MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — Quiz Date: May 9, 2002
Prof. Alan Guth — Post Date: May 1, 2004

QUIZ 3

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USEFUL INFORMATION:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2\ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or: $$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\text{vac}} = -\rho_{\text{vac}} c^2 \qquad \rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$G = 6.673 \times 10^{-8}\ \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\,\text{erg/K}$$
$$= 8.617 \times 10^{-5}\,\text{eV/K}\ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27}\ \text{erg-sec}$$
$$= 6.582 \times 10^{-16}\ \text{eV-sec}\ ,$$

$$c = 2.998 \times 10^{10}\ \text{cm/sec}$$

$$1\ \text{yr} = 3.156 \times 10^7\ \text{s}$$

$$1\ \text{eV} = 1.602 \times 10^{-12}\ \text{erg}\ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G} \right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

PARTICLE PROPERTIES:

While working on this exam you may refer to any of the tables in Lecture Notes 11. Please bring your copy of Lecture Notes 11 with you to the exam.

PROBLEM 1: SHORT ANSWERS *(40 points)*

(a) *(6 points)* In chapter 6 of *The First Three Minutes*, Steven Weinberg posed the question, "Why was there no systematic search for this [cosmic background] radiation, years before 1965?" In discussing this issue, he contrasted it with the history of two different elementary particles, each of which were predicted approximately 20 years before they were first detected. Name one of these two elementary particles. (If you name them both correctly, you will get 3 points extra credit. However, one right and one wrong will get you 4 points for the question, compared to 6 points for just naming one particle and getting it right.)

Answer: ________________________
2nd Answer (optional): ________________________

(b) *(6 points)* In Chapter 6 of **The First Three Minutes**, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)

(i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.

(ii) There was a breakdown in communication between theorists and experimentalists.

(iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.

(iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

(v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.

(vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

— Problem 1 Continues —

Continuation of Problem 1:

(c) *(8 points)* In Chapter 6 of Rowan-Robinson's *Cosmology*, he discusses the observational evidence which indicates that galaxies and clusters of galaxies contain large amounts of dark matter — matter which is not seen, but which is detected indirectly through its gravitational effects. State whether each of the following statements is true or false. (2 points for each right answer, no penalty for guessing wrong, and remember that a statement that is partly true and partly false counts as false.)

(i) Measurements of the rotation curves (rotational velocity as a function of distance from the center) of spiral galaxies are found to be approximately flat, even when extended to large radii. If, however, the mass were concentrated where the light is seen, these rotation curves would fall off with distance from the center. **T** or **F** .

(ii) For elliptical galaxies, measurements of the Doppler spreading of the emission lines are used to determine the typical speeds of stars and gas, which can be related by the virial theorem to the gravitational potential energy. This Doppler spreading is much smaller than would be expected in the absence of dark matter, indicating that about 90% of the total mass of the galaxy is in the form of a halo of dark matter. **T** or **F** .

(iii) The masses of rich clusters can be estimated by observing the pattern of X-ray emission. **T** or **F** .

(iv) The masses of individual stars can be determined by their spectral characteristics, and these masses can be added to find the mass of the galaxy. **T** or **F** .

Continuation of Problem 1:

(d) *(6 points)* On the graph below, sketch the potential energy function $V(\phi)$ (at zero temperature) that is assumed in the new inflationary universe theory. Label the location of the true vacuum and false vacuum.

$V(\phi)$

ϕ

(e) *(6 points)* The word "supersymmetry" refers to a symmetry that relates the behavior of one certain class of particles with the behavior of another class. What are the names of these two classes *(2 points)*? What physical property distinguishes particles of one class from the particles of the other *(4 points)*?

1st Class: ____________ 2nd Class: ____________

Physical distinction: ____________

(f) *(8 points)* Grand unified theories unify three of the four known classes of particle interactions. For 2 points each, name these three, and also name the one that is left out.

Included: ____________

Included: ____________

Included: ____________

Excluded: ____________

PROBLEM 2: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS *(20 points)*

The following problem was on the Review Problems for Quiz 3, 2002.

(a) *(5 points)* When the temperature of the early universe was 5×10^{10} K, what was the ratio of neutrons to protons? You may assume thermal equilibrium, and that the mass difference is given by $(m_n - m_p)c^2 = 1.293$ MeV.

Questions (b), (c), and (d) all refer to calculations that describe a hypothetical world, which differs from the real world in a specified way. In each case you are asked about the calculation of the predicted helium abundance for this hypothetical world. Each of these three parts are to be answered independently; that is, in each part you are to consider a hypothetical world that differs from the real world only by the characteristics stated in that part.

(b) *(5 points)* Suppose the proton-neutron mass difference were larger than the actual value of 1.293 MeV/c^2. Would the predicted helium abundance be larger or smaller than in the standard calculation? Explain your answer in a sentence, or in a few sentences.

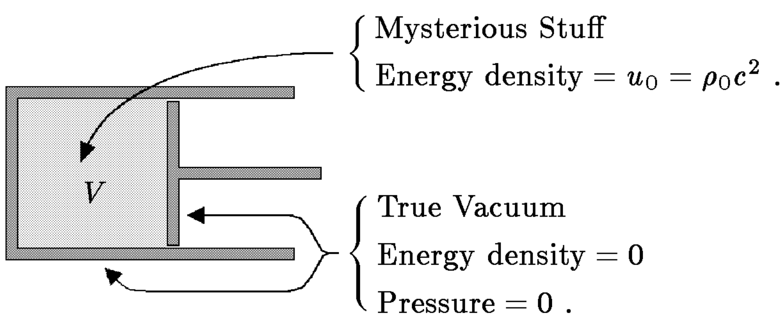
(c) *(5 points)* Suppose that the nucleosynthesis calculations were carried out with an electron mass given by $m_e c^2 = 1$ KeV, instead of the physical value of 0.511 MeV. This change would affect the production of helium in several ways. Describe one way in which the helium production process would be affected, and explain in a few sentences whether this change would increase or decrease the predicted helium abundance.

(d) *(5 points)* Suppose, due to some significant difference in the nuclear reaction rates, that nucleosynthesis occurred suddenly at a temperature of 5×10^{10} K. In that case, what would be the predicted value of Y, the fraction of the baryonic mass density of the universe which is helium?

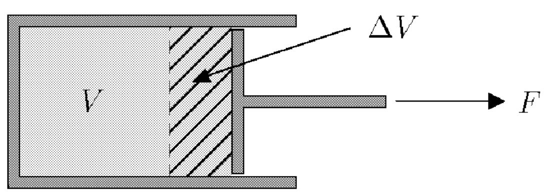
PROBLEM 3: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

In Lecture Notes 13, a thought experiment involving a piston was used to show that $p = -\rho c^2$ for any substance for which the energy density remains constant under expansion. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔV to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

PROBLEM 4: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

Consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and the same ***mysterious stuff*** that was introduced in the previous problem. Since the mass density of mysterious stuff falls off as $1/\sqrt{V}$, where V is the volume, it follows that in an expanding universe the mass density of mysterious stuff falls off as $1/R^{3/2}(t)$.

Suppose that you are given the present value of the Hubble parameter H_0, and also the present values the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\mathrm{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe t_0 in terms of these quantities.

(a) *(8 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{R(t)}{R(t_0)}$$

for an arbitrary time t. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(b) *(7 points)* Write an integral expression for the age of the universe t_0. The expression should depend only on H_0 and the various contributions to Ω_0 listed above ($\Omega_{m,0}$, $\Omega_{r,0}$, etc.), but it might include x as a variable of integration.

Extra Credit for Super-Sharpies (no partial credit): For 5 points extra credit, you can calculate the angular diameter $\Delta\theta$ of the image of a spherical object at redshift z which had a physical diameter w at the time of emission. You should assume that $\Omega_{\mathrm{tot}} < 1$, and also that $\Delta\theta \ll 1$. The calculation is to be carried out for the same model universe described above.