MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 13, 2004
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QUIZ 2

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USEFUL INFORMATION:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$
$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$
$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$
$$\kappa \equiv -k\ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

PROBLEM 1: EVOLUTION OF MODEL UNIVERSES *(30 points)*

This problem is based on Chapter 5 of Ryden. Since her notation is a little different from mine, I am presenting the problem in both notations, and you can answer it in the notation of your choice.

The evolution of a homogeneous, isotropic universe is governed by the following three independent equations:

The Friedmann equation,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \quad \Leftarrow \text{or} \Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2} \;, \tag{1}$$

the fluid equation,

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \quad \Leftarrow \text{or} \Rightarrow \quad \dot{\varepsilon} = -3\frac{\dot{a}}{a}(\varepsilon + P) \;, \tag{2}$$

and the equation of state,

$$p = w\rho c^2 \quad \Leftarrow \text{or} \Rightarrow \quad P = w\varepsilon \;. \tag{3}$$

In Eq. (3) we assume that w is a constant. In this problem we will examine the time evolution of the scale factor, $R(t)$ [or $a(t)$], for different assumptions about the nature of the matter and its equation of state.

(a) *(8 points)* First consider an empty universe ($\rho = \varepsilon = 0$). What are the possible forms for the function $R(t)$ [or $a(t)$], and is the universe open, closed, or flat in each case?

For the rest of the problem we consider a ***flat*** universe, made up of "stuff" that has some constant w relating the pressure and the mass density (according to the equation of state above).

(b) *(8 points)* What value of w corresponds to

(i) nonrelativistic matter?

(ii) relativistic matter (i.e., radiation)?

and

(iii) the cosmological constant?

For this part you may simply state the answers without doing any calculations.

— *Problem 1 Continues* —

Problem 1 Continued —

(c) *(6 points)* In such a universe, $\rho \propto R^{-b}$ [or $\varepsilon \propto a^{-b}$], where b is a constant that depends only on w. Find b. For full credit, your answer should show how to derive the expression for b using only mathematics and Eqs. (1), (2), and (3) above.

(d) *(6 points)* Using $\rho \propto R^{-b}$ [or $\varepsilon \propto a^{-b}$] , determine $R(t)$ [or $a(t)$] for both $b = 0$ and $b \neq 0$. For $b \neq 0$ you should express $R(t)$ in terms of t and b. For the case $b = 0$ you should express your answer in terms of the present value of the Hubble constant, H_0. In both cases your answer can contain a "proportional to" sign ($\propto$), or you can introduce an arbitrary constant of proportionality. Again, to obtain full credit you must show how to derive the answer from Eqs. (1), (2), and (3) above.

PROBLEM 2: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(35 points)*

The following problem was Problem 1 of the Review Problems for Quiz 2.

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

(a) *(9 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi =$ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(9 points)* Write an expression for the physical horizon distance ℓ_{phys} at time t. You should leave your answer in the form of a definite integral.

The form of $R(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.,* dominated by nonrelativistic matter), then $R(t)$ is described by the parametric equations

$$ct = \alpha(\theta - \sin\theta) ,$$

$$R = \alpha(1 - \cos\theta) ,$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{c^2} .$$

These equations are identical to those on the front of the exam, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where θ is the parameter used to describe the evolution.

(d) *(7 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

PROBLEM 3: ROTATING FRAMES OF REFERENCE *(35 points)*

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}t^2 - \Big[\mathrm{d}r^2 + r^2\,\left(\mathrm{d}\phi + \omega\,\mathrm{d}t\right)^2 + \mathrm{d}z^2\Big]\ , \tag{1}$$

which corresponds to a coordinate system rotating about the z-axis, where ϕ is the azimuthal angle around the z-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \le r < \infty$, $-\infty < z < \infty$, and $0 \le \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

EXTRA INFORMATION

To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (1) was obtained by starting with a Minkowski metric in cylindrical coordinates $\bar{t}$, $\bar{r}$, $\bar{\phi}$, and $\bar{z}$,

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar{t}^{\,2} - \left[\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\bar{\phi}^2 + \mathrm{d}\bar{z}^2\right]\ ,$$

and then introducing new coordinates t, r, ϕ, and z that are related by

$$\bar{t} = t, \qquad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z\ ,$$

so $\mathrm{d}\bar{t} = \mathrm{d}t$, $\mathrm{d}\bar{r} = \mathrm{d}r$, $\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \omega\,\mathrm{d}t$, and $\mathrm{d}\bar{z} = \mathrm{d}z$.

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu\ ,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, g_{11} (which can also be called g_{rr}) is equal to -1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned} g_{11} &\equiv g_{rr} = -1 \\ g_{00} &\equiv g_{tt} = ? \\ g_{20} \equiv g_{02} \equiv g_{\phi t} &\equiv g_{t\phi} = ? \\ g_{22} &\equiv g_{\phi\phi} = ? \\ g_{33} &\equiv g_{zz} = ? \end{aligned} \tag{2}$$

— *Problem 3 Continues* —

Problem 3 Continued —

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned} g_{11} &\equiv g_{rr} = -1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_1(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) \ , \end{aligned} \tag{3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

explicitly write the equation that results when the free index μ is equal to 1, corresponding to the coordinate r.

(c) *(7 points)* Explicitly write the equation that results when the free index μ is equal to 2, corresponding to the coordinate ϕ.

(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants c and ω. Be sure to note that your answer should depend on the derivatives of t, ϕ, and z with respect to t, not τ. *(Hint: first find an expression for $\mathrm{d}\tau/\mathrm{d}t$, in terms of the quantities indicated, and then ask yourself how this result can be used to find $\mathrm{d}t/\mathrm{d}\tau$.)*