MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 30, 2004
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QUIZ 1 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) In 1826, the astronomer Heinrich Olber wrote a paper on a paradox regarding the night sky. What is Olber's paradox? What is the primary resolution of it?

(Ryden, Chapter 2, Pages 6-8)

Ans: Olber's paradox is that the night sky appears to be dark, instead of being uniformly bright. The primary resolution is that the universe has a finite age, and so the light from stars beyond the horizon distance has not reached us yet. (However, even in the steady-state model of the universe, the paradox is resolved because the light from distant stars will be red-shifted beyond the visible spectrum).

(b) What is the value of the Newtonian gravitational constant G in Planck units? The Planck length is of the order of 10^{-35} m, 10^{-15} m, 10^{15} m, or 10^{35} m?

(Ryden, Chapter 1, Page 3)

Ans: $G = 1$ in Planck units, by definition.

The Planck length is of the order of 10^{-35} m. (Note that this answer could be obtained by a process of elimination as long as you remember that the Planck length is much smaller than 10^{-15} m, which is the typical size of a nucleus).

(c) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it?

(Weinberg, Chapter 2, Pages 21-23; Ryden, Chapter 2, Page 11)

Ans: The Cosmological Principle states that there is nothing special about our location in the universe, i.e. the universe is homogeneous and isotropic.

Yes, the Hubble expansion is consistent with it (since there is no center of expansion).

(d) In the "Standard Model" of the universe, when the universe cooled to about 3×10^a K, it became transparent to photons, and today we observe these as the Cosmic Microwave Background (CMB) at a temperature of about 3×10^b K. What are the integers a and b?

(Weinberg, Chapter 3; Ryden, Chapter 2, Page 22)

$a = 3$, $b = 0$.

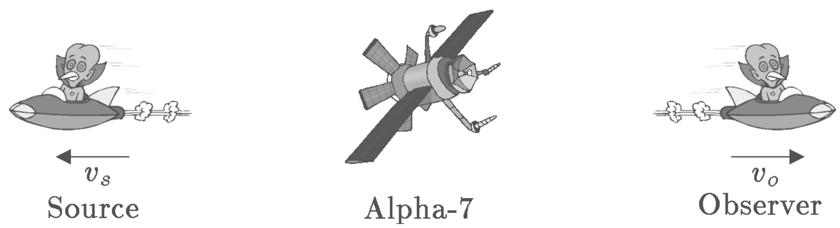
(e) What did the universe primarily consist of at about 1/100th of a second after the Big Bang? Include any constituent that is believed to have made up more than 1% of the mass density of the universe.

(Weinberg, Chapter 1, Page 5)

Ans: Electrons, positrons, neutrinos, and photons.

PROBLEM 2: SPECIAL RELATIVITY DOPPLER SHIFT *(20 points)*

Consider the Doppler shift of radio waves, for a case in which both the source and the observer are moving. Suppose the source is a spaceship moving with a speed v_s relative to the space station Alpha-7, while the observer is on another spaceship, moving in the opposite direction from Alpha-7 with speed v_o relative to Alpha-7.



(a) *(10 points)* Calculate the Doppler shift z of the radio wave as received by the observer. (Recall that radio waves are electromagnetic waves, just like light except that the wavelength is longer.)

(b) *(10 points)* Use the results of part (a) to determine $v_{\rm tot}$, the velocity of the source spaceship as it would be measured by the observer spaceship. *(8 points will be given for the basic idea, whether or not you have the right answer for part (a), and 2 points will be given for the algebra.)*

ANSWER:

(a) The easiest way to solve this problem is by a double application of the standard special-relativity Doppler shift formula, which was given on the front of the exam:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \ , \tag{2.1}$$

where $\beta = v/c$. Remembering that the wavelength is stretched by a factor $1+z$, we find immediately that the wavelength of the radio wave received at Alpha-7 is given by

$$\lambda_{\text{Alpha}-7} = \sqrt{\frac{1+v_s/c}{1-v_s/c}}\, \lambda_{\text{emitted}} \ . \tag{2.2}$$

The photons that are received by the observer are in fact never received by Alpha-7, but the wavelength found by the observer will be the same as if Alpha-7 acted as a relay station, receiving the photons and retransmitting them at the received wavelength. So, applying Eq. (2.1) again, the wavelength seen by the observer can be written as

$$\lambda_{\text{observed}} = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\, \lambda_{\text{Alpha}-7} \ . \tag{2.3}$$

Combining Eqs. (2.2) and (2.3),

$$\lambda_{\text{observed}} = \sqrt{\frac{1+v_o/c}{1-v_o/c}} \sqrt{\frac{1+v_s/c}{1-v_s/c}}\, \lambda_{\text{emitted}} \ , \tag{2.4}$$

so finally

$$\boxed{z = \sqrt{\frac{1+v_o/c}{1-v_o/c}} \sqrt{\frac{1+v_s/c}{1-v_s/c}} - 1 \ .} \tag{2.5}$$

(b) Although we used the presence of Alpha-7 in determining the redshift z of Eq. (2.5), the redshift is not actually affected by the space station. So the special-relativity Doppler shift formula, Eq. (2.1), must directly describe the redshift resulting from the relative motion of the source and the observer. Thus

$$\sqrt{\frac{1+v_{\text{tot}}/c}{1-v_{\text{tot}}/c}} - 1 = \sqrt{\frac{1+v_o/c}{1-v_o/c}} \sqrt{\frac{1+v_s/c}{1-v_s/c}} - 1 \ . \tag{2.6}$$

The equation above determines $v_{\rm tot}$ in terms of v_o and v_s, so the rest is just algebra. To simplify the notation, let $\beta_{\rm tot} \equiv v_{\rm tot}/c$, $\beta_o \equiv v_o/c$, and $\beta_s \equiv v_s/c$. Then

$$1 + \beta_{\rm tot} = \frac{1+\beta_o}{1-\beta_o}\,\frac{1+\beta_s}{1-\beta_s}\,(1-\beta_{\rm tot})$$

$$\beta_{\rm tot}\left[1 + \frac{1+\beta_o}{1-\beta_o}\,\frac{1+\beta_s}{1-\beta_s}\right] = \frac{1+\beta_o}{1-\beta_o}\,\frac{1+\beta_s}{1-\beta_s} - 1$$

$$\beta_{\rm tot}\left[\frac{(1-\beta_o-\beta_s+\beta_o\beta_s)+(1+\beta_o+\beta_s+\beta_o\beta_s)}{(1-\beta_o)(1-\beta_s)}\right] =$$

$$\frac{(1+\beta_o+\beta_s+\beta_o\beta_s)-(1-\beta_o-\beta_s+\beta_o\beta_s)}{(1-\beta_o)(1-\beta_s)}$$

$$\beta_{\rm tot}[2(1+\beta_o\beta_s)] = 2(\beta_o+\beta_s)$$

$$\beta_{\rm tot} = \frac{\beta_o+\beta_s}{1+\beta_o\beta_s}$$

$$\boxed{v_{\rm tot} = \frac{v_o+v_s}{1+\dfrac{v_o v_s}{c^2}}\ .} \qquad (2.7)$$

The final formula is the relativistic expression for the addition of velocities. Note that it guarantees that $|v_{\rm tot}| \le c$ as long as $|v_o| \le c$ and $|v_s| \le c$.

PROBLEM 3: EVOLUTION OF A FLAT UNIVERSE WITH $R(t) = bt^{1/4}$ *(35 points)*

The following questions all pertain to a flat universe, with a scale factor given by

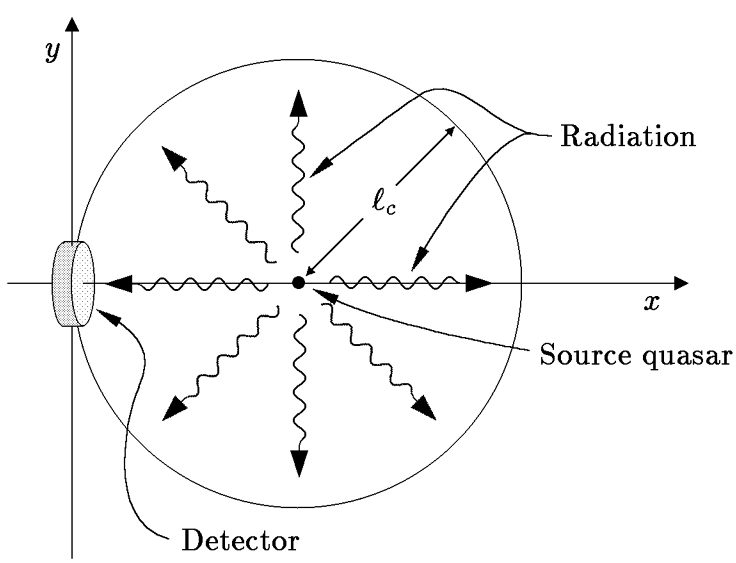
$$R(t) = bt^{1/4} ,$$

where b is a constant and t denotes cosmic time.

(a) *(10 points)* If a light pulse is emitted by a quasar at time t_e and observed here on Earth at time t_o, find the physical separation $\ell_p(t_e)$ between the quasar and the Earth, at the time of ***emission.***

(b) *(10 points)* At what time is the light pulse equidistant from the quasar and the Earth?

(c) *(15 points)* At the time of emission, the quasar had a power output P (measured, say, in ergs/sec), which was radiated uniformly in all directions. What is the radiation energy flux J from this quasar at the Earth today? Energy flux (which might be measured in ergs-cm^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow. You may find it useful to think of the detector as a small part of a sphere that is centered on the source, as shown in the following diagram:



ANSWER:

(a) Since light travels at a coordinate speed $c/R(t)$, the coordinate distance between the quasar and Earth can be found by integrating this speed over the time period of travel. Note that the speed changes with time, so one cannot simply multiply the speed by the time. Integrating,

$$\ell_c = \int_{t_e}^{t_o} \frac{c}{R(t)}\,\mathrm{d}t = \int_{t_e}^{t_o} \frac{c}{bt^{1/4}}\,\mathrm{d}t = \frac{4c}{3b}\left(t_o^{3/4} - t_e^{3/4}\right) \ .$$

The physical distance is given by

$$\ell_p(t) = R(t)\ell_c \ ,$$

so the physical distance at the time of emission is

$$\ell_p(t_e) = R(t_e)\ell_c = \boxed{\frac{4ct_e}{3}\left[\left(\frac{t_o}{t_e}\right)^{3/4} - 1\right] \ .}$$

(b) If the light pulse is equidistant from the quasar and the Earth in physical distance, it is also equidistant in coordinate distance. So, if we let t_{eq} denote the time that the light pulse is equidistant, t_{eq} must satisfy the relation

$$\int_{t_e}^{t_{\mathrm{eq}}} \frac{c}{R(t)}\,\mathrm{d}t = \int_{t_{\mathrm{eq}}}^{t_o} \frac{c}{R(t)}\,\mathrm{d}t \ ,$$

so

$$\frac{4c}{3b}\left(t_{\mathrm{eq}}^{3/4} - t_e^{3/4}\right) = \frac{4c}{3b}\left(t_o^{3/4} - t_{\mathrm{eq}}^{3/4}\right) \quad \Longrightarrow \quad t_{\mathrm{eq}}^{3/4} - t_e^{3/4} = t_o^{3/4} - t_{\mathrm{eq}}^{3/4} \ .$$

Solving for t_{eq},

$$\boxed{t_{\mathrm{eq}} = \left(\frac{t_o^{3/4} + t_e^{3/4}}{2}\right)^{4/3} \ .}$$

(c) In the diagram shown, photons are emitted at the center at time t_e and arrive at Earth, and at all points on the spherical surface shown, at time t_o. If the photons are emitted at some rate r, then the rate at which they arrive as the spherical surface is redshifted, in exactly the same way that the frequency of an electromagnetic wave is redshifted. Thus the rate is reduced by a factor

$$1 + z = \frac{R(t_o)}{R(t_e)} = \left(\frac{t_o}{t_e}\right)^{1/4} \ .$$

In addition to the reduction in rate, the energy of each photon will be shifted by an amount proportional to its frequency, since the energy of a photon is given by $E = h\nu$, where ν is the frequency and h is Planck's constant. Thus, the power P received on the spherical surface at time t_o is given by

$$P_{\text{spherical surface}} = \frac{P}{(1+z)^2} \; .$$

If the detector has physical area ΔA, then its area in coordinate units is $\Delta A/R^2(t_o)$. (Since lengths scale as $R(t)$, areas scale as $R^2(t)$.) The total area of the spherical surface, in comoving coordinates, is $4\pi\ell_c^2$, so the fraction of the area subtended by the detector is given by

$$\text{fraction} = \frac{A}{4\pi R^2(t_o)\ell_c^2} \; .$$

Since the power is distributed uniformly over the spherical surface, the power striking the detector is equal to this fraction of the total:

$$P_{\text{detector}} = \frac{A}{4\pi R^2(t_o)\ell_c^2} P_{\text{spherical surface}} \; .$$

Putting together the pieces and using the expression from part (a) for ℓ_c,

$$\begin{aligned} P_{\text{detector}} &= \frac{A}{4\pi b^2 t_o^{1/2}\left[\frac{4c}{3b}\left(t_o^{3/4} - t_e^{3/4}\right)\right]^2} P \left(\frac{t_e}{t_o}\right)^{1/2} \\ &= \frac{9A}{64\pi c^2 t_o^2 \left[1 - \left(\frac{t_e}{t_o}\right)^{3/4}\right]^2} \left(\frac{t_e}{t_o}\right)^{1/2} P \; . \end{aligned}$$

The flux J is the power per area hitting the detector, so

$$\boxed{J = \frac{P_{\text{detector}}}{A} = \left(\frac{t_e}{t_o}\right)^{1/2} \frac{9P}{64\pi c^2 t_o^2 \left[1 - \left(\frac{t_e}{t_o}\right)^{3/4}\right]^2} \; .}$$

PROBLEM 4: EVOLUTION OF A CLOSED, MATTER-DOMINATED UNIVERSE *(20 points)*

The following problem was Problem Set 2, Problem 4.

It was shown in Lecture Notes 5 that the evolution of a closed, matter-dominated universe can be described by introducing the time-parameter θ, with

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

where α is a constant with the units of length.

(a) *(10 points)* Use these expressions to find H, the Hubble "constant," as a function of α and θ. *(Hint: You can use the first of the equations above to calculate $d\theta/dt$.)*

(b) *(5 points)* Find ρ, the mass density, as a function of α and θ. *(4 points will be given for a correct answer, with 1 additional point if the answer is algebraically simplified.)*

(c) *(5 points)* Find Ω, where $\Omega \equiv \rho/\rho_c$, as a function of α and θ.

ANSWER:

(a) Using chain rule, the standard formula for the Hubble constant can be rewritten as

$$H(\theta) = \frac{1}{R}\frac{dR}{dt} = \frac{1}{R}\frac{dR}{d\theta}\frac{d\theta}{dt} \ .$$

By differentiating the parametric equations for R and t, one finds

$$\frac{dR}{d\theta} = \alpha\sqrt{k}\sin\theta \ ,$$

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) = \frac{1}{d\theta/dt} \ .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{k}\alpha(1 - \cos\theta)}\right]\left[\alpha\sqrt{k}\sin\theta\right]\left[\frac{c}{\alpha(1 - \cos\theta)}\right]$$

$$= \boxed{\frac{c\sin\theta}{\alpha(1 - \cos\theta)^2} \ .}$$

(b) The evolution equation for a homogeneous isotropic universe can be written as

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} .$$

Then, solving for ρ gives,

$$\rho = \frac{3}{8\pi G}\left(H^2 + \frac{kc^2}{R^2}\right) .$$

Using the answer from (a), and the parametric expression for $R/\sqrt{k}$, one has

$$\begin{aligned}\rho &= \frac{3}{8\pi G}\left[\frac{c^2 \sin^2\theta}{\alpha^2(1-\cos\theta)^4} + \frac{c^2}{\alpha^2(1-\cos\theta)^2}\right] \\ &= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{\sin^2\theta}{(1-\cos\theta)^2} + 1\right]\end{aligned}$$

This expression is greatly simplified by using the following trigonometric identity

$$\sin^2\theta = 1 - \cos^2\theta = (1-\cos\theta)(1+\cos\theta)$$

Using this in our expression for ρ we have

$$\begin{aligned}\rho &= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{(1+\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1-\cos\theta)} + 1\right] \\ &= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{1+\cos\theta}{1-\cos\theta} + 1\right] \\ &= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{2}{1-\cos\theta}\right] \\ &= \boxed{\frac{3c^2}{4\pi G\alpha^2(1-\cos\theta)^3} .}\end{aligned}$$

(c) Using the answer from (a) and the standard expression for ρ_c, one has

$$\rho_c = \frac{3H^2}{8\pi G} = \frac{3c^2 \sin^2\theta}{8\pi G\alpha^2(1-\cos\theta)^4} .$$

Then

$$\Omega \equiv \rho/\rho_c = \frac{2(1-\cos\theta)}{\sin^2\theta} = \boxed{\frac{2}{1+\cos\theta} .}$$