MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — May 10, 2004
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QUIZ 3 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

(a) No, we believe that there is more matter than antimatter in the universe. One piece of evidence is that the cosmic rays that we observe on earth come from large distances in our galaxy and are observed to be made up mostly of matter. Another piece of evidence is that we do not observe the high energy photons that would be produced by the annihilation of matter and antimatter had they been present in comparable amounts in the universe.

(b) The deuterium nucleus became stable about three minutes after the big bang (hence the title of Weinberg's book). After the deuterium bottleneck was passed, nucleosynthesis began and the neutrons rapidly became bound into stable helium nuclei.

(c) If stars and gas were the only matter present, then the orbital speeds would decrease in the Keplerian form $v \propto 1/\sqrt{R}$ at radii larger than the visible center. However, we observe that typically the orbital velocity does not fall off at large radii, indicating the presence of a dark halo which prevents the high-speed stars from becoming gravitationally unbound.

(d) The location of the first peak depends on the curvature of the universe, and the fact that it is at $l \sim 200$ indicates that the universe is flat.

PROBLEM 2: A NEW SPECIES OF LEPTON *(25 points)*

a) The number density is given by the formula at the start of the exam,

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \;.$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$\begin{aligned} n &= 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\ \cancel{\text{MeV}}}{6.582\times 10^{-16}\ \cancel{\text{eV}}\text{-}\cancel{\text{sec}} \times 2.998\times 10^{10}\ \cancel{\text{cm}}\text{-}\cancel{\text{sec}}^{-1}}\right)^3 \\ &\quad \times \left(\frac{10^6\ \cancel{\text{eV}}}{1\ \cancel{\text{MeV}}}\right)^3 \times \left(\frac{10^2\ \cancel{\text{cm}}}{1\ \text{m}}\right)^3 \\ &= 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\times 10^6 \times 10^2}{6.582\times 10^{-16}\times 2.998\times 10^{10}}\right)^3 \text{m}^{-3} \;. \end{aligned}$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\times 10^6 \times 10^2}{6.582\times 10^{-16}\times 2.998\times 10^{10}}\right)^3 \;.}$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot R}{R}\right)^2 = \frac{8\pi}{3} G\rho \;.$$

During the radiation-dominated era $R(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot R}{R} = \frac{1}{2t} \;.$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3} G\rho \;.$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \quad .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}$ cm^3 sec^{-2} g^{-1}, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732 \times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9$ g/cm^3.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \quad .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion − anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\text{tot}} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2} \frac{1}{g_{\text{tot}}} \hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/R$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\text{rest}} T^3(t) R^3(t)$$

where $g_{\rm rest} = g_{\rm tot} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 R^3(t) = g_{\rm rest} T^3(t) R^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\rm rest}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 3: EVOLUTION OF FLATNESS *(15 points)*

(a) We start with the Friedmann equation from the formula sheet on the quiz:

$$H^2 = \left(\frac{\dot R}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \ .$$

The critical density is the value of ρ corresponding to $k = 0$, so

$$H^2 = \frac{8\pi}{3} G\rho_c \ .$$

Using this expression to replace H^2 on the left-hand side of the Friedmann equation, and then dividing by $8\pi G/3$, one finds

$$\rho_c = \rho - \frac{3kc^2}{8\pi G R^2} \ .$$

Rearranging,

$$\frac{\rho - \rho_c}{\rho} = \frac{3kc^2}{8\pi G R^2 \rho} \ .$$

On the left-hand side we can divide the numerator and denominator by ρ_c, and then use the definition $\Omega \equiv \rho/\rho_c$ to obtain

$$\frac{\Omega - 1}{\Omega} = \frac{3kc^2}{8\pi G R^2 \rho} \ . \qquad (1)$$

For a matter-dominated universe we know that $\rho \propto 1/R^3(t)$, and so

$$\frac{\Omega - 1}{\Omega} \propto R(t) \ .$$

If the universe is nearly flat we know that $R(t) \propto t^{2/3}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t^{2/3} \ .}$$

(b) Eq. (1) above is still true, so our only task is to re-evaluate the right-hand side. For a radiation-dominated universe we know that $\rho \propto 1/R^4(t)$, so

$$\frac{\Omega - 1}{\Omega} \propto R^2(t) \ .$$

If the universe is nearly flat then $R(t) \propto t^{1/2}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t \ .}$$

PROBLEM 4: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

(a) Since $\Omega_m + \Omega_\Lambda = 0.35 + 0.65 = 1$, the universe is flat. It therefore obeys a simple form of the Friedmann equation,

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G(\rho_m + \rho_\Lambda) ,$$

where the overdot indicates a derivative with respect to t, and the term proportional to k has been dropped. Using the fact that $\rho_m \propto 1/R^3(t)$ and $\rho_\Lambda =$ const, the energy densities on the right-hand side can be expressed in terms of their present values $\rho_{m,0}$ and $\rho_\Lambda \equiv \rho_{\Lambda,0}$. Defining

$$x(t) \equiv \frac{R(t)}{R(t_0)} ,$$

one has

$$\begin{aligned}\left(\frac{\dot{x}}{x}\right)^2 &= \frac{8\pi}{3} G \left(\frac{\rho_{m,0}}{x^3} + \rho_\Lambda\right) \\ &= \frac{8\pi}{3} G\rho_{c,0} \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \\ &= H_0^2 \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) .\end{aligned}$$

Here we used the facts that

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}}; \qquad \Omega_{\Lambda,0} \equiv \frac{\rho_\Lambda}{\rho_{c,0}} ,$$

and

$$H_0^2 = \frac{8\pi}{3} G\rho_{c,0} .$$

The equation above for $(\dot{x}/x)^2$ implies that

$$\dot{x} = H_0\, x \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}} ,$$

which in turn implies that

$$dt = \frac{1}{H_0} \frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .$$

Using the fact that x changes from 0 to 1 over the life of the universe, this relation can be integrated to give

$$t_0 = \int_0^{t_0} dt = \boxed{\frac{1}{H_0} \int_0^1 \frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

The answer can also be written as

$$\boxed{t_0 = \frac{1}{H_0} \int_0^1 \frac{x\, dx}{\sqrt{\Omega_{m,0}\, x + \Omega_{\Lambda,0}\, x^4}}}$$

or

$$\boxed{t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}} ,}$$

where in the last answer I changed the variable of integration using

$$x = \frac{1}{1+z} ; \qquad dx = -\frac{dz}{(1+z)^2} .$$

Note that the minus sign in the expression for $\mathrm{d}x$ is canceled by the interchange of the limits of integration: $x = 0$ corresponds to $z = \infty$, and $x = 1$ corresponds to $z = 0$.

Your answer should look like one of the above boxed answers. You were not expected to complete the numerical calculation, but for pedagogical purposes I will continue. The integral can actually be carried out analytically, giving

$$\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}} + \sqrt{\Omega_{\Lambda,0}}}{\sqrt{\Omega_m}}\right) .$$

Using

$$\frac{1}{H_0} = \frac{9.778 \times 10^9}{h_0} \text{ yr} ,$$

where $H_0 = 100\,h_0\,\text{km-sec}^{-1}\text{-Mpc}^{-1}$, one finds for $h_0 = 0.65$ that

$$\frac{1}{H_0} = 15.043 \times 10^9 \text{ yr} .$$

Then using $\Omega_m = 0.35$ and $\Omega_{\Lambda,0} = 0.65$, one finds

$$\boxed{t_0 = 13.88 \times 10^9 \text{ yr} .}$$

So the SDSS people were right on target.

(b) Having done part (a), this part is very easy. The dynamics of the universe is of course the same, and the question is only slightly different. In part (a) we found the amount of time that it took for x to change from 0 to 1. The light from the quasar that we now receive was emitted when

$$x = \frac{1}{1+z} ,$$

since the cosmological redshift is given by

$$1 + z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} .$$

Using the expression for $\mathrm{d}t$ from part (a), the amount of time that it took the universe to expand from $x = 0$ to $x = 1/(1+z)$ is given by

$$t_e = \int_0^{t_e} \mathrm{d}t = \boxed{\frac{1}{H_0}\int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Again one could write the answer other ways, including

$$\boxed{t_0 = \frac{1}{H_0}\int_z^{\infty} \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0}}} .}$$

Again you were expected to stop with an expression like the one above. Continuing, however, the integral can again be done analytically:

$$\int_0^{x_{\max}} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}x_{\max}^3} + \sqrt{\Omega_{\Lambda,0}}\,x_{\max}^{3/2}}{\sqrt{\Omega_m}}\right) .$$

Using $x_{\max} = 1/(1 + 5.82) = .1466$ and the other values as before, one finds

$$\boxed{t_e = \frac{0.06321}{H_0} = 0.9509 \times 10^9 \text{ yr} .}$$

So again the SDSS people were right.

(c) To find the physical distance to the quasar, we need to figure out how far light can travel from $z = 5.82$ to the present. Since we want the present distance, we multiply the coordinate distance by $R(t_0)$. For the flat metric

$$ds^2 = -c^2\, d\tau^2 = -c^2 dt^2 + R^2(t)\left\{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} ,$$

the coordinate velocity of light (in the radial direction) is found by setting $ds^2 = 0$, giving

$$\frac{dr}{dt} = \frac{c}{R(t)} .$$

So the total coordinate distance that light can travel from t_e to t_0 is

$$\ell_c = \int_{t_e}^{t_0} \frac{c}{R(t)}\, dt .$$

This is not the final answer, however, because we don't explicitly know $R(t)$. We can, however, change variables of integration from t to x, using

$$dt = \frac{dt}{dx}\, dx = \frac{dx}{\dot{x}} .$$

So

$$\ell_c = \frac{c}{R(t_0)} \int_{x_e}^{1} \frac{dx}{x\,\dot{x}} ,$$

where x_e is the value of x at the time of emission, so $x_e = 1/(1+z)$. Using the equation for $\dot{x}$ from part (a), this integral can be rewritten as

$$\ell_c = \frac{c}{H_0 R(t_0)} \int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .$$

Finally, then

$$\boxed{\ell_{\text{phys},0} = R(t_0)\,\ell_c = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Alternatively, this result can be written as

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{dx}{\sqrt{\Omega_{m,0}\, x + \Omega_{\Lambda,0}\, x^4}} ,}$$

or by changing variables of integration to obtain

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_{0}^{z} \frac{dz'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}} .}$$

Continuing for pedagogical purposes, this time the integral has no analytic form, so far as I know. Integrating numerically,

$$\int_0^{5.82} \frac{dz'}{\sqrt{0.35\,(1+z')^3 + 0.65}} = 1.8099 ,$$

and then using the value of $1/H_0$ from part (a),

$$\boxed{\ell_{\text{phys},0} = 27.23 \text{ light-yr} .}$$

Right again.

(d) $\ell_{\text{phys},e} = R(t_e)\ell_c$, so

$$\ell_{\text{phys},e} = \frac{R(t_e)}{R(t_0)}\,\ell_{\text{phys},0} = \boxed{\frac{\ell_{\text{phys},0}}{1+z} .}$$

Numerically this gives

$$\boxed{\ell_{\text{phys},e} = 3.992 \times 10^9 \text{ light-yr} .}$$

The SDSS announcement is still okay.

(e) The speed defined in this way obeys the Hubble law exactly, so

$$v = H_0\,\ell_{\text{phys},0} = c \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}} .$$

Then

$$\boxed{\frac{v}{c} = \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}} .}$$

Numerically, we have already found that this integral has the value

$$\boxed{\frac{v}{c} = 1.8099 .}$$

The SDSS people get an A.