MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 9, 2002
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QUIZ 2

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USEFUL INFORMATION:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):
$$ct = \alpha(\theta - \sin\theta) \ ,$$
$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{k^{3/2} c^2}$$

Open ($\Omega < 1$):
$$ct = \alpha\left(\sinh\theta - \theta\right)$$
$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{\kappa^{3/2} c^2} \ ,$$
$$\kappa \equiv -k \ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or:
$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

PHYSICAL CONSTANTS:

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\,\text{erg/K}$$
$$= 8.617 \times 10^{-5}\,\text{eV/K} \ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27} \text{ erg-sec}$$
$$= 6.582 \times 10^{-16} \text{ eV-sec} ,$$
$$c = 2.998 \times 10^{10} \text{ cm/sec}$$
$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg} .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3} , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions} , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3c^5}{16\pi^3 gG}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

PROBLEM 1: DID YOU DO THE READING? *(30 points)*

The following questions are worth 6 points each.

(a) Which one of the following is the correct statement of Birkhoff's theorem, a result from general relativity theory?

- (i) In a frame of reference that falls freely, the effects of gravity disappear provided that the speed is small compared to that of light.
- (ii) The interval between two nearby points in spacetime is always either a quadratic or cubic form in the coordinate differentials.
- (iii) The gravitational effect of a uniform medium external to a spherical cavity is zero.
- (iv) A light ray passing nearby a massive body is deflected through half the angle that would be predicted by Newtonian physics (assuming that the gravitational field is not very strong).

(b) What is the most likely explanation for the apparent predominance of matter over antimatter in the present day universe?

- (i) The number of baryons in the early universe exceeded the number of antibaryons by about 1 part in 10^9. The overwhelming majority of these baryon-antibaryon pairs annihilated, leaving an excess of baryons and a contribution to the large ratio of photons to baryons.
- (ii) There were the same number of baryons as antibaryons in the very early universe, but during nucleosynthesis the antibaryons were preferentially bound into helium.
- (iii) The universe is segregated into seperate domains of pure matter or antimatter, and we happen to live in a matter domain.
- (iv) Antimatter is preferentially accreted by black holes.

(c) The collisions of neutrons and protons with electrons, neutrinos, and their antiparticles ceased to be important by $t \approx 3$ minutes, but there was still one process that continued to cause the ratio of neutrons to protons to change. What was this process?

- (i) Since the thermal equilibrium ratio of neutrons to protons depends on temperature, the ratio of neutrons to protons continued to decrease as the universe cooled, so that thermal equilibrium was maintained.
- (ii) The neutron can decay into an electron, proton, and antineutrino. This process continued until the temperature was low enough for deuterium to form.
- (iii) The proton can decay into a positron, neutron, and neutrino. This process continued until the temperature was low enough for deuterium to form.
- (iv) Since protons are charged particles, they interacted much more strongly with the photons than the neutrons did, and hence their number was reduced.

— Problem 1 Continues —

(d) The spectrum of the cosmic background radiation is distorted very slightly by the Sunyaev-Zeldovich effect. Which of the following statements is the best description of this effect?

(i) The quadrupole anisotropy in the microwave background, produced in a universe which is expanding anisotropically (rotating or shearing), causes the peak of the spectrum to be suppressed.
(ii) The dipole anisotropy in the microwave background, due to the Doppler shift produced by the Earth's motion with respect to the fundamental cosmological frame of reference, causes the spectrum to be shifted slightly toward the red.
(iii) When cosmic background radiation photons traverse a hot cluster of galaxies, they are scattered by the electons in the hot gas within the cluster. The scattering on average increases the energy of the photons, with the result that the background looks cooler than average at long wavelengths and hotter than average at short wavelengths.
(iv) The light produced by the first generation of stars is absorbed by dust grains, leading to re-radiation in the infrared which is redshifted into the microwave band.

(e) Which one of the following processes in the early history of the universe was the last to occur?

(i) Electron-positron annihilation.
(ii) Muon-antimuon annihilation.
(iii) "Recombination" (or the "epoch of decoupling of radiation and matter"), i.e. the formation of neutral hydrogen and helium atoms for the first time.
(iv) Primordial nucleosynthesis.
(v) Decoupling of electron neutrinos.

PROBLEM 2: PROPERTIES OF BLACK-BODY RADIATION *(30 points)*

The following problem was Problem 4, Quiz 3, 1998, and this year was one of the Review Problems for Quiz 2.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since this time you were asked to bring calculators, you should evaluate each numerical answer to 3 significant figures.

(a) *(6 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

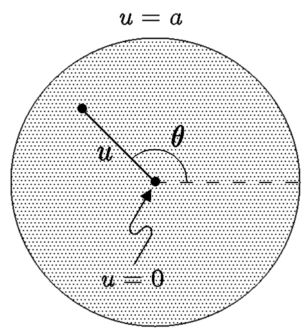
(b) *(6 points)* For the same radiation, what is the average entropy per photon?

(c) *(6 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(6 points)* Now consider the black-body radiation of electron neutrinos. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(6 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

PROBLEM 3: A TWO-DIMENSIONAL CURVED SPACE *(40 points)*

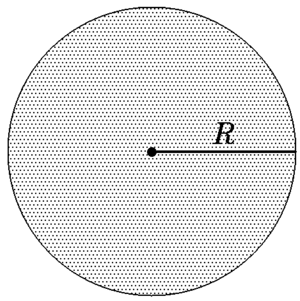


Consider a two-dimensional curved space described by polar coordinates u and θ, where $0 \le u \le a$ and $0 \le \theta \le 2\pi$, and $\theta = 2\pi$ is as usual identified with $\theta = 0$. The metric is given by

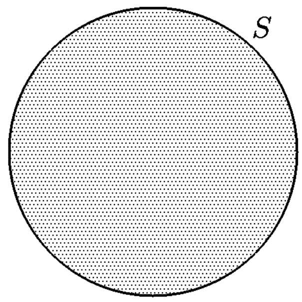
$$\mathrm{d}s^2 = \frac{a\,\mathrm{d}u^2}{4u(a-u)} + u\,\mathrm{d}\theta^2 \ .$$

A diagram of the space is shown at the right, but you should of course keep in mind that the diagram does not accurately reflect the distances defined by the metric.

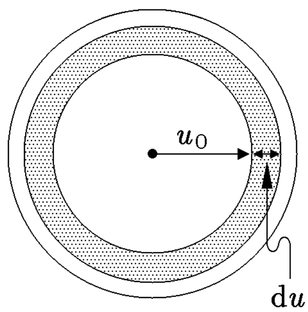
(a) *(6 points)* Find the radius R of the space, defined as the length of a radial (i.e., $\theta = constant$) line. You may express your answer as a definite integral, which you need not evaluate. Be sure, however, to specify the limits of integration.



(b) *(6 points)* Find the circumference S of the space, defined as the length of the boundary of the space at $u = a$.



(c) *(7 points)* Consider an annular region as shown, consisting of all points with a u-coordinate in the range $u_0 \le u \le u_0 + \mathrm{d}u$. Find the physical area $\mathrm{d}A$ of this region, to first order in $\mathrm{d}u$.



(d) *(3 points)* Using your answer to part (c), write an expression for the total area of the space.

— *Problem 3 Continues* —

(e) *(10 points)* Consider a geodesic curve in this space, described by the functions $u(s)$ and $\theta(s)$, where the parameter s is chosen to be the arc length along the curve. Find the geodesic equation for $u(s)$, which should have the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[F(u,\theta)\,\frac{\mathrm{d}u}{\mathrm{d}s}\right] = \ldots \ ,$$

where $F(u,\theta)$ is a function that you will find. (Note that by writing F as a function of u and θ, we are saying that it *could* depend on either or both of them, but we are not saying that it *necessarily* depends on them.) You need not simplify the left-hand side of the equation.

(f) *(8 points)* Similarly, find the geodesic equation for $\theta(s)$, which should have the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[G(u,\theta)\,\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = \ldots \ ,$$

where $G(u,\theta)$ is a function that you will find. Again, you need not simplify the left-hand side of the equation.