MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 4, 2002
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QUIZ 1 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) The radio emission from the Milky Way is primarily produced by cosmic ray electrons spiralling in the Galaxy's magnetic field.

(b) The ***Hertzsprung-Russell*** diagram, on which stars burning hydrogen like our sun lie on a line called the ***main sequence***, is a plot of the luminosity of a star against color or (surface) temperature.

(c) The oldest stars in our galaxy lie in globular clusters.

(d) Newton proposed that one could test whether or not stars are distributed with uniform number density throughout the universe by counting the number of stars as a function of their observed flux.

(e) Cosmologists now believe that the universe today is dominated by "dark energy" (i.e., energy density of the vacuum, or some form of peculiar matter that behaves very similarly).

(f) The horizontal (wavelength) axis of the graph of the spectrum of the cosmic background radiation from Weinberg's book is calibrated in centimeters. The peak is at about 0.2 cm.

(g) The curve falls off at long wavelengths because it is hard to fit radiation into any volume whose dimensions are smaller than the wavelength.

(h) The curve falls off at short wavelengths because the energy of any photon is inversely proportional to the wavelength, so at a given temperature there will not be enough energy to produce many photons of very short wavelength.

PROBLEM 2: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(40 points)*

a) *(5 points)* The cosmological redshift is given by the usual form,

$$1 + z = \frac{R(t_0)}{R(t_e)} \quad .$$

For light emitted by an object at time t_e, the redshift of the received light is

$$1 + z = \frac{R(t_0)}{R(t_e)} = \left(\frac{t_0}{t_e}\right)^{\gamma} \quad .$$

So,

$$z = \left(\frac{t_0}{t_e}\right)^{\gamma} - 1 \quad .$$

b) *(5 points)* The coordinates t_0 and t_e are cosmic time coordinates. The "look-back" time as defined in the exam is then the interval $t_0 - t_e$. We can write this as

$$t_0 - t_e = t_0\left(1 - \frac{t_e}{t_0}\right) \quad .$$

We can use the result of part (a) to eliminate t_e/t_0 in favor of z. From (a),

$$\frac{t_e}{t_0} = (1+z)^{-1/\gamma} \quad .$$

Therefore,

$$t_0 - t_e = t_0\left[1 - (1+z)^{-1/\gamma}\right] \quad .$$

c) *(10 points)* The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0) = R(t_0)\int_{t_e}^{t_0} \frac{c}{R(t)}dt \quad .$$

Calculating this integral gives

$$\ell_p(t_0) = \frac{ct_0^{\gamma}}{1-\gamma}\left[\frac{1}{t_0^{\gamma-1}} - \frac{1}{t_e^{\gamma-1}}\right] \quad .$$

Factoring $t_0^{\gamma-1}$ out of the parentheses gives

$$\ell_p(t_0) = \frac{ct_0}{1-\gamma}\left[1 - \left(\frac{t_0}{t_e}\right)^{\gamma-1}\right] \quad .$$

This can be rewritten in terms of z and H_0 using the result of part (a) as well as,

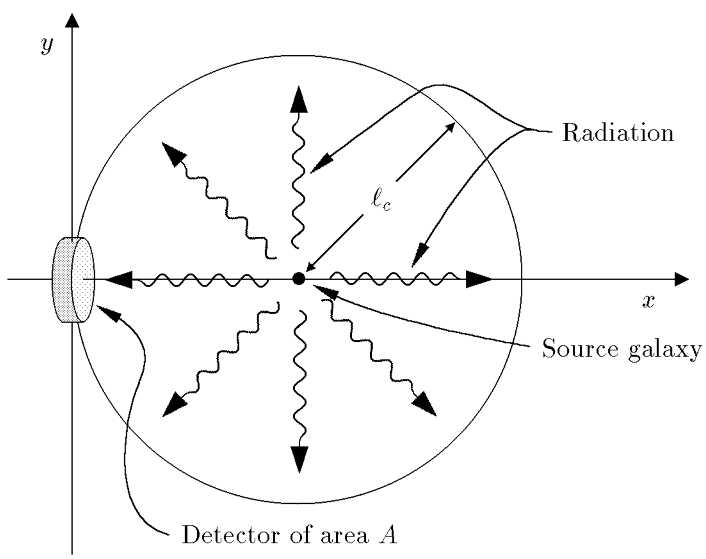
$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{\gamma}{t_0} \quad .$$

Finally then,

$$\boxed{\ell_p(t_0) = cH_0^{-1}\frac{\gamma}{1-\gamma}\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right] \quad .}$$

d) *(10 points)* A nearly identical problem was worked through in Problem 8 of Problem Set 1.

The energy of the observed photons will be redshifted by a factor of $(1+z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of photon emmission, reducing the flux by another factor of $(1+z)$. Consequently, the observed power will be redshifted by two factors of $(1+z)$ to $P/(1+z)^2$.



Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance ℓ_c. Now consider the photons passing through a patch of the sphere with physical area A. In comoving coordinates the present area of the patch is $A/R(t_0)^2$. Since the object radiates uniformly in all directions, the patch will intercept a fraction $(A/R(t_0)^2)/(4\pi\ell_c^2)$ of the photons passing through the sphere. Thus the power hitting the area A is

$$\frac{(A/R(t_0)^2)}{4\pi\ell_c^2}\frac{P}{(1+z)^2} \quad .$$

The radiation energy flux J, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4\pi\ell_p(t_0)^2}\frac{P}{(1+z)^2}$$

where we used $\ell_p(t_0) = R(t_0)\ell_c$. Using the result of part (c) to write J in terms of P, H_0, z, and γ gives,

$$\boxed{J = \frac{H_0^2}{4\pi c^2}\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{P}{(1+z)^2\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right]^2} \;.}$$

e) *(10 points)* Following the solution of Problem 1 of Problem Set 1, we can introduce a fictitious relay station that is at rest relative to the galaxy, but located just next to the jet, between the jet and Earth. As in the previous solution, the relay station simply rebroadcasts the signal it receives from the source, at exactly the instant that it receives it. The relay station therefore has no effect on the signal received by the observer, but allows us to divide the problem into two simple parts.

The distance between the jet and the relay station is very short compared to cosmological scales, so the effect of the expansion of the universe is negligible. For this part of the problem we can use special relativity, which says that the period with which the relay station measures the received radiation is given by

$$\Delta t_{\text{relay station}} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \times \Delta t_{\text{source}} \;.$$

Note that I have used the formula from the front of the exam, but I have changed the size of v, since the source in this case is moving toward the relay station, so the light is blue-shifted. To observers on Earth, the relay station is just a source at rest in the comoving coordinate system, so

$$\Delta t_{\text{observed}} = (1+z)\Delta t_{\text{relay station}} \;.$$

Thus,

$$\begin{aligned} 1 + z_J \equiv \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{source}}} &= \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{relay station}}}\frac{\Delta t_{\text{relay station}}}{\Delta t_{\text{source}}} \\ &= (1+z)\big|_{\text{cosmological}} \;\times\; (1+z)\big|_{\text{special relativity}} \\ &= (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \;. \end{aligned}$$

Thus,

$$\boxed{z_J = (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} - 1 \;.}$$

Note added: In looking over the solutions to this problem, I found that a substantial number of students wrote solutions based on the incorrect assumption that the Doppler shift could be treated as if it were entirely due to motion. These students used the special relativity Doppler shift formula to convert the redshift z of the galaxy to a velocity of recession, then subtracted from this the speed v of the jet, and then again used the special relativity Doppler shift formula to find the Doppler shift corresponding to this composite velocity. However, as discussed at the end of Lecture Notes 3, the cosmological Doppler shift is given by

$$1 + z \equiv \frac{\Delta t_o}{\Delta t_e} = \frac{R(t_o)}{R(t_e)} \;, \tag{3.11}$$

and is not purely an effect caused by motion. It is really the combined effect of the motion of the distant galaxies and the gravitational field that exists between the galaxies, so the special relativity formula relating z to v does not apply.

PROBLEM 3: PARTICLE TRAJECTORIES IN NEWTONIAN COSMOLOGY *(35 points)*

(a) *(10 points)* The particle will feel the gravitational field of all those particles in the model universe whose radius is less than $|\vec{r}_A|$. (The spherical shell of matter outside of radius $|\vec{r}_A|$ does not create any gravitational field inside the shell.) The total mass enclosed within a radius $|\vec{r}_A|$ at time t is given by

$$M = \frac{4\pi}{3}\,|\vec{r}_A|^3\,\rho(t) \;.$$

The acceleration caused by this mass is

$$\vec{a}_A = -\frac{GM}{|\vec{r}_A|^2}\,\hat{r}_A = -\frac{4\pi}{3}\,G\,\rho(t)|\vec{r}_A|\,\hat{r}_A$$

$$= \boxed{-\frac{4\pi}{3}\,G\,\rho(t)\,\vec{r}_A \;.}$$

The acceleration is radially inward, but the vector expression above already includes this information.

(b) *(8 points)* The Hubble velocity is equal to the Hubble parameter $H(t)$ times the distance from the center, $|\vec{r}_A|$, directed radially outward. As a vector expression, this can be written simply as

$$\boxed{\vec{v}_H = H(t)\,\vec{r}_A \;.}$$

(c) *(9 points)* Starting from the definition of the proper velocity,

$$\vec{v}_p = \vec{v}_A - \vec{v}_H \ ,$$

where $\vec{v}_H$ is given in the previous part, we can differentiate to find

$$\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} = \frac{\mathrm{d}\vec{v}_A}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\left[H(t)\,\vec{r}_A\right] = \frac{\mathrm{d}\vec{v}_A}{\mathrm{d}t} - \dot{H}\,\vec{r}_A - H\vec{v}_A \ .$$

Then note that $\mathrm{d}\vec{v}_A/\mathrm{d}t$ can be written from part (a) as

$$\frac{\mathrm{d}\vec{v}_A}{\mathrm{d}t} = -\frac{4\pi}{3}G\rho(t)\vec{r}_A \ ,$$

and

$$\dot{H} = \frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{R}}{R}\right) = \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} = -\frac{4\pi}{3}G\rho(t) - H^2(t) \ ,$$

where in the last substitution we used the $\ddot{R}$ Friedmann equation,

$$\ddot{R} = -\frac{4\pi}{3}G\rho R \ ,$$

which was listed on the front of the exam. Substituting these expressions into the previous expression for $\mathrm{d}\vec{v}_p/\mathrm{d}t$, one finds

$$\begin{aligned}\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} &= -\frac{4\pi}{3}G\rho(t)\vec{r}_A - \dot{H}\,\vec{r}_A - H\,\vec{v}_A \\ &= -\frac{4\pi}{3}G\rho(t)\vec{r}_A + \frac{4\pi}{3}G\rho(t)\vec{r}_A + H^2\,\vec{r}_A - H\vec{v}_A \ .\end{aligned}$$

The first two terms cancel — physically, this cancellation is just the statement that that the test particle A and the comoving particles that it is passing are experiencing the same acceleration. From part (b) we recall that $H^2\,\vec{r}_A = H\,\vec{v}_H$, so

$$\boxed{\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} = H(\vec{v}_H - \vec{v}_A) = -H\,\vec{v}_p \ .}$$

This agrees with the specified form,

$$\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} = -\lambda H(t)\vec{v}_p \ ,$$

provided that $\boxed{\lambda = 1}$.

(d) *(8 points)* From the given equation

$$\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} = -\lambda H(t)\vec{v}_p \ ,$$

one can start manipulating by replacing $H(t)$ by $\dot{R}/R$. One can then specialize the vector equation to the component $v_{p,x}$, and one can then manipulate the equation to put everything that depends on $v_{p,x}$ on the left, and everything that depends on t or $R(t)$ on the right. That is,

$$\frac{\mathrm{d}v_{p,x}}{\mathrm{d}t} = -\lambda\, H(t)\, v_{p,x}$$

can be rewritten as

$$\frac{\mathrm{d}v_{p,x}}{v_{p,x}} = -\lambda \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\,\mathrm{d}t = -\lambda \frac{\mathrm{d}R}{R} \ .$$

Integrating both sides,

$$\ln v_{p,x} = -\lambda \ln R + \mathrm{const} \ .$$

Exponentiating both sides,

$$v_{p,x} = e^{\mathrm{const}}\, R^{-\lambda}(t) \ .$$

If we now define the constant

$$v_0 \equiv e^{\mathrm{const}} \ ,$$

we get the desired the final result,

$$\boxed{v_{p,x} = R^{-\lambda}(t)\, v_0 \ .}$$

This matches the desired form, provided that

$$\boxed{n = -\lambda \ .}$$