MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe April 29, 2004
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 3

Corrected Version: May 1, 2004

QUIZ DATE: Thursday, May 6, 2004

COVERAGE: Lecture Notes 7, 8, and 10; Problem Sets 4 and 5; Weinberg, Chapters 4 and 5; Ryden, Chapters 8 and 9, and the Epilogue. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems.** The starred problems are the ones that I recommend that you review most carefully: Problems 1, 2, 3, 4, 6, and 7.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, and 2000. The last quiz from 2002 will be posted shortly. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match the coverage of any of the quizzes from previous years.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" at the beginning. For the second quiz, this useful information will be the following:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

EVOLUTION OF A FLAT ($\boldsymbol{\Omega \equiv \rho/\rho_c = 1}$) UNIVERSE:

$R(t) \propto t^{2/3}$ (matter-dominated)

$R(t) \propto t^{1/2}$ (radiation-dominated)

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\, \rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{k^{3/2} c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{\kappa^{3/2} c^2}\ ,$$

$$\kappa \equiv -k\ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \; ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{ g_{ij} \frac{dx^j}{ds} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell}\right) \frac{dx^k}{ds} \, \frac{dx^\ell}{ds}$$

or: $$\frac{d}{d\tau}\left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma}\right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\rm vac} = -\rho_{\rm vac} c^2 \qquad \rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$G = 6.673 \times 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16} \, \text{erg/K}$$
$$= 8.617 \times 10^{-5} \, \text{eV/K} \; ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27} \text{ erg-sec}$$
$$= 6.582 \times 10^{-16} \text{ eV-sec} \; ,$$

$$c = 2.998 \times 10^{10} \text{ cm/sec}$$

$$1 \text{ yr} = 3.156 \times 10^7 \text{ s}$$

$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg} \; .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\;, \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\;, \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\;.$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3c^5}{16\pi^3 gG}\right)^{1/4}\,\frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

∗PROBLEM 1: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

Today the temperature of the cosmic microwave background radiation is 2.7°K. Calculate the number density of photons in this radiation. What is the number density of thermal neutrinos left over from the big bang?

∗ PROBLEM 2: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

∗ PROBLEM 3: A NEW SPECIES OF LEPTON

The following problem was Problem 2, Quiz 3, 1992, worth 25 points.

Suppose the calculations describing the early universe were modified by including an additional, hypothetical lepton, called an 8.286ion. The 8.286ion has roughly the same properties as an electron, except that its mass is given by $mc^2 = 0.750$ MeV.

Parts (a)-(c) of this question require numerical answers, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included. (For example, if you were asked how many meters a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$.)

a) *(5 points)* What would be the number density of 8.286ions, in particles per cubic meter, when the temperature T was given by $kT = 3$ MeV?

b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the value of the mass density at $t = .01$ sec? You may assume that 0.75 MeV $\ll$

$kT \ll 100$ MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, e^+-e^- pairs, and 8.286ion-anti8286ion pairs. Express your answer in the units of gm-cm^{-3}.

c) *(5 points)* Under the same assumptions as in (b), what would be the value of kT, in MeV, at $t = .01$ sec?

d) *(5 points)* When nucleosynthesis calculations are modified to include the effect of the 8.286ion, is the production of helium increased or decreased? Explain your answer in a few sentences.

e) *(5 points)* Suppose the neutrinos decouple while $kT \gg 0.75$ MeV. If the 8.286ions are included, what does one predict for the value of T_ν/T_γ today? (Here T_ν denotes the temperature of the neutrinos, and T_γ denotes the temperature of the cosmic background radiation photons.)

∗ PROBLEM 4: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(20 points)*

The following problem was Problem 3, Quiz 3, 1998.

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(10 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $R(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{R^5(t)} \ .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(5 points)* Find the behavior of the scale factor $R(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $R(t)$ up to a constant factor.

(c) *(5 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{6}\rho c^2 \ .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{R^n(t)} \ .$$

Find the power n.

PROBLEM 5: DID YOU DO THE READING?

The following problem was Problem 1 on Quiz 3, 2000, where it was worth 25 points. It is based on Chapters 4 and 5 of Rowan-Robinson's book, 3rd edition. For 2004 you should be able to answer parts (a), (c), and (e) on the basis of reading in Weinberg's book. You should also understand the physics, although possibly not the vocabulary, of parts (b) and (d).

(a) *(5 points)* What does Birkhoff's theorem state?

- (i) A uniform medium outside a spherical cavity has no gravitational effect inside the cavity.
- (ii) A redshifted version of a blackbody spectrum remains a blackbody spectrum, but at a lower temperature.
- (iii) The universe is homogeneous and isotropic.
- (iv) The effect of a uniform gravitational field is indistinguishable from the effect of a uniform acceleration.
- (v) A Hubble flow is the only global motion allowed in a completely homogeneous and isotropic universe.

(b) *(5 points)* Two special-case cosmological models are the "Milne model" and the "Einstein de-Sitter model". Pick one of them and briefly describe its distinguishing characteristics from other common models (be sure in your answer to specify which one you are describing).

(c) *(5 points)* The observation that only about one-quarter of the primordial gas in the universe is helium means that the Big Bang appears to have produced about seven times as many protons as neutrons. What can help explain this asymmetry?

- (i) The very energetic conditions of the early universe forced the GUT proton decay process to run in reverse.
- (ii) The early population of muons preferentially decayed into protons, boosting their density.
- (iii) The neutron is heavier than the proton, causing the weak reaction rates to shift as the temperature dropped.
- (iv) The rest of the neutrons formed into neutron stars, and thus aren't observed in the primordial gas at all.
- (v) The same asymmetry which gave us more particles than antiparticles also produced more down quarks than up quarks.

(d) *(5 points)* What causes the dipole anisotropy in the cosmic microwave background radiation?

(e) *(5 points)* Place the following events in order in the standard Big Bang picture, from earliest to latest. A valid answer would read, for instance: v, iv, iii, ii, i.

(i) Primordial nucleosynthesis

(ii) Decoupling of electron neutrinos

(iii) Quark confinement

(iv) Recombination

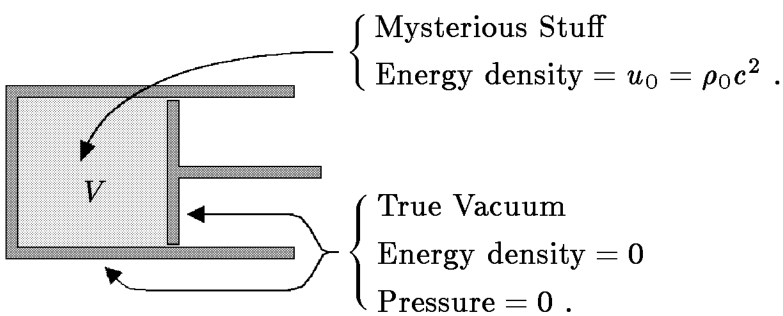
(v) Muon annihiliation

∗ PROBLEM 6: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

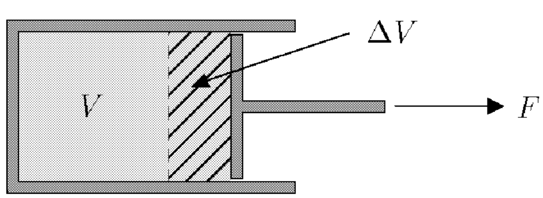
The following problem was Problem 3, Quiz 3, 2002. Although it is couched in the language of Lecture Notes 13, the physics is really the same as the pressure calculations in Lecture Notes 7, so a modified form of this problem would be fair for the coming quiz.

In Lecture Notes 13, a thought experiment involving a piston was used to show that $p = -\rho c^2$ for any substance for which the energy density remains constant under expansion. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

∗ PROBLEM 7: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

Consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and the same ***mysterious stuff*** that was introduced in the previous problem. Since the mass density of mysterious stuff falls off as $1/\sqrt{V}$, where V is the volume, it follows that in an expanding universe the mass density of mysterious stuff falls off as $1/R^{3/2}(t)$.

Suppose that you are given the present value of the Hubble parameter H_0, and also the present values of the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\text{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe t_0 in terms of these quantities.

(a) *(8 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{R(t)}{R(t_0)}$$

for an arbitrary time t. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(b) *(7 points)* Write an integral expression for the age of the universe t_0. The expression should depend only on H_0 and the various contributions to Ω_0 listed above ($\Omega_{m,0}$, $\Omega_{r,0}$, etc.), but it might include x as a variable of integration.

Extra Credit for Super-Sharpies (no partial credit): For 5 points extra credit, you can calculate the angular diameter $\Delta\theta$ of the image of a spherical object at redshift z which had a physical diameter w at the time of emission. You should assume that $\Omega_{\text{tot}} < 1$, and also that $\Delta\theta \ll 1$. The calculation is to be carried out for the same model universe described above.

PROBLEM 8: TIME SCALES IN COSMOLOGY

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-37} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the beginning of the processes involved in big bang nucleosynthesis;

(b) the end of the processes involved in big bang nucleosynthesis;

(c) the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{16}$ GeV;

(d) "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

PROBLEM 9: DID YOU DO THE READING? *(30 points)*

The following problem was Problem 1, Quiz 4, 2000. For 2004, you should be able to answer parts (a), (d), and (e).

(a) *(5 points)* When orbital velocities of stars in spiral galaxies are measured, we find that they are mostly constant over a large range in radius. What explanation is usually given to understand these flat rotation curves?

(i) The density waves producing the spiral arms perturb the stellar orbits.

(ii) A flat rotation curve is exactly what you'd expect from Kepler's laws applied to the observed mass profile of spiral galaxies.

(iii) The measurements are dominated by bright young stars in the spiral arms, so we're mistaking the wave velocity of the arms for the rotation of the galaxy as a whole.

(iv) Spiral galaxies contain a halo of dark matter in addition to their normal disk mass.

(v) The stellar orbits aren't circular, so we're measuring stars with more and more elliptical orbits at larger radii.

(b) *(5 points)* Briefly describe the distinguishing characteristics of the Eddington-Lemaître cosmological models. (Hint: they are related to Einstein's static closed universe model.)

(c) *(5 points)* What is the Jeans length?

(i) The size at which the sound-crossing time is equal to the age of the universe

(ii) The minimum size of density fluctuations which are unstable to gravitational collapse

(iii) The size of the first peak in the power spectrum of the cosmic microwave background fluctuations

(iv) The size where we expect the effects of quantum gravity to have a significant influence

(v) Approximately equal to the Jeans waist size

(d) *(5 points)* By what factor does the lepton number per comoving volume of the universe change between temperatures of $kT = 10$ MeV and $kT = 0.1$ MeV? You should assume the existence of the normal three species of neutrinos for your answer. *[Note: this question is based on Chapter 4 of Weinberg, and so it would not be appropriate for Quiz 3 of 2002.]*

(e) *(5 points)* Measurements of the primordial deuterium abundance would give good constraints on the baryon density of the universe. However, this abundance is hard to measure accurately. Which of the following is NOT a reason why this is hard to do? *[Note: this question is based on Chapter 4 of Weinberg, and so it would not be appropriate for Quiz 3 of 2002.]*

(i) The neutron in a deuterium nucleus decays on the time scale of 15 minutes, so almost none of the primordial deuterium produced in the Big Bang is still present.

(ii) The deuterium abundance in the Earth's oceans is biased because, being heavier, less deuterium than hydrogen would have escaped from the Earth's surface.

(iii) The deuterium abundance in the Sun is biased because nuclear reactions tend to destroy it by converting it into helium-3.

(iv) The spectral lines of deuterium are almost identical with those of hydrogen, so deuterium signatures tend to get washed out in spectra of primordial gas clouds.

(v) The deuterium abundance is so small (a few parts per million) that it can be easily changed by astrophysical processes other than primordial nucleosynthesis.

(f) *(5 points)* Give three examples of hadrons.

PROBLEM 10: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS *(20 points)*

The following problem was on Quiz 4, 2000, except that part (c) has been modified. For 2004 a problem about nucleosynthesis, like this one, would be considered a difficult problem, since the topic was covered only in your readings of Weinberg. If I were to use a question like this on the coming quiz, I would probably try to make it easier by adding some hints.

(a) *(5 points)* When the temperature of the early universe was 5×10^{10} K, what was the ratio of neutrons to protons? You may assume thermal equilibrium, and that the mass difference is given by $(m_n - m_p)c^2 = 1.293$ MeV.

Questions (b), (c), and (d) all refer to calculations that describe a hypothetical world, which differs from the real world in a specified way. In each case you are asked about the calculation of the predicted helium abundance for this hypothetical world. Each of these three parts are to be answered independently; that is, in each part you are to consider a hypothetical world that differs from the real world only by the characteristics stated in that part.

(b) *(5 points)* Suppose the proton-neutron mass difference were larger than the actual value of 1.293 MeV/c^2. Would the predicted helium abundance be larger or smaller than in the standard calculation? Explain your answer in a sentence, or in a few sentences.

(c) *(5 points)* Suppose that the nucleosynthesis calculations were carried out with an electron mass given by $m_e c^2 = 1$ KeV, instead of the physical value of 0.511 MeV. This change would affect the production of helium in several ways. Describe one way in which the helium production process would be affected, and explain in a few sentences whether this change would increase or decrease the predicted helium abundance.

(d) *(5 points)* Suppose, due to some significant difference in the nuclear reaction rates, that nucleosynthesis occurred suddenly at a temperature of 5×10^{10} K. In that case, what would be the predicted value of Y, the fraction of the baryonic mass density of the universe which is helium?

SOLUTIONS

PROBLEM 1: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

In general, the number density of a particle in the black-body radiation is given by

$$n = g^* \frac{\xi(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3$$

For photons, one has $g^* = 2$. Then

$$\left.\begin{aligned} k &= 1.381 \times 10^{-16}\text{erg}/^\circ\text{K} \\ T &= 2.7\,^\circ\text{K} \\ \hbar &= 1.055 \times 10^{-27}\text{erg-sec} \\ c &= 2.998 \times 10^{10}\text{cm/sec} \end{aligned}\right\} \quad \Longrightarrow \quad \left(\frac{kT}{\hbar c}\right)^3 = 1.638 \times 10^3\text{cm}^{-3} \ .$$

Then using $\xi(3) \simeq 1.202$, one finds

$$\boxed{n_\gamma = 399/\text{cm}^3 \ .}$$

For the neutrinos,

$$g^*_\nu = 2 \times \frac{3}{4} = \frac{3}{2} \quad \text{per species.}$$

The factor of 2 is to account for ν and $\bar{\nu}$, and the factor of 3/4 arises from the Pauli exclusion principle. So for three species of neutrinos one has

$$g^*_\nu = \frac{9}{2} \ .$$

Using the result

$$T^3_\nu = \frac{4}{11} T^3_\gamma$$

from Problem 8 of Problem Set 3 (2000), one finds

$$\begin{aligned} n_\nu &= \left(\frac{g^*_\nu}{g^*_\gamma}\right)\left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma \\ &= \left(\frac{9}{4}\right)\left(\frac{4}{11}\right) 399\text{cm}^{-3} \\ \Longrightarrow &\quad \boxed{n_\nu = 326/\text{cm}^3 \text{ (for all three species combined).}} \end{aligned}$$

PROBLEM 2: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)}\,kT\ .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\,kT\ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{2\pi^4}{45\zeta(3)}\,k\ .}$$

Numerically, this gives $3.602\,k$, where k is the Boltzman constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is 7/8 times the number of spin states, and g^* is 3/4 times the number of spin states. So the average energy per particle is

$$E = \frac{g \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8} \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{180\zeta(3)} kT\ .}$$

Numerically, $E = 3.1514\, kT$.

Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$S = \frac{g \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8} \dfrac{2\pi^2}{45} \dfrac{k^4 T^3}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{135\zeta(3)} k \ .}$$

Numerically, this gives $S = 4.202\, k$.

PROBLEM 3: A NEW SPECIES OF LEPTON

a) The number density is given by the formula at the start of the exam,

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \ .$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$\begin{aligned} n = 3\frac{\zeta(3)}{\pi^2} \times & \left(\frac{3 \ \cancel{\text{MeV}}}{6.582 \times 10^{-16} \ \cancel{\text{eV}}\text{-}\cancel{\text{sec}} \times 2.998 \times 10^{10} \ \cancel{\text{cm}}\text{-}\cancel{\text{sec}}^{-1}} \right)^3 \\ & \times \left(\frac{10^6 \ \cancel{\text{eV}}}{1 \ \cancel{\text{MeV}}} \right)^3 \times \left(\frac{10^2 \ \cancel{\text{cm}}}{1 \text{ m}} \right)^3 \qquad z \\ = 3\frac{\zeta(3)}{\pi^2} \times & \left(\frac{3 \times 10^6 \times 10^2}{6.582 \times 10^{-16} \times 2.998 \times 10^{10}} \right)^3 \text{m}^{-3} \ . \end{aligned}$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3 \times 10^6 \times 10^2}{6.582 \times 10^{-16} \times 2.998 \times 10^{10}} \right)^3 \ .}$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho \quad .$$

During the radiation-dominated era $R(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot{R}}{R} = \frac{1}{2t} \quad .$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3} G\rho \quad .$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \quad .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}$ cm^3 sec^{-2} g^{-1}, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732 \times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9$ g/cm^3.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \quad .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion – anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\rm tot} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2}\frac{1}{g_{\rm tot}}\hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/R$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\rm rest} T^3(t) R^3(t)$$

where $g_{\text{rest}} = g_{\text{tot}} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 R^3(t) = g_{\text{rest}} T^3(t) R^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\text{rest}}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 4: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION

(a) This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by $dU = -p\,dV$. Using the fact that the energy density u is equal to ρc^2, the energy conservation relation can be written

$$\frac{dU}{dt} = -p\frac{dV}{dt} \quad \Longrightarrow \quad \frac{d}{dt}\left(\rho c^2 R^3\right) = -p\frac{d}{dt}\left(R^3\right) \quad . \tag{1}$$

Setting

$$\rho = \frac{\alpha}{R^5} \tag{2}$$

for some constant α, the conservation of energy formula becomes

$$\frac{d}{dt}\left(\frac{\alpha c^2}{R^2}\right) = -p\frac{d}{dt}\left(R^3\right) \quad ,$$

which implies

$$-2\frac{\alpha c^2}{R^3}\frac{dR}{dt} = -3pR^2\frac{dR}{dt} \quad .$$

Thus

$$p = \frac{2}{3}\frac{\alpha c^2}{R^5} = \boxed{\frac{2}{3}\rho c^2 \ .}$$

For those students who could not reconstruct Eq. (1) or some equivalent equation from memory, the conservation of energy equation could be derived from the formulas for cosmological evolution on the front of the exam:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \tag{3}$$

$$\ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R \; . \tag{4}$$

By rewriting Eq. (3) as

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 - kc^2 \; ,$$

the time derivative becomes

$$2\dot{R}\,\ddot{R} = \frac{8\pi}{3}G\dot{\rho}R^2 + \frac{16\pi}{3}G\rho R\dot{R} \; .$$

This equation can be solved for $\dot{\rho}$ to give

$$\dot{\rho} = \frac{3}{4\pi G}\frac{\dot{R}\,\ddot{R}}{R^2} - 2\frac{\dot{R}}{R}\rho \; .$$

Using Eq. (4) to replace $\ddot{R}$, one finds

$$\dot{\rho} = -\frac{\dot{R}}{R}\left(\rho + \frac{3p}{c^2}\right) - 2\frac{\dot{R}}{R}\rho = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \; . \tag{5}$$

It is easy to show that Eq. (5) is equivalent to Eq. (1), but it is not necessary to do so. The question can be answered directly from Eq. (5), by substituting Eq. (2) and manipulating.

(b) For a flat universe, Eq. (3) reduces to

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho \; .$$

Using Eq. (2), this implies that

$$\dot{R} = \frac{\beta}{R^{3/2}} \; ,$$

for some constant β. Rewriting this as

$$R^{3/2}\,dR = \beta\,dt \; ,$$

we can integrate the equation to give

$$\frac{2}{5} R^{5/2} = \beta t + \text{const} \ ,$$

where the constant of integration has no effect other than to shift the origin of the time variable t. Using the standard big bang convention that $R = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$\boxed{R \propto t^{2/5} \ .} \tag{6}$$

The arbitrary constant of proportionality in Eq. (6) is consistent with the wording of the problem, which states that "You should be able to determine the function $R(t)$ up to a constant factor." Note that we could have expressed the constant of proportionality in terms of the constant α in Eq. (2), but there would not really be any point in doing that. The constant α was not a given variable. If the comoving coordinates are measured in "notches," then R is measured in meters per notch, and the constant of proportionality in Eq. (6) can be changed by changing the arbitrary definition of the notch.

(c) Combining Eq. (1) with $p = \frac{1}{6}\rho c^2$, one has

$$\frac{d}{dt}\left(\rho c^2 R^3\right) = -\frac{1}{6}\rho c^2 \frac{d}{dt}\left(R^3\right) \ ,$$

or equivalently

$$\frac{d}{dt}\left(\rho R^3\right) + \frac{1}{6}\rho \frac{d}{dt}\left(R^3\right) = 0 \ . \tag{7}$$

There are various ways to proceed from here. Since the problem told us that

$$\rho = \frac{\text{const}}{R^n} \ ,$$

the most straightforward approach would be to use this expression to replace ρ in Eq. (7), and then solve the equation for n. A cleverer approach would be to multiply Eq. (7) by $R^{1/2}$, and then rewrite it as

$$\frac{d}{dt}\left(\rho R^{7/2}\right) = 0 \ ,$$

from which one can see immediately that

$$\boxed{\rho(t) \propto \frac{1}{R^{7/2}(t)} \ ,}$$

and therefore

$$\boxed{n = 7/2 \ .}$$

PROBLEM 5: DID YOU DO THE READING?

The solutions to this problem were written by Edward Keyes.

(a) Birkhoff's theorem

Birkhoff's theorem states that "the gravitational effect of a uniform medium external to a spherical cavity is zero." This is a theorem from general relativity, and necessary to know in order to extrapolate our Newtonian cosmology results to the whole universe: it might have been the case that the global curvature of space would have interfered with our Newtonian results. The other choices in the question were generally true statements from other areas of cosmology.

(b) Special-case cosmological models

The Einstein-de Sitter model is not, as some answered, Einstein's original, static universe with a cosmological constant. Instead, this model describes a flat ($k = 0$) universe with a critical density of ordinary matter ($\rho = \rho_c$). As we showed earlier in the class, this means that its scale factor grows as $R(t) \propto t^{2/3}$.

The Milne model describes an empty universe: it is open ($k = -1$) and has no matter or radiation in it ($\rho = 0$). Its scale factor grows linearly with time, since there's no matter to slow down the Hubble expansion. (One normally includes "test" particles in the description of the Milne universe, so that we can talk about their motion. But the mass of these test particles is taken to be arbitrarily small, so we completely ignore any gravitational field that they might produce.)

As an interesting aside, we might ask why the Milne model has $k = -1$. Since there is no matter, there shouldn't be any general relativity effects, and so we would ordinarily expect that the metric should be the normal, flat, Minkowski special relativity metric. Why is this space hyperbolic instead?

The answer is an illustration of the subtleties that can arise in changing coordinate systems. In fact, the metric of the Milne universe can be viewed as ***either*** a flat, Minkowski metric, ***or*** as the negatively curved metric of an open universe, depending on what coordinate system one uses. If one uses coordinates for time and space as they would be measured by a single inertial observer, then one finds a Minkowski metric; in this way of describing the model, it is clear that special relativity is sufficient, and general relativity plays no role. In this coordinate system all the test particles start at the origin at time $t = 0$, and they move outward from the origin at speeds ranging from zero, up to (but not including) the speed of light.

On the other hand, we can describe the same universe in a way that treats all the test particles on an equal footing. In this description we define time not as it would be measured by a single observer, but instead we define the time at each location as the time that would be measured by observers riding with the test particles at that location. This definition is what we have been calling "cosmic time" in our description of cosmology. One can also introduce a comoving spatial coordinate system that expands with the motion of the particles. With a particular definition of these spatial coordinates, one can show that the metric is precisely that of an open Robertson-Walker universe with $R(t) = t$.

The derivation is left as an exercise for the curious student. You should find that the normal special-relativistic time dilation and Lorentz contraction formulas, when applied to the velocities of a Hubble expansion to construct the comoving coordinate system, introduce the negative curvature to the metric.

(c) Neutron-proton ratio

In the early universe, neutrons and protons first formed when the temperature dropped far enough to keep them from being torn apart into their constituent quarks. This happened around a microsecond after the Big Bang, and at this time there were roughly equal numbers of neutrons and protons.

In fact, neutrons could be converted into protons and vice-versa in several weak reactions with electrons, positrons, and neutrinos:

$$n + \nu_e \longleftrightarrow p + e^- \quad , \quad n + e^+ \longleftrightarrow p + \bar{\nu}_e \quad , \quad n \longleftrightarrow p + e^- + \bar{\nu}_e$$

Since these are weak-force reactions, though, their rates are strongly dependent on the temperature. Once T drops below 10^{10} K, the neutrinos stop interacting with matter, and these reactions freeze, except for the forward direction of the third reaction, which describes free neutron decay (this process has a half-life of 15 minutes, so it doesn't affect things very much).

Before the freeze-out, which essentially fixes the neutron/proton ratio, the reaction rates shift as the temperature changes. The neutron is 1.3 MeV heavier than the proton, while the mass/energy of an electron is only 0.5 MeV. This means that the conversion of a neutron to a proton and electron is energetically favorable, while the reverse process costs energy. As the temperature drops so that kT is of the order of 1 MeV, these energy differences become significant compared to the available free thermal energy, and the reaction rates shift so that thermal equilibrium favors protons over neutron by an increasing margin.

When the weak reactions freeze out, this unequal ratio of neutrons and protons is preserved. Since essentially all of the neutrons end up in helium after nucleosynthesis, this also fixes the ratio of hydrogen to helium formed by the Big Bang.

(d) The dipole anisotropy

When we look at the temperature of the cosmic microwave background radiation, to first order it appears uniform across the sky. When we look closer, though, we see that it is hotter in one direction and smoothly shades into cooler in the opposite direction, at a level of about one part in 1000. This is the dipole anisotropy.

The explanation is quite simple: the Earth is not at rest with respect to the cosmic background radiation. The motion of our Sun around the center of the Galaxy, and the motion of our Galaxy towards the Virgo Cluster, etc., all give us a net velocity of around 600 km/sec, which causes us to see blueshifted CMB photons in one direction, and redshifted ones in the opposite direction. As we learned earlier in the class, a redshifted blackbody spectrum just shifts its temperature, so we see the effects of this motion as a smooth temperature variation across the sky.

(e) Events in the early universe

The correct order is:

(iii) Quark confinement, at $t \sim 10^{-6}$ sec.

(v) Muon annihilation, at $t \sim 10^{-4}$ sec.

(ii) Decoupling of electron neutrinos, at $t \sim 1$ sec.

(i) Primordial nucleosynthesis, at $t \sim 10^{2}$ sec.

(iv) Recombination, at $t \sim 10^{5}$ years.

A surprising number of students did not realize that recombination is the final stage of the early universe. After this event takes place, the universe is transparent to photons and the temperature has dropped to just a few thousand K. Nothing interesting happens after this until the processes of structure formation begins.

PROBLEM 6: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} \; .$$

(The above expression is proportional to $1/\sqrt{V + \Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1 + \frac{\Delta V}{V}}} = \frac{u_0}{1 + \frac{1}{2}\frac{\Delta V}{V}} = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) \; .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2} u_0 \ .}$$

PROBLEM 7: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF

(a) The critical density ρ_c is defined as that density for which $k = 0$, where the Friedmann equation from the front of the exam implies that

$$H^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \ .$$

Thus the critical density today is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} \ .$$

The mass density today of any species X is then related to $\Omega_{X,0}$ by

$$\rho_{X,0} = \rho_c \Omega_{X,0} = \frac{3H_0^2 \Omega_{X,0}}{8\pi G} \ .$$

The total mass density today is then expressed in terms of its four components as

$$\rho_0 = \frac{3H_0^2}{8\pi G}\left[\Omega_{m,0} + \Omega_{r,0} + \Omega_{v,0} + \Omega_{\mathrm{ms},0}\right] \ .$$

But we also know how each of these contributions to the mass density scales with $x(t)$: $\rho_m \propto 1/x^3$, $\rho_r \propto 1/x^4$, $\rho_v \propto 1$, and $\rho_{\mathrm{ms}} \propto 1/\sqrt{V} \propto 1/x^{3/2}$. Inserting these factors,

$$\rho(t) = \frac{3H_0^2}{8\pi G}\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}}\right] \ .$$

(b) The Friedmann equation then becomes

$$\left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G}{3}\frac{3H_0^2}{8\pi G}\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}}\right] - \frac{kc^2}{R^2} \ .$$

Defining

$$H_0^2\Omega_{k,0} = -\frac{kc^2}{R^2(t_0)} \ ,$$

so

$$-\frac{kc^2}{R^2(t)} = -\frac{kc^2}{R^2(t_0)}\frac{1}{x^2} = \frac{H_0^2\Omega_{k,0}}{x^2} \ ,$$

and then the Friedmann equation becomes

$$\left(\frac{\dot{x}}{x}\right)^2 = H_0^2\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}\right] \ .$$

Applying this equation today, when $\dot{x}/x = H_0$, one finds that

$$\boxed{\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{v,0} - \Omega_{\mathrm{ms},0} \ .}$$

Rearranging the equation for $(\dot{x}/x)^2$ above,

$$H_0\,\mathrm{d}t = \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \ .$$

The age of the universe is found by integrating over the full range of x, which starts from 0 when the universe is born, and is equal to 1 today. So

$$\boxed{t_0 = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \ .}$$

Extra Credit for Super-Sharpies (no partial credit):

Since $\Omega_{\rm tot} < 1$, we use the Robertson-Walker open universe form

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have started with the general form from the front of the exam, and replaced k by -1. To discuss the radial transmission of light rays it is useful to define a new radial coordinate

$$r = \sinh\psi ,$$

so

$$dr = \cosh\psi\,d\psi = \sqrt{1+r^2}\,d\psi ,$$

where I used the identity that $\cosh^2\psi = 1 + \sinh^2\psi$. The metric can then be rewritten as

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{d\psi^2 + \sinh^2\psi\,\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Light rays then travel with $d\tau^2 = 0$, so

$$\frac{d\psi}{dt} = \frac{c}{R(t)} .$$

If a light ray leaves the object at time t_e and arrives at Earth today, then it will travel an interval of ψ given by

$$\psi = \int_{t_e}^{t_0} \frac{c}{R(t')}\,dt' .$$

We will need to know ψ, but we don't know either t_e or $R(t)$. So we need to manipulate the right-hand side of the above equation to express it in terms of things that we do know. Changing integration variables from t' to x, where $x = R(t')/R(t_0)$, one finds $dx = \dot{x}\,dt'$, and then

$$\psi = \int_{x_e}^{1} \frac{c}{R(t_0)}\frac{1}{x}\frac{dx}{\dot{x}} .$$

Using $H = \dot{x}/x$,

$$\psi = \frac{c}{R(t_0)}\int_{x_e}^{1} \frac{dx}{x^2 H} .$$

Now use the formula for $H = \dot{x}/x$ from part (b), so

$$\psi = \frac{c}{R(t_0)H_0} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \ .$$

Here

$$\boxed{x_e = \frac{R(t_e)}{R(t_0)} = \frac{1}{1+z} \ ,}$$

and the coefficient in front of the integral can be evaluated using the Friedman equation for $k = -1$:

$$H_0^2 = \frac{8\pi}{3} G\rho_0 + \frac{c^2}{R^2(t_0)} = H_0^2\Omega_0 + \frac{c^2}{R^2(t_0)} \ ,$$

so

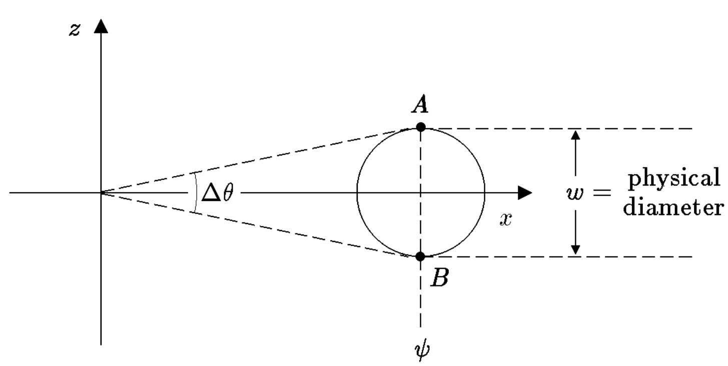
$$\frac{c^2}{R^2(t_0)\,H_0^2} = 1 - \Omega_0 = \Omega_{k,0} \ .$$

Finally, then, the expression for ψ can be written

$$\boxed{\psi = \sqrt{\Omega_{k,0}} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \ ,}$$

where x_e is given by the boxed equation above.

Once we know ψ, the rest is straightforward. We draw a picture in comoving coordinates of the light rays leaving the object and arriving at Earth:



In this picture $\Delta\theta$ is the angular size that would be measured. Using the $\mathrm{d}\theta^2$ part of the metric,

$$\mathrm{d}s^2 = R^2(t)\sinh^2\psi\,\mathrm{d}\theta^2 \ ,$$

we can relate w, the physical size of the object at the time of emission, to $\Delta\theta$:

$$w = R(t_e)\sinh\psi\,\Delta\theta \ .$$

To evaluate $R(t_e)$ we can use

$$R(t_e) = x_e R(t_0) = \frac{x_e c}{H_0\sqrt{\Omega_{k,0}}} \ .$$

Finally, then,

$$\boxed{\Delta\theta = \frac{w H_0\sqrt{\Omega_{k,0}}}{x_e c\,\sinh\psi} \ ,}$$

where ψ is given by the boxed equation above.

PROBLEM 8: TIME SCALES IN COSMOLOGY

(a) 1 sec. *[This is the time at which the weak interactions begin to "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) 10^{-37} sec. *[We learned in Lecture Notes 7 that kT was about 1 MeV at $t = 1$ sec. Since 1 GeV = 1000 MeV, the value of kT that we want is 10^{19} times higher. In the radiation-dominated era $T \propto R^{-1} \propto t^{-1/2}$, so we get 10^{-38} sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) 10^{-5} sec. *[As in (c), we can use $t \propto T^{-2}$, with $kT \approx 1$ MeV at $t = 1$ sec.]*

PROBLEM 9: DID YOU DO THE READING? *(30 points)*

(a) The correct answer, of course, is (iv). The other items are supposed to be plausible-sounding but flawed explanations. See Rowan-Robinson section 6.4, pages 102-3.

(b) The key fact is that they have a critical value for the cosmological constant, leading to an asymptotic period of near-static evolution. Technically one asymptotes to the Einstein static model at infinite negative time and then expands (this has no Big Bang), and the other asymptotes to the Einstein case at infinite positive time after a Big Bang and initial period of expansion. See Rowan-Robinson section 8.2, pages 133-4.

(c) The correct answer is, of course, (ii). Answer (i) is intended to be evil and tricky, but the others are merely wrong, and (v) is just a joke to compensate for confusing the students with (i). See Rowan-Robinson section 6.1, pages 97-9.

(d) This is a total trick question. Lepton number is, of course, conserved, so the factor is just 1. See Weinberg chapter 4, pages 91-4.

(e) The correct answer is (i). The others are all real reasons why it's hard to measure, although Weinberg's book emphasizes reason (v) a bit more than modern astrophysicists do: astrophysicists have been looking for other ways that deuterium might be produced, but no significant mechanism has been found. See Weinberg chapter 5, pages 114-7.

(f) The most obvious answers would be proton, neutron, and pi meson. However, any of the particles listed as baryons or mesons in Lecture Notes 11 would be correct. See Weinberg chapter 7, pages 136-8.

PROBLEM 10: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS

(a) In thermal equilibrium, the ratio of neutrons to protons is given by a Boltzmann factor,

$$\boxed{\frac{n_n}{n_p} = e^{-\Delta m\, c^2/kT} \ ,}$$

where $\Delta m = (m_n - m_p)$. For $\Delta m\, c^2 = 1.293 \times 10^6$ eV, $k = 8.617 \times 10^{-5}$ eV/K, and $T = 5 \times 10^{10}$ K, this gives

$$\frac{n_n}{n_p} = \exp\left\{-1.293 \times 10^6/(8.617 \times 10^{-5} \times 5 \times 10^{10})\right\} = \boxed{0.741 \ .}$$

Caveat (for stat mech experts): The above formula would be a precise consequence of statistical mechanics if the neutron and proton were two possible energy levels of the same system. In this case one would describe the system using the *canonical ensemble*, which implies that the probability of the system existing in any specific state i is proportional to $\exp(-E_i/kT)$, where E_i is the energy of the state. However, the neutron and proton are not really different energy levels of the same system, because the conversion between neutrons and protons involves other particles as well; a sample conversion reaction would be

$$n + \nu_e \longleftrightarrow p + e^- \ ,$$

where ν_e is the electron neutrino, and e^- is the electron. This means that if the universe contained a very large density of electron neutrinos, then n-ν_e collisions would occur more frequently, and the reaction would be driven in the forward direction. Thus, a large density of electron neutrinos would lead to a lower ratio of neutrons to protons than the Boltzmann factor given above. Similarly, if the universe contained a large density of electrons, then the reaction would be driven in the reverse direction, and the ratio of neutrons to protons would be higher than the Boltzmann factor. A complete statistical mechanical treatment of this situation would use the *grand canonical ensemble*, which describes systems in which the number of particles of a given type can change by chemical reactions. In this formalism the density of each type of particle is related to a quantity called the *chemical potential* μ, where in general the relationship is given by

$$n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp\left[(E-\mu)/(kT)\right] \pm 1} E \, \mathrm{d}E$$

where the $+$ sign holds for Fermi particles, the $-$ sign holds for Bose particles, and the factor g has the same meaning as in Lecture Notes 7. The ratio of neutrons to protons is then given by

$$\frac{n_n}{n_p} = e^{-(\Delta m \, c^2 + \mu_\nu - \mu_e)/kT} \ ,$$

where μ_ν and μ_e represent the chemical potentials for electron neutrinos and electrons, respectively. In the early universe, however, the standard theories imply that the chemical potentials for electrons and neutrinos were both negligible.

(b) A larger Δm would mean that the Boltzmann factor described in the previous answer would be smaller, so that there would be fewer neutrons at any given temperature. Fewer neutrons implies less helium, since essentially all the neutrons that exist when the temperature falls enough for deuterium to become stable become bound into helium.

(c) There are at least four effects that occur when the electron mass/energy is taken as 1 KeV instead of 0.511 MeV:

(i) For the real mass/energy of 0.511 MeV the electron-positron pairs freeze out before nucleosynthesis, but a mass/energy of 1 KeV would mean that electron-positron pairs would behave as massless particles throughout the nucleosynthesis process. Just like adding an extra species of neutrino, this additional massless particle would mean that the expansion rate would be larger, since for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho \ ,$$

and

$$\rho = \frac{u}{c^2} = g \frac{\pi^2}{30} \frac{(kT)^4}{\hbar^3 c^5} \ .$$

Faster expansion means that the weak interactions "freeze out" earlier, since the freeze-out point is the time at which the interactions can no longer maintain equilibrium as the universe expands. An earlier freeze-out means a higher temperature of freeze-out and hence more neutrons at the time of freeze-out. In addition, the faster expansion rate means faster cooling, which means less time before the temperature of nucleosynthesis is reached, and therefore less time for neutrons to decay. Thus, faster expansion means more neutrons. Since essentially all the neutrons present when the deuterium bottleneck breaks are collected into helium, this implies more helium.

(ii) The most important reactions that keep protons and neutrons in thermal equilibrium all involve electrons and positrons:

$$n + e^+ \longleftrightarrow p + \bar{\nu}_e$$
$$n + \nu_e \longleftrightarrow p + e^- \ .$$

If the electron-positron mass/energy were smaller, then the rates of all of these reactions would be enhanced. The reactions in which an e^+ or e^- appears in the initial state will be enhanced by the presence of more e^+'s and e^-'s, and the reactions in which they appear in the final state will be enhanced because a lighter final state is easier to produce. The enhanced rate for these reactions will keep neutrons and protons in thermal equilibrium longer, and hence to lower temperatures, and this would decrease the final abundance of neutrons. Thus this effect will go in the opposite direction as effect (i), leading to the production of less helium.

(iii) If the electron mass is decreased, then the neutron decay

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

becomes more exothermic, so it will happen more quickly. Thus more neutrons can decay, leading to less helium.

(iv) As mentioned in (i), lowering the mass/energy of electron-positron pairs to 1 KeV would mean that their freeze-out would not occur until after nucleosynthesis is over. In the real case, however, with $m_e c^2 = 0.511$ MeV, the electron-positron pairs start to freeze out at $t \approx 10$ sec. The energy released by this freeze-out heats the photons, protons, and neutrons, and this extra heat delays the time when the universe cools enough to break the deuterium bottleneck so that helium production can proceed. The delay allows more time for the neutrons to decay, resulting in less helium. Since the freeze-out that occurs for $m_e c^2 = 0.511$ MeV results in less helium, the absence of this freeze-out if $m_e c^2 = 1$ KeV would result in more helium.

Since the effects point in different directions, there is no easy way to know what the net effect will be. I (AHG) tried carrying out a full numerical integration, using the equations from P.J.E. Peebles, "Primordial helium abundance and the primordial fireball II," *Astrophysical Journal* **146**, 542-552 (1966). I found that the net effect of changing $m_e c^2$ to 1 KeV was to produce less helium. Apparently effects (ii) and (iii) above are the most significant. Of course I did not expect students to figure this out in doing their problem sets.

(d) Part (a) asked for the ratio of neutrons to protons, so its answer is

$$A = \frac{n_{\rm neutron}}{n_{\rm proton}} \ .$$

The fraction of the baryonic mass in neutrons is then

$$\frac{n_{\rm neutron}}{n_B} = \frac{n_{\rm neutron}}{n_{\rm neutron} + n_{\rm proton}} = \frac{\frac{n_{\rm neutron}}{n_{\rm proton}}}{\frac{n_{\rm neutron}}{n_{\rm proton}} + 1} = \frac{A}{1+A} \ .$$

The fraction of the baryonic mass in helium is twice this number, since after nucleosynthesis essentially all neutrons are in helium, and the mass of each helium nucleus is twice the mass of the neutrons within it. Thus

$$Y = \frac{2A}{1+A} \ .$$

This gives $\boxed{Y = 0.851.}$