MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 24, 2002
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QUIZ 2 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(30 points)*

(a) Which one of the following is the correct statement of Birkhoff's theorem, a result from general relativity theory?

The correct answer was (iii), "The gravitational effect of a uniform medium external to a spherical cavity is zero" (see Rowan-Robinson, p.63).

Birkhoff's theorem in general relativity is the analogue of Newton's "iron sphere" theorems for non-relativistic gravity. Newton showed that the gravitational field outside of a hollow spherical shell of matter is the same as if all the matter were concentrated in a point at the center of the sphere, and also that the field inside the shell is zero (these follow simply from Gauss' law). We derived the equations which determine the evolution of the scale factor with time in a Newtonian model using these facts. The analogous derivation of these equations in full general relativity invokes Birkhoff's theorem (and remarkably, apart from the relativistic correction for pressure, the result is exactly the same as the Newtonian result).

(b) What is the most likely explanation for the apparent predominance of matter over antimatter in the present day universe?

The correct answer was (i), "The number of baryons in the early universe exceeded the number of antibaryons by about 1 part in 10^9. The overwhelming majority of these baryon-antibaryon pairs annihilated, leaving an excess of baryons and a contribution to the large ratio of photons to baryons" (see Weinberg p.95-98).

There is no known natural physical mechanism which seperates matter from antimatter. And if the observable universe does contain seperate domains of antimatter, we might also expect to see some signature of the 511 keV annihilation radiation produced at the boundaries between matter and antimatter domains (although if the nearest domain boundary is

beyond the local supercluster, the radiation would probably be too weak to detect).

Most cosmologists are therefore inclined to think that the visible universe has a real matter-antimatter asymmetry, which presumably arises from some process in the very early universe. In inflationary models in particular, there is no way that this asymmetry could have been part of the initial conditions, since in inflationary models (as we will see) the matter of the universe is generated as it evolves. This led cosmologists to theories of baryogenesis, in which the universe started out baryon symmetric but was driven into an asymmetric state (usually by invoking the physics of grand unified theories (GUTs), which come into play at extreme temperatures).

(c) The collisions of neutrons and protons with electrons, neutrinos, and their antiparticles ceased to be important by $t \approx 3$ minutes, but there was still one process that continued to cause the ratio of neutrons to protons to change. What was this process?

The correct answer was (ii), "The neutron can decay into an electron, proton, and antineutrino. This process continued until the temperature was low enough for deuterium to form", (see R-R p.88, Weinberg p.109).

After a short time the universe had cooled enough to prevent the baryons from breaking up into free quarks, but was hot enough (1 MeV $\ll k_B T \ll$ 1 GeV) for neutrons (n), protons (p), electrons (e^-), positrons (e^+), muons (μ^-), and antimuons (μ^+) to spontaneously form and react with one another. After a while the temperature dropped below the rest mass of neutrons, protons and muons, and the typical reactions occurring were

$$\begin{array}{cclll}
n & \longrightarrow & p^+ + e^- + \bar\nu_e & & \text{(beta decay)} \\
n + e^+ & \longleftrightarrow & p^+ + \bar\nu_e & & (e^+ \text{ capture}) \\
n + \nu_e & \longleftrightarrow & p^+ + e^- & & (e^- \text{ capture}) \\
e^+ + e^- & \longleftrightarrow & 2\gamma & & \text{(ann./pair-prod.)} \\
e^+ + e^- & \longleftrightarrow & \nu_e + \bar\nu_e & & \text{(weak reactions)}
\end{array}$$

These reactions kept all species in equilibrium. The rate of beta decay is fixed at $\lambda_{np} \approx 887s$, but the rate of the e^+/e^- capture reactions has a strong temperature dependance $\lambda_c \propto T^5$. The capture reactions therefore eventually "freeze-out" as the universe cools and the reaction rate drops below the rate of expansion $\dot{R}/R$. Before this occurs, thermal equilibrium ensures that the ratio of the (comoving) number density of neutrons (denoted by n) to the number density of protons (denoted by p) is given by the Boltzmann factor

$$(n/p)_{\text{eq}} = \exp(-Q/k_B T) \quad \text{where} \quad Q \equiv (m_n - m_p)c^2 \ .$$

After freezout of the capture reactions it is still possible for neutrons to beta decay. ("Beta decay" refers to the reaction $n \longrightarrow p^+ + e^- + \bar{\nu}_e$ which occurs via the weak force, and the name originates from early 20th century nuclear physics, when the electrons emitted in the radioactive decay (via the weak force) of the neutrons in certain nuclei were referred to as "beta radiation").

This further reduces the neutron to proton ratio, until the temperature becomes low enough for the deuteron (^{2}H) to form, at about $10^{8.9} K$, corresponding to 0.068 MeV (much less than the deuteron binding energy 2.22 MeV, since even at temperatures much lower than the binding energy there may still be plenty of photons in the tail of the blackbody distribution with greater than binding energy). This temperature was reached when the time was $t \approx 180$ s (Hence "the first three minutes").

After the temperature had dropped enough for the deuteron to form, the following reactions can occur

$$\begin{aligned} n + p &\longleftrightarrow d + \gamma \\ d + d &\longrightarrow t + p \\ d + d &\longrightarrow {}^3\text{He} + n \\ d + t &\longrightarrow {}^4\text{He} + n \\ d + {}^3\text{He} &\longrightarrow {}^4\text{He} + p \ , \end{aligned}$$

where d is the deuteron (^{2}H) and t is tritium (^{3}H). Hardly any heavier elements and isotopes are produced until the stars form, much later, after recombination (but some ^{7}Li is produced, and can be observed).

(d) The spectrum of the cosmic background radiation is distorted very slightly by the Sunyaev-Zeldovich effect. Which of the following statements is the best description of this effect?

The correct answer was (iii), "When cosmic background radiation photons traverse a hot cluster of galaxies, they are scattered by the electrons in the hot gas within the cluster. The scattering on average increases the energy of the photons, with the result that the background looks cooler than average at long wavelengths and hotter than average at short wavelengths" (see R-R p.86). This effect has already been observed in many clusters of galaxies. Its use lies in the fact that the observed distortion of the microwave background spectrum, combined with independent measurements of the x-ray emission from the hot gas, allows determination of both the baryon density and temperature within the cluster. This in turn allows measurement of the Hubble constant, since the intrinsic x-ray luminosity of the cluster can be determined from its temperature.

(e) Which one of the following processes in the early history of the universe was the last to occur?

The correct answer was (iii), "Recombination" (or the "epoch of decoupling of radiation and matter"), i.e. the formation of neutral hydrogen and helium atoms for the first time (see R-R, p.78, Weinberg, p.112).

Here is a rough description of the sequence of events in the early universe.

The muons (and taus) annihilated earliest when the temperature was about $T \approx 10^{12.1K}$ ($T \approx 10^{13.3}$ for taus). At temperatures above $T \approx 10^{10.5} K$, all of the following reactions occurred maintaining all species in equilibrium.

$$\begin{array}{cccl} n & \longrightarrow & p^+ + e^- + \bar{\nu}_e & \text{(beta decay)} \\ n + e^+ & \longleftrightarrow & p^+ + \bar{\nu}_e & (e^+ \text{ capture}) \\ n + \nu_e & \longleftrightarrow & p^+ + e^- & (e^- \text{ capture}) \\ e^+ + e^- & \longleftrightarrow & 2\gamma & \text{(ann./pair-prod)} \\ e^+ + e^- & \longleftrightarrow & \nu_e + \bar{\nu}_e & \text{(weak reactions)} \ . \end{array}$$

As the temperature dropped below $T \approx 10^{10.5} K$, the reaction $e^+ + e^- \longleftrightarrow \nu_e + \bar{\nu}_e$ froze out2, and neutrinos were no longer in thermal equilibrium after this (neutrino decoupling).

When the temperature further dropped to $T \approx 10^{10.1} K$ the reactions $n + e^+ \longleftrightarrow p^+ + \bar{\nu}_e$, $n + \nu_e \longleftrightarrow p^+ + e^-$ froze out, and the neutrons were left to beta decay.

Slightly later, when $T \approx 10^{9.7} K$, the electrons and positrons annihilated, enhancing the temperature of the photon background relative to the neutrino background (since the neutrinos had decoupled).

The neutron beta decay $n \longrightarrow p^+ + e^- + \bar{\nu}_e$ continued for a few minutes, further reducing the neutron to proton ratio, until the temperature was low enough for the deuteron to exist stably, $T \approx 10^{8.9} K$. Deuterium then fused to produce stable helium nuclei.

Much later, when the universe had cooled to $T \approx 1000K$, there was an insufficient number of photons with $E > 13.6$ eV to prevent neutral hydrogen from forming for the first time (actually neutral helium formed slightly earlier than the neutral hydrogen, since helium has a higher ionization energy, $E \approx 24.6$ eV). After this "epoch of recombination" the universe became transparent to the CMB photons, which did not interact further with matter until later, at relatively low redshift $z = 30 - 100$, the universe was re-ionized by the appearance of the first stars.

PROBLEM 2: PROPERTIES OF BLACK-BODY RADIATION *(30 points)*

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so

$g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)} kT \ .}$$

Numerically, this gives

$$\boxed{E = 2.701\, kT \ .}$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate. By substituting k = Boltzmann's constant $= 1.381 \times 10^{-16}$ erg/K $= 8.617 \times 10^{-5}$ eV/K, one has

$$\boxed{E = \left(3.730 \times 10^{-16} \text{ erg/K}\right) T = \left(2.327 \times 10^{-4} \text{ eV/K}\right) T \ .}$$

A note about style: The official convention is to use K and not °K, but the ° symbol is still used with °C and °F. Note also that the K in the denominator of the answer is necessary: the symbol T is a temperature, not a pure number, so "T ergs" would have the units of K·erg, and not ergs. Finally, a conceivable way to write the answer would be

$$E = 3.730 \times 10^{-16} T(\text{in K}) \text{ erg} \ .$$

This is intelligible, but style guides such as NIST (National Institute of Standards and Technology) Special Publication 811 strongly discourage this format. Another acceptable format would be

$$E = 3.730 \times 10^{-16} (T/\text{K}) \text{ erg} \ .$$

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g\dfrac{2\pi^2}{45}\,\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\,\dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{2\pi^4}{45\zeta(3)}\,k\ .}$$

Numerically, this gives

$$\boxed{S = 3.602\,k = 4.974\times10^{-16}\ \mathrm{erg/K} = 3.104\times10^{-4}\ \mathrm{eV/K}\ ,}$$

where k is the Boltzman constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is 7/8 times the number of spin states, and g^* is 3/4 times the number of spin states. So the average energy per particle is

$$E = \frac{g\dfrac{\pi^2}{30}\,\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\,\dfrac{(kT)^3}{(\hbar c)^3}} = \frac{\dfrac{7}{8}\dfrac{\pi^2}{30}\,\dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\,\dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{7\pi^4}{180\zeta(3)}\,kT\ .}$$

Numerically, $E = 3.1514\,kT$.

Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

Completing the numerics,

$$\boxed{E = 3.151\,kT = \left(4.352\times10^{-16}\ \mathrm{erg/K}\right)T = \left(2.716\times10^{-4}\ \mathrm{eV/K}\right)T\ .}$$

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

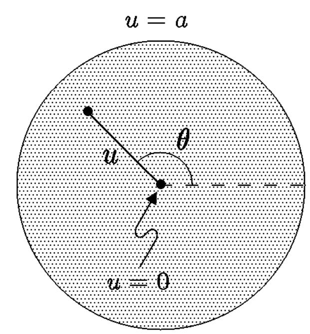
(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$
\begin{aligned}
S &= \frac{g\dfrac{2\pi^2}{45}\,\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\,\dfrac{(kT)^3}{(\hbar c)^3}} \\
&= \frac{\dfrac{7}{8}\,\dfrac{2\pi^2}{45}\,\dfrac{k^4T^3}{(\hbar c)^3}}{\dfrac{3}{4}\,\dfrac{\zeta(3)}{\pi^2}\,\dfrac{(kT)^3}{(\hbar c)^3}} \\
&= \boxed{\frac{7\pi^4}{135\zeta(3)}\,k\ .}
\end{aligned}
$$

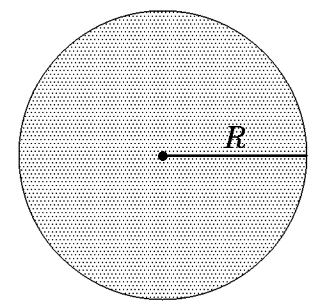
Numerically, this gives

$$\boxed{S = 4.202\,k = 5.803 \times 10^{-16}\ \mathrm{erg/K} = 3.621 \times 10^{-4}\ \mathrm{eV/K}\ .}$$

PROBLEM 3: A TWO-DIMENSIONAL CURVED SPACE *(40 points)*



(a) For $\theta = constant$, the expression for the metric reduces to



$$ds^2 = \frac{a\,\mathrm{d}u^2}{4u(a-u)} \quad \Longrightarrow$$

$$ds = \frac{1}{2}\sqrt{\frac{a}{u(a-u)}}\,\mathrm{d}u\ .$$

To find the length of the radial line shown, one must integrate this expression from the value of u at the center, which is 0, to the value of u at the outer edge, which is a. So

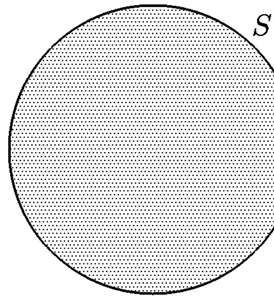
$$\boxed{R = \frac{1}{2}\int_0^a \sqrt{\frac{a}{u(a-u)}}\,\mathrm{d}u\ .}$$

You were not expected to do it, but the integral can be carried out, giving $R = (\pi/2)\sqrt{a}$.

(b) For $u = constant$, the expression for the metric reduces to

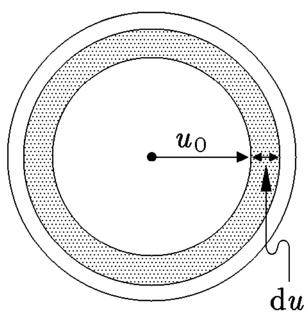
$$ds^2 = u\,\mathrm{d}\theta^2 \quad \Longrightarrow \quad ds = \sqrt{u}\,\mathrm{d}\theta\ .$$

Since θ runs from 0 to 2π, and $u = a$ for the circumference of the space,



$$\boxed{S = \int_0^{2\pi} \sqrt{a}\,\mathrm{d}\theta = 2\pi\sqrt{a}\ .}$$

(c) To evaluate the answer to first order in $\mathrm{d}u$ means to neglect any terms that would be proportional to $\mathrm{d}u^2$ or higher powers. This means that we can treat the annulus as if it were arbitrarily thin, in which case we can imagine bending it into a rectangle without changing its area. The area is then equal to the circumference times the width. Both the circumference and the width must be calculated by using the metric:



$$\begin{aligned}\mathrm{d}A &= \text{circumference} \times \text{width} \\ &= [2\pi\sqrt{u_0}\,] \times \left[\frac{1}{2}\sqrt{\frac{a}{u_0(a-u_0)}}\,\mathrm{d}u\right] \\ &= \boxed{\pi\sqrt{\frac{a}{(a-u_0)}}\,\mathrm{d}u\ .}\end{aligned}$$

(d) We can find the total area by imagining that it is broken up into annuluses, where a single annulus starts at radial coordinate u and extends to $u + \mathrm{d}u$. As in part (a), this expression must be integrated from the value of u at the center, which is 0, to the value of u at the outer edge, which is a.

$$\boxed{A = \pi \int_0^a \sqrt{\frac{a}{(a-u)}}\,\mathrm{d}u\ .}$$

You did not need to carry out this integration, but the answer would be $A = 2\pi a$.

(e) From the list at the front of the exam, the general formula for a geodesic is written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{k\ell}}{\partial x^i}\frac{\mathrm{d}x^k}{\mathrm{d}s}\frac{\mathrm{d}x^\ell}{\mathrm{d}s}\ .$$

The metric components g_{ij} are related to $\mathrm{d}s^2$ by

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\ ,$$

where the Einstein summation convention (sum over repeated indices) is assumed. In this case

$$g_{11} \equiv g_{uu} = \frac{a}{4u(a-u)}$$

$$g_{22} \equiv g_{\theta\theta} = u$$

$$g_{12} = g_{21} = 0\ ,$$

where I have chosen $x^1 = u$ and $x^2 = \theta$. The equation with $\mathrm{d}u/\mathrm{d}s$ on the left-hand side is found by looking at the geodesic equations for $i = 1$. Of course j, k, and ℓ must all be summed, but the only nonzero contributions arise when $j = 1$, and k and ℓ are either both equal to 1 or both equal to 2:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{uu}\frac{\mathrm{d}u}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{uu}}{\partial u}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial u}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 .$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{a}{4u(a-u)}\frac{\mathrm{d}u}{\mathrm{d}s}\right] &= \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{a}{4u(a-u)}\right)\right]\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}u}(u)\right]\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \\ &= \frac{1}{2}\left[\frac{a}{4u(a-u)^2} - \frac{a}{4u^2(a-u)}\right]\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \\ &= \boxed{\frac{1}{8}\frac{a(2u-a)}{u^2(a-u)^2}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 .}\end{aligned}$$

(f) This part is solved by the same method, but it is simpler. Here we consider the geodesic equation with $i = 2$. The only term that contributes on the left-hand side is $j = 2$. On the right-hand side one finds nontrivial expressions when k and ℓ are either both equal to 1 or both equal to 2. However, the terms on the right-hand side both involve the derivative of the metric with respect to $x^2 = \theta$, and these derivatives all vanish. So

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{uu}}{\partial \theta}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial \theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 ,$$

which reduces to

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left[u\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = 0 .}$$