MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 30, 2004
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QUIZ 2 SOLUTIONS

PROBLEM 1: EVOLUTION OF MODEL UNIVERSES *(30 points)*

(a) For an empty universe, the Friedmann equation is

$$\left(\frac{\dot{R}}{R}\right)^2 = -\frac{kc^2}{R^2} \ .$$

Since the left-hand side cannot be negative, an empty universe cannot have $k > 0$, i.e. it cannot be closed.

Now consider $k = 0$, i.e. a flat universe. In this case the above equation has the solution

$$R(t) = R_0 \ ,$$

where R_0 is independent of time. So an empty universe can be flat as long as it is static.

Finally consider $k < 0$, i.e. an open universe. From the Friedmann equation we get

$$\dot{R} = \sqrt{|k|}c \quad \Longrightarrow \quad R(t) = \sqrt{|k|}ct + \text{const} \ ,$$

where the constant of integration *const* can be set to zero by using the convention that $t = 0$ when $R(t) = 0$. So, an empty universe can be open with a scale factor that increases linearly with time.

(b)

(i) Nonrelativistic matter: $w = 0$.

(ii) Relativistic matter: $w = 1/3$.

(iii) The cosmological constant: $w = -1$.

(c) The fluid equation is

$$\dot{\rho} = -3\frac{\dot{R}}{R}(1+w)\rho \ ,$$

where we have used the equation of state to express p in terms of ρ. Now we're given that $\rho \propto R^{-b}$ and so $\dot{\rho} \propto -bR^{-b-1}\dot{R}$, with the same constant of proportionality. Plugging these expressions into the fluid equation,

$$-bR^{-b-1}\dot{R} = -3\frac{\dot{R}}{R}(1+w)R^{-b}$$

$$\Longrightarrow \quad \boxed{b = 3(1+w) \ .}$$

(d) We can use the Friedmann equation for a flat universe to determine $R(t)$:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho \ .$$

For $\rho \propto R^{-b}$, the above equation can be written as

$$\left(\frac{\dot{R}}{R}\right)^2 \propto R^{-b} \ .$$

First consider the case $b = 0$, for which we find

$$\frac{\dot{R}}{R} = \text{const} = H_0 \quad \Longrightarrow \quad \frac{\mathrm{d}R}{R} = H_0\,\mathrm{d}t \ .$$

Integrating,

$$\ln R = H_0 t + \text{const} \quad \Longrightarrow \quad \boxed{R(t) \propto e^{H_0 t} \ .}$$

Next consider the case $b \neq 0$, for which we find

$$\frac{\dot{R}}{R} \propto R^{-b/2} \quad \Longrightarrow \quad R^{b/2-1}\,\mathrm{d}R \propto \mathrm{d}t \ .$$

Integrating,

$$R^{b/2} \propto t + \text{const} \quad \Longrightarrow \quad R^{b/2} \propto t \quad \Longrightarrow \quad \boxed{R(t) \propto t^{2/b} \ ,}$$

where again the constant of integration was set to zero by our convention for choosing the origin of time t.

— *Problem and solution written by Vishesh Khemani.*

PROBLEM 2: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(35 points)*

(a) Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\,dt^2 + R^2(t)\,d\psi^2 \ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{R^2(t)} \ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm\frac{c}{R(t)} \ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\rm hor} = \int_0^t \frac{c}{R(t')}dt' \ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\rm hor}$, which according to the metric is given by

$$\ell_{\rm phys}(t) = \int_{\psi=0}^{\psi=\psi_{\rm hor}} ds = \int_0^{\psi_{\rm hor}} R(t)\,d\psi = \boxed{R(t)\int_0^t \frac{c}{R(t')}dt' \ .}$$

(c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{R(t)} \ .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) \ .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1 - \cos\theta)}{R(t)} \ .$$

Then using $R = \alpha(1 - \cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1 \ .}$$

(d) This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ

must also change by 2π. From $R = \alpha(1 - \cos\theta)$, one can see that R returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1 \ .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 6. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

$$x^2 + y^2 + z^2 + w^2 = a^2 \ ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0, 0, 0, 1)$ can be called the north pole. Each point (x, y, z, w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x, y, z, w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0, 0, 0, -1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: In the past (not this year!) some students answered this question by saying that the photon would return in the lifetime of the universe, but they reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point

before the big crunch. Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\text{Crunch}}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the space-time. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\text{Crunch}} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\text{Crunch}}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\text{Crunch}} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost $1/2$, and it would approach $1/2$ as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 3: ROTATING FRAMES OF REFERENCE *(35 points)*

(a) The metric was given as

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}t^2 - \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] \ ,$$

and the metric coefficients are then just read off from this expression:

$$
\begin{aligned}
&g_{11} \equiv g_{rr} = -1 \\
&g_{00} \equiv g_{tt} = \text{coefficient of } \mathrm{d}t^2 = c^2 - r^2\omega^2 \\
&g_{20} \equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \frac{1}{2} \times \text{coefficient of } \mathrm{d}\phi\,\mathrm{d}t = -r^2\omega^2 \\
&g_{22} \equiv g_{\phi\phi} = \text{coefficient of } \mathrm{d}\phi^2 = -r^2 \\
&g_{33} \equiv g_{zz} = \text{coefficient of } \mathrm{d}z^2 = -1 \,.
\end{aligned}
$$

Note that the off-diagonal term $g_{\phi t}$ must be multiplied by $1/2$, because the expression

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu$$

includes the two equal terms $g_{20}\,\mathrm{d}\phi\,\mathrm{d}t + g_{02}\,\mathrm{d}t\,\mathrm{d}\phi$, where $g_{20} \equiv g_{02}$.

(b) Starting with the general expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \,,$$

we set $\mu = r$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{r\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_r g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \,.$$

When we sum over ν on the left-hand side, the only value for which $g_{r\nu} \neq 0$ is $\nu = 1 \equiv r$. Thus, the left-hand side is simply

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) = -\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} \,.$$

The RHS includes every combination of λ and σ for which $g_{\lambda\sigma}$ depends on r, so that $\partial_r\, g_{\lambda\sigma} \neq 0$. This means g_{tt}, $g_{\phi\phi}$, and $g_{\phi t}$. So,

$$
\begin{aligned}
RHS &= \frac{1}{2}\partial_r(c^2 - r^2\omega^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r(-r^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + \partial_r(-r^2\omega)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} \\
&= -r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 - 2r\omega\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} \\
&= -r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \,.
\end{aligned}
$$

Note that the final term in the first line is really the sum of the contributions from $g_{\phi t}$ and $g_{t\phi}$, where the two terms were combined to cancel the factor of $1/2$ in the general expression. Finally,

$$\boxed{\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \,.}$$

If one expands the RHS as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2r\omega\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} \,,$$

then one can identify the term proportional to ω^2 as the centrifugal force, and the term proportional to ω as the Coriolis force.

(c) Substituting $\mu = \phi$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\phi\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\phi g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \,.$$

But none of the metric coefficients depend on ϕ, so the right-hand side is zero. The left-hand side receives contributions from $\nu = \phi$ and $\nu = t$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + g_{\phi t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} - r^2\omega\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0 \,,$$

so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}+r^2\omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)=0\ .}$$

Note that one cannot "factor out" r^2, since r can depend on τ. If this equation is expanded to give an equation for $\mathrm{d}^2\phi/\mathrm{d}\tau^2$, the term proportional to ω would be identified as the Coriolis force. There is no term proportional to ω^2, since the centrifugal force has no component in the ϕ direction.

(d) If Eq. (1) of the problem is divided by $c^2\mathrm{d}t^2$, one obtains

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2=1-\frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2+r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}+\omega\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]\ .$$

Then using

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}=\frac{1}{\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)}\ ,$$

one has

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau}=\frac{1}{\sqrt{1-\frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2+r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}+\omega\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]}}\ .}$$

Note that this equation is really just

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}=\frac{1}{\sqrt{1-v^2/c^2}}\ ,$$

adapted to the rotating cylindrical coordinate system.