent the chemical potentials for electron neutrinos and electrons, respectively. In the early universe, however, the standard theories imply that the chemical potentials for electrons and neutrinos were both negligible.

(b) A larger $\Delta m$ would mean that the Boltzmann factor described in the previous answer would be smaller, so that there would be fewer neutrons at any given temperature. Fewer neutrons implies less helium, since essentially all the neutrons that exist when the temperature falls enough for deuterium to become stable become bound into helium.

(c) There are at least four effects that occur when the electron mass/energy is taken as 1 KeV instead of 0.511 MeV:

(i) For the real mass/energy of 0.511 MeV the electron-positron pairs freeze out before nucleosynthesis, but a mass/energy of 1 KeV would mean that electron-positron pairs would behave as massless particles throughout the nucleosynthesis process. Just like adding an extra species of neutrino, this additional massless particle would mean that the expansion rate would be larger, since for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho ,$$

and

$$\rho = \frac{u}{c^2} = g\frac{\pi^2}{30}\frac{(kT)^4}{\hbar^3 c^5} .$$

Faster expansion means that the weak interactions "freeze out" earlier, since the freeze-out point is the time at which the interactions can no longer maintain equilibrium as the universe expands. An earlier freeze-out means a higher temperature of freeze-out and hence more neutrons at the time of freeze-out. In addition, the faster expansion rate means faster cooling, which means less time before the temperature of nucleosynthesis is reached, and therefore less time for neutrons to decay. Thus, faster expansion means more neutrons. Since essentially all the neutrons present when the deuterium bottleneck breaks are collected into helium, this implies more helium.

(ii) The most important reactions that keep protons and neutrons in thermal equilibrium all involve electrons and positrons:

$$n + e^{+} \longleftrightarrow p + \bar{\nu}_e$$
$$n + \nu_e \longleftrightarrow p + e^{-} \ .$$

If the electron-positron mass/energy were smaller, then the rates of all of these reactions would be enhanced. The reactions in which an $e^{+}$ or $e^{-}$ appears in the initial state will be enhanced by the presence of more $e^{+}$'s and $e^{-}$'s, and the reactions in which they appear in the final state will be enhanced because a lighter final state is easier to produce. The enhanced rate for these reactions will keep neutrons and protons in thermal equilibrium longer, and hence to lower temperatures, and this would decrease the final abundance of neutrons. Thus this effect will go in the opposite direction as effect (i), leading to the production of less helium.

(iii) If the electron mass is decreased, then the neutron decay

$$n \longrightarrow p + e^{-} + \bar{\nu}_e$$

becomes more exothermic, so it will happen more quickly. Thus more neutrons can decay, leading to less helium.

(iv) As mentioned in (i), lowering the mass/energy of electron-positron pairs to 1 KeV would mean that their freeze-out would not occur until after nucleosynthesis is over. In the real case, however, with $m_e c^2 = 0.511$ MeV, the electron-positron pairs start to freeze out at $t \approx 10$ sec. The energy released by this freeze-out heats the photons, protons, and neutrons, and this extra heat delays the time when the universe cools enough to break the deuterium bottleneck so that helium production can proceed. The delay allows more time for the neutrons to decay, resulting in less helium. Since the freeze-out that occurs for $m_e c^2 = 0.511$ MeV results in less helium, the absence of this freeze-out if $m_e c^2 = 1$ KeV would result in more helium.

Since the effects point in different directions, there is no easy way to know what the net effect will be. I (AHG) tried carrying out a full numerical integration, using the equations from P.J.E. Peebles, "Primordial helium abundance and the primordial fireball II," *Astrophysical Journal* **146**, 542-552 (1966). I found that the net effect of changing $m_e c^2$ to 1 KeV was to produce less helium. Apparently effects (ii) and (iii) above are the most significant. Of course I did not expect students to figure this out in doing their problem sets.

(d) Part (a) asked for the ratio of neutrons to protons, so its answer is

$$A = \frac{n_{\text{neutron}}}{n_{\text{proton}}} \ .$$

The fraction of the baryonic mass in neutrons is then

$$\frac{n_{\text{neutron}}}{n_B} = \frac{n_{\text{neutron}}}{n_{\text{neutron}} + n_{\text{proton}}} = \frac{\frac{n_{\text{neutron}}}{n_{\text{proton}}}}{\frac{n_{\text{neutron}}}{n_{\text{proton}}} + 1} = \frac{A}{1+A} \ .$$

The fraction of the baryonic mass in helium is twice this number, since after nucleosynthesis essentially all neutrons are in helium, and the mass of each helium nucleus is twice the mass of the neutrons within it. Thus

$$Y = \frac{2A}{1+A} \ .$$

This gives $\boxed{Y = 0.851.}$

**PROBLEM 5: INFLATION AND THE HORIZON PROBLEM** *(30 points)*

(a) As described in Lecture Notes 10, the photons of the cosmic background radiation have a uniform temperature (and hence energy) to an accuracy of $\boxed{\text{about one part in } 10^5.}$ (In addition, the motion of the earth through the cosmic background radiation produces an anisotropy of about one part in $10^3$, but this is not really an anisotropy of the radiation itself.)

(b) Again, as described in Lecture Notes 10, the number is $\boxed{\text{about } 90.}$

(c) In an inflationary model, at very early times the region from which the observed universe evolves can be much smaller than in conventional cosmology. The uniformity can be established before inflation, when the region was incredibly small. Then inflation can expand this tiny region to become large enough to easily encompass everything that we see.

(d) To estimate the time $t_{\text{GUT}}$ at which $kT$ was equal to $E_{\text{GUT}}$, we use the formula from the front of the exam for a flat radiation-dominated universe:

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}} \ .$$

For a typical grand unified theory, we will use $g_{\text{GUT}} \approx 100$ (as stated in Lecture Notes 12). So

$$t_{\text{GUT}} \approx \frac{1}{E_{\text{GUT}}^2} \sqrt{\frac{45\hbar^3 c^5}{16\pi^3 g_{\text{GUT}} G}} \ .$$

Numerically,

$$t_{\rm GUT} \approx$$
$$\frac{1}{(10^{25}\ {\rm eV})^2}\sqrt{\frac{45\cdot(6.582\times 10^{-16}\ {\rm eV}\cdot{\rm s})^3\cdot(2.998\times 10^{10}\ {\rm cm}\cdot{\rm s}^{-1})^5}{16\pi^3\cdot 100\cdot 5.573\times 10^{-8}\ {\rm cm}^3\cdot{\rm g}^{-1}\cdot{\rm s}^{-2}}}$$
$$\times\sqrt{\frac{1\ {\rm eV}}{1.602\times 10^{-12}{\rm erg}}\frac{1\ {\rm erg}}{{\rm g}\cdot{\rm cm}^2\cdot{\rm s}^{-2}}}$$
$$= 2.42\times 10^{-39}\ {\rm s}\ .$$

For a radiation-dominated universe the physical horizon length $\ell_{\rm hor}$ is $2ct$, so

$$\begin{aligned}\ell_{\rm hor}(t_{\rm GUT}) &= 2ct_{\rm GUT}\\ &\approx 2\cdot 2.998\times 10^{10}\ {\rm cm}\cdot{\rm s}^{-1}\cdot 2.42\times 10^{-39}\ {\rm s}\\ &= 1.45\times 10^{-28}\ {\rm cm}\ .\end{aligned}$$

We are asked to compare the horizon length $\ell_{\rm hor}(t_{\rm GUT})$ with the diameter at that time of the region that will evolve to become the presently observed universe. The radius of the presently observed universe is about equal to the horizon distance calculated for a matter-dominated universe,

$$\ell_{\rm hor}(t_0) \approx 3ct_0\ .$$

At $t_{\rm GUT}$, the diameter of this region was

$$d(t_{\rm GUT}) \approx 6ct_0\,\frac{R(t_{\rm GUT})}{R(t_0)}\ .$$

To determine this ration of scale factors, one can use the fact that entropy is roughly conserved between $t_{\rm GUT}$ and the present. (It is this statement that would be radically changed in an inflationary model, in which a huge amount of entropy is produced by the reheating at the end of the period of inflation.) Since entropy density $s$ is proportional to $gT^3$, and conservation of entropy implies that $R^3 s = const$, one has

$$R^3(t_{\rm GUT})g_{\rm GUT}T^3_{\rm GUT} \approx R^3(t_0)g_0T_0^3\ .$$

Here $g_0$ is the present value of $g$, which we will take to be about equal to 10. Then

$$d(t_{\rm GUT}) \approx 6ct_0\left(\frac{g_0}{g_{\rm GUT}}\right)^{1/3}\frac{T_0}{T_{\rm GUT}} = 6ct_0\left(\frac{g_0}{g_{\rm GUT}}\right)^{1/3}\frac{kT_0}{E_{\rm GUT}}\ .$$

Numerically,

$$\begin{aligned} d(t_{\rm GUT}) \approx\ & 6 \cdot (2.998 \times 10^{10}\ \text{cm} \cdot \text{s}^{-1}) \cdot (1.5 \times 10^{10}\ \text{yr}) \\ & \times (3.156 \times 10^{7}\ \text{s} \cdot \text{yr}^{-1}) \cdot \left(\frac{10}{100}\right)^{1/3} \\ & \times \frac{(8.617 \times 10^{-5}\ \text{eV} \cdot \text{K}^{-1}) \cdot 2.7\ \text{K}}{10^{25}\ \text{eV}} \\ \approx\ & 0.920\ \text{cm}\ . \end{aligned}$$

Finally, the ratio is given by

$$\frac{d(t_{\rm GUT})}{\ell_{\rm hor}(t_{\rm GUT})} \approx \frac{0.920\ \text{cm}}{1.45 \times 10^{-28}\ \text{cm}} \approx 6.34 \times 10^{27} \sim \boxed{10^{28}}\ .$$

That is, at the time when $kT$ was equal to $T_{\rm GUT}$, the diameter of the region that will evolve to become the presently observed universe was about $10^{28}$ times larger than the horizon length at that time.