REVIEW PROBLEMS FOR QUIZ 1

QUIZ DATE: Tuesday, October 18, 2005

NEW READING ASSIGNMENT: Steven Weinberg, The First Three Minutes, Chapters 4 and 5.

QUIZ COVERAGE: Lecture Notes 1 (sections on the Doppler shift only); Lecture Notes 3, 4, and 5; Problem Sets 1 and 2; Weinberg, Chapters 1-5, Ryden, Chapters 1-3.

One of the problems on the quiz will be taken verbatim (or at least almost verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems. The starred problems are the ones that I recommend that you review most carefully: Problems 2, 4, 7, 12, 14, 15, and 18. The starred problems do not include any reading questions, but parts of the reading questions in these Review Problems may also recur on the upcoming quiz.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. Except for a few parts which are clearly marked, they are all problems that I would consider fair for the coming quiz. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course website the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, and 2004. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. Since the schedule and the number of quizzes has varied over the years, the coverage of this quiz will not necessarily be the same as Quiz 1 from previous years. In fact, the first quiz this year covers the same material as the first two quizzes in 2000, and the same as Quiz 1 from 2002 and 2004.

REVIEW SESSION: To help you study for the quiz, Jesse Shelton will hold a review session on Sunday, October 16, at 6:00 p.m. The location will be announced on the course website.
PROBLEM 1: DID YOU DO THE READING? (35 points)

The following problem was Problem 1, Quiz 1, 2000. The parts were each worth 5 points.

a) The Doppler effect for both sound and light waves is named for Johann Christian Doppler, a professor of mathematics at the Realschule in Prague. He predicted the effect for both types of waves in 1842. What are the two digits?

b) When the sky is very clear (i.e., almost night in Boston), one can see a white band of light across the middle sky that appears stationary relative to the Earth. This band of light is known as the Milky Way. Explain in a sentence or two how this band of light is related to the shape of the galaxy in which we live, which is also called the Milky Way.

c) The statement that the distant galaxies are on average receding from us with a speed proportional to their distance was first published by Edwin Hubble in 1929, and has become known as Hubble’s law. Was Hubble’s original paper based on the study of 2, 18, 180, or 1,800 galaxies?

d) The following diagram, labeled Homogeneity and the Hubble Law, was used by Weinberg to explain how Hubble’s law is consistent with the homogeneity of the universe. The arrows and labels from the “Velocities seen by B” and the “Velocities seen by C” rows have been deleted from this copy of the figure, and it is your job to sketch the figure in your exam book with these arrows and labels included.

(Actually, in Weinberg’s diagram these arrows were not labeled, but the labels are required here so that the grader does not have to judge the precise length of hand-drawn arrows.)

e) The horizon is the present distance of the most distant objects from which light has had time to reach us since the beginning of the universe. The horizon changes with time, but of course so does the size of the universe as a whole.

During a time interval in which the linear size of the universe grows by 1%, does the horizon distance (i) grow by more than 1%, or (ii) grow by less than 1%, or (iii) grow by the same 1%?

f) Name the two men who in 1964 discovered the cosmic background radiation.

With what institution were they affiliated?

g) At a temperature of 3000K, the nuclei and electrons that filled the universe combined to form neutral atoms, which interact very weakly with the photons of the background radiation. After this process, known as “recombination,” the background radiation expanded freely. Since recombination, how have each of the following quantities varied as the size of the universe has changed? (Your answers should resemble statements such as “proportional to the size of the universe,” or “inversely proportional to the square of the size of the universe.” The word “size” will be interpreted to mean linear size, not volume.)

(i) the average distance between photons
(ii) the typical wavelength of the radiation
(iii) the number density of photons in the radiation
(iv) the energy density of the radiation
(v) the temperature of the radiation

The steady-state theory of the universe was proposed in the late 1940s by Hermann Bondi, Thomas Gold, and Fred Hoyle, and was considered a viable model for the universe until the cosmic background radiation was discovered and its properties confirmed.

a) (10 points) The steady-state theory proposes that the universe is homogeneous and the Hubble Law was used by Hoyle. Find the most general form for the scale factor function which is consistent with this hypothesis.

The horizon distance is the present distance of the most distant objects from which light has had time to reach us since the beginning of the universe. The horizon changes with time, but of course so does the size of the universe as a whole.

PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY (25 points)

The following problem was Problem 2, Quiz 1, 2000.

The steady-state theory of the universe was proposed in the late 1940s by Hermann Bondi, Thomas Gold, and Fred Hoyle, and was considered a viable model for the universe until the cosmic background radiation was discovered and its properties confirmed. As the name suggests, this theory is based on the hypothesis that the large-scale properties of the universe do not change with time. The expansion of the universe was an established fact when the steady-state theory was invented, but the steady-state theory reconciles the expansion with a steady-state density of matter by proposing that new matter is created as the universe expands, so that the matter density remains constant. The universe is considered to be infinite, and so the steady-state theory predicts that the universe is observationally uniform in all directions. Since the universe is considered to be infinite, there is no edge, and so if one were to travel in a straight line through space, one would eventually return to the point of origin.

a) (10 points) The steady-state theory proposes that the Hubble constant, like other cosmological parameters, does not change with time, so \( H = H_0 \). Find the most general form for the scale factor function which is consistent with this hypothesis.

The horizon distance is the present distance of the most distant objects from which light has had time to reach us since the beginning of the universe. The horizon changes with time, but of course so does the size of the universe as a whole.

b) (15 points) The steady-state theory also proposes that the Hubble constant is constant in space. This implies that the universe is homogeneous in space, and that the Hubble constant is the same everywhere. Explain why this is true.
PROBLEM 4: AN EXPONENTIALLY EXPANDING UNIVERSE

(a) The assumptions of homogeneity and isotropy greatly simplify the description of the observable universe. We find that there are three possibilities for a homogeneous and isotropic universe: an open universe, a flat universe, and a closed universe. What quantity or condition distinguishes between these three cases: the density of matter and radiation domination, or redshift?

(b) What is the temperature, in Kelvin, of the cosmic microwave background to first order? Estimate the diameter and thickness of the disk of the Milky Way galaxy. Any numbers within a factor of 2 of those given in Weinberg's book will be accepted.

(c) Estimate the diameter and thickness of the disk of the Milky Way galaxy. Any numbers within a factor of 2 of those given in Weinberg's book will be accepted.

(d) The mathematical theory of an expanding universe was first published in 1922 by the Russian mathematician Alexandre Friedmann, the Dutch Astronomer Willem de Sitter, the American astronomer Edwin Hubble, or the Belgian cleric Georges Lemaître?

(e) After discovering an inexplicable hiss coming from their radio telescope, Arno Penzias and Robert Wilson of Bell Laboratories learned that P.J.E. Peebles, a Princeton theorist, had calculated that the big bang would produce a background of cosmic radiation with a temperature today of 10−5 K. What implication does this have for the theory of an expanding universe?

PROBLEM 5: "DID YOU DO THE READING?"

(a) In 1750 the English instrumentmaker Thomas Wright published a book, A New Hypothesis of the Universe or New Hypothesis of the Universe which contained an incorrect notion of the position of the Crab Nebula, the solar system, the Milky Way, or the local supercluster?

(b) In 1755 Immanuel Kant published his book "On the Future of Mankind," which contained a chapter called "The hypothesis of the universe." What is the hypothesis of the universe?

(c) Which of the following supports the hypothesis that the universe is isotropic: the ground, clustering of galaxies on large scales, or the age and distribution of globular clusters?

(d) The distance to the Andromeda Nebula (roughly) 10 kpc, 5 billion lightyears, 200 million light years, or 3 light years?
Consider a flat universe which is filled with some peculiar form of matter, so

\[
\text{TIME EVOLUTION}
\]

The following problem was Problem 3, Quiz 1, 2000.

**Problem 8: Another Flat Universe with an Unusual Energy Flow**

\[
\text{TIME EVOLUTION}
\]

The following problem was Problem 3, Quiz 2, 1988.

**Problem 6: A Flat Universe with Unusual Time Evolution**

Consider a flat universe which is filled with some peculiar form of matter, so
b) The Andromeda Nebula was shown conclusively to lie outside our own galaxy when astronomers acquired telescopes powerful enough to resolve the individual stars of Andromeda. Was this feat ... in 1755, by Henrietta Swan Leavitt in 1912, by Edwin Hubble in 1923, or by Walter Baade and Allan Sandage in the 1950s?

c) Some of the earliest measurements of the cosmic background radiation were made indirectly, by observing interstellar clouds of a molecule called cyanogen (CN). State whether each of the following statements is true or false (1 point each):

(i) The first measurements of the temperature of the interstellar cyanogen were made over twenty years before the cosmic background radiation was directly observed.

(ii) Cyanogen helps to measure the cosmic background radiation by reflecting it toward the Earth, so that it can be received with microwave detectors.

(iii) One reason why the cyanogen observations were important was that they gave the first measurements of the equivalent temperature of the cosmic background radiation at wavelengths shorter than the peak of the black-body spectrum.

(iv) By measuring the spectrum of visible starlight that passes through the cyanogen clouds, astronomers can infer the intensity of the microwave radiation that bathes the clouds.

(v) By observing chemical reactions in the cyanogen clouds, astronomers can infer the temperature of the microwave radiation in which they are bathed.

d) In about 280 B.C., a Greek philosopher proposed that the Earth and the other planets revolve around the sun. What was the name of this person?

[Note for 2004: this question was based on readings from Joseph Silk’s *The Big Bang*, and therefore is not appropriate for Quiz 1 of this year.]

e) In 1832 Heinrich Wilhelm Olbers presented what we now know as “Olbers’ Paradox,” although a similar argument had been discussed as early as 1610 by Johannes Kepler. Olbers argued that if the universe were finite and static, then his paradox “would not be a paradox anymore, as it is now,” that is, if the universe were finite and static, then one of the following consequences would result:

(i) The brightness of the night sky would be infinite.

(ii) Any patch of the night sky would look as bright as the surface of the sun.

(iii) The brightness of the night sky would be infinite.

(iv) Any patch of the night sky would look as bright as the sun as the distance of the

sun.

The following problem was Problem 3, Quiz 1, 1996:

Consider a flat universe which is filled with some peculiar form of matter, so that the Robertson-Walker scale factor behaves as \( R(t) = bt^{3/5} \), where \( b \) is a constant.

a) Find the Hubble constant \( H \) at an arbitrary time \( t \).

b) What is the physical horizon distance at time \( t \)?

c) Suppose a light pulse leaves galaxy \( A \) at time \( t_A \) and arrives at galaxy \( B \) at time \( t_B \). What is the coordinate distance between these two galaxies?

d) What is the physical separation between galaxy \( A \) and galaxy \( B \) at time \( t_A \)? At time \( t_B \)?

e) At what time is the light pulse equidistant from the two galaxies?

f) What is the speed of \( B \) relative to \( A \) at the time \( t_A \)? (By “speed,” I mean the rate of change of the physical distance with respect to cosmic time, \( dL/dt \).)

g) For observations made at time \( t \), what is the present value of the physical distance as a function of the redshift \( z \) (and the time \( t \))? What physical distance corresponds to \( z = \infty \)? How does this compare with the horizon distance?

h) Returning to the discussion of the galaxies \( A \) and \( B \) which were considered in parts (c)-(f), suppose the radiation from galaxy \( A \) is emitted with total power \( P \). What is the power per area received at galaxy \( B \)?

i) When the light pulse is received by galaxy \( B \), a pulse is immediately sent back toward galaxy \( A \). At what time does this second pulse arrive at galaxy \( A \)?
decays which corresponds to \( k = \frac{\rho}{\rho_0} \) is the critical mass density (i.e., that mass
where \( \rho = \rho_0 \)).

You have forgotten. \( \rho \) is defined by the equation

\[
\frac{d\rho}{dt} = 0
\]

... Find the relationship between \( t \) and \( \ell \) for a matter-dominated universe. In the case

\[
\frac{d\ell}{dt} = b
\]

Expansion. The parameter \( \ell \) is a direct measure of the slowing down of the cosmic

\begin{align*}
\text{acceleration parameter} \qquad \text{for a universe which is filled with some peculiar form of matter, so}
\end{align*}

Many standard references in cosmology define a quantity called the deceleration parameter \( q \), which is a direct measure of the slowing down of the cosmic

\[
q = \frac{\ddot{R}}{R^2}
\]

When the response is received by galaxy A, the radio waves will be

\[
\text{frequency} = \text{frequency}
\]

in terms of \( \frac{c}{t} \).

Upon receipt of the message, the creatures on galaxy B immediately

\[
\text{express your answer in terms of } c \quad \text{and } t
\]

You must decode all the information in the message before you can

\[
\text{express your answer in terms of } c \quad \text{and } t
\]

and 

\[
\text{express your answer in terms of } c \quad \text{and } t
\]

Tell the Robertson-Walker scale factor behaves as

\[
\frac{\dot{R}}{R} = \frac{\ddot{R}}{R} + \frac{\dot{R}}{c^2}
\]

Energy density

\[
2nT = 0
\]

Number density

\[
0
\]

The following questions were taken from Problem 2, Quiz 2, 1992, where it counted 10 points out

\begin{align*}
\text{of 100}
\end{align*}

The following questions are worth 5 points each.

\section*{Problem 12: The Deceleration Parameter}

\begin{align*}
\text{(a)} \quad & \text{The early universe is believed to have been filled with thermal, or black-body,}
\end{align*}

\begin{align*}
\text{radiation. For such radiation the number density of photons and the energy density are each proportional to powers of the absolute temperature}
\end{align*}

\[
\text{number density} \propto \frac{T^4}{c^3}
\]

\begin{align*}
\text{(b)} \quad & \text{At about 3,000 K the matter in the universe underwent a certain chemical}
\end{align*}

\begin{align*}
\text{transition. For such a transition the number density of photons and the energy}
\end{align*}

\[
\text{density} \propto \frac{T^4}{c^3}
\]

\begin{align*}
\text{(c)} \quad & \text{The Hubble constant } H \text{ is an arbitrary time function.}
\end{align*}

\begin{align*}
\text{Find the Hubble constant } H \text{ at an arbitrary time } t.
\end{align*}

\begin{align*}
\text{where } H \text{ is a constant.}
\end{align*}

\begin{align*}
\text{a)} \quad & \text{In case you have forgotten, } H(t) \text{ is the Hubble constant at an arbitrary time } t.
\end{align*}

\begin{align*}
\text{b)} \quad & \text{Consider a universe which is filled with some peculiar form of matter, so}
\end{align*}

\begin{align*}
\text{that the Robertson-Walker scale factor behaves as}
\end{align*}

\[
\frac{\dot{R}}{R} = \frac{\ddot{R}}{R} + \frac{\dot{R}}{c^2}
\]

\begin{align*}
\text{The following was Problem 3, Quiz 1, 1998:}
\end{align*}

\begin{align*}
\text{Problem 11: Another Flat Universe with Critical Density}
\end{align*}

\begin{align*}
\text{The following was Problem 2, Quiz 2, 1992, where it counted 10 points out}
\end{align*}

\begin{align*}
\text{of 100}
\end{align*}

\begin{align*}
\text{d)} \quad & \text{At about 3,000 K the matter in the universe underwent a certain chemical}
\end{align*}

\begin{align*}
\text{transition. For such a transition the number density of photons and the energy}
\end{align*}

\[
\text{density} \propto \frac{T^4}{c^3}
\]

\begin{align*}
\text{The following questions were worth 5 points total:}
\end{align*}

\begin{align*}
\text{a)} \quad & \text{At about 3,000 K the matter in the universe underwent a certain chemical}
\end{align*}

\begin{align*}
\text{transition. For such a transition the number density of photons and the energy}
\end{align*}

\[
\text{density} \propto \frac{T^4}{c^3}
\]

\begin{align*}
\text{b)} \quad & \text{The following questions were worth 5 points total:}
\end{align*}

\begin{align*}
\text{a)} \quad & \text{At about 3,000 K the matter in the universe underwent a certain chemical}
\end{align*}

\begin{align*}
\text{transition. For such a transition the number density of photons and the energy}
\end{align*}

\[
\text{density} \propto \frac{T^4}{c^3}
\]
PROBLEM 13: A RADIATION-DOMINATED FLAT UNIVERSE

We have learned that a matter-dominated homogeneous and isotropic universe can be described by a scale factor \( R(t) \) obeying the equation

\[
\frac{1}{2} \left( \frac{dR}{dt} \right)^2 = \frac{8\pi}{3} G \rho - \frac{k c^2}{R^2}.
\]

This equation in fact applies to any form of mass density, so we can apply it to a universe in which the mass density is dominated by the energy of photons. Recall that the mass density of nonrelativistic matter falls off as \( 1/R^3 \) as the universe expands; the mass of each particle remains constant, and the density of particles falls off as \( 1/R^3 \) because the volume increases as \( R^3 \). For the photon-dominated universe, the density of photons falls off as \( 1/R^3 \), but in addition the frequency (and hence the energy) of each photon redshifts in proportion to \( 1/R \). Since mass and energy are equivalent, the mass density of the gas of photons falls off as \( 1/R^4 \).

For a flat (i.e., \( k = 0 \)) matter-dominated universe we learned that the scale factor \( R(t) \) is proportional to \( t^{2/3} \). How does \( R(t) \) behave for a photon-dominated universe?

PROBLEM 14: EVOLUTION OF AN OPEN UNIVERSE

The following problem was taken from Quiz 2, 1990, where it counted 10 points out of 100.

Consider an open, matter-dominated universe, as described by the evolution equations on the front of the quiz. Find the time \( t \) at which \( R/\sqrt{\kappa} = 2 \alpha \).

PROBLEM 15: ANTICIPATING A BIG CRUNCH

Suppose that we lived in a closed, matter-dominated universe, as described by the equations on the front of the quiz. Suppose further that we measured the mass density parameter \( \Omega \) to be \( \Omega_0 = 2 \), and we measured the Hubble “constant” to have some value \( H \). How much time would we have before our universe ended in a big crunch?

Consider an open, matter-dominated universe, as described by the equations on the front of the quiz. Find the time \( t \) at which \( R/\sqrt{\kappa} = 2 \alpha \).

We have learned that a closed, matter-dominated universe cannot collapse to a point.

PROBLEM 16: A POSSIBLE MODIFICATION OF NEWTON’S LAW

The following problem was Problem 2, Quiz 2, 2000.

In Lecture Notes 4 we developed a Newtonian model of cosmology by considering a uniform sphere of mass, centered at the origin, with initial mass density \( \rho_i \) and an initial pattern of velocities corresponding to Hubble expansion:

\[
\vec{v}_i = H \vec{r}.
\]

We denoted the radius at time \( t \) of a particle which started at radius \( r_i \) by the function \( r(r_i, t) \). Assuming Newton’s law of gravity, we concluded that each particle would experience an acceleration given by

\[
\vec{g} = -\frac{G M(r_i)}{r_i^2} \hat{r}.
\]

where \( M(r_i) \) denotes the total mass contained initially in the region \( r < r_i \), given by

\[
M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i.
\]

Suppose that the law of gravity is modified to contain a repulsive term, producing an acceleration which grows as the power of the distance, with a strength that is independent of the mass. That is, suppose \( g \) is given by

\[
\vec{g} = \frac{G M(r_i)}{r_i^2} \hat{r} + \gamma r_n(r_i, t) \hat{r},
\]

where \( \gamma \) is a constant. The function \( r(r_i, t) \) then obeys the differential equation

\[
\ddot{r} = -\frac{G M(r_i)}{r_i^2} + \gamma r_n(r_i, t).
\]

This equation in fact applies to any form of mass density, so we can apply it to a universe in which the mass density is dominated by the energy of photons. Recall that mass and energy are equivalent, the mass density of the gas of photons falls off as \( 1/R^4 \), but in addition the frequency (and hence the energy) of each photon redshifts in proportion to \( 1/R \). Since mass and energy are equivalent, the mass density of the gas of photons falls off as \( 1/R^4 \). How does \( R(t) \) behave for a photon-dominated universe?

For a flat (i.e., \( k = 0 \)) matter-dominated universe we learned that the scale factor \( R(t) \) is proportional to \( t^{2/3} \). How does \( R(t) \) behave for a photon-dominated universe?
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a) (6 points) As done in the lecture notes, we define
\[ u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \]
Write the differential equation obeyed by \( u \).

b) (6 points) For what value of the power \( n \) is the differential equation found in part (a) independent of \( r_i \)?

c) (8 points) Write the initial conditions for \( u \) which, when combined with the differential equation found in (a), uniquely determine the function \( u \).

d) (15 points) If all is going well, then you have learned that for a certain value of \( n \), the function \( u(r_i, t) \) will in fact not depend on \( r_i \), so we can define \( R(t) \equiv u(r_i, t) \).
Show that the differential equation for \( R \) can be integrated once, as described in the previous paragraph.

PROBLEM 17: DID YOU DO THE READING?
The following problem was taken from Problem 1, Quiz 1, 2004, where each part counted 5 points, for a total of 25 points. The reading assignment included the first three chapters of Ryden, Introduction to Cosmology, and the first three chapters of Weinberg, The First Three Minutes.

(a) In 1826, the astronomer Heinrich Olber wrote a paper on a paradox regarding the night sky. What is Olber's paradox? What is the primary resolution of it?

(b) What is the value of the Newtonian gravitational constant \( G \) in Planck units? The Planck length is of the order of \( 10^{-35} \) m, \( 10^{-15} \) m, \( 10^{15} \) m, or \( 10^{35} \) m?

(c) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it? (For the latter question, a simple "yes" or "no" will suffice.)

(d) In the "Standard Model" of the universe, what is the universe's density at about 1/100th of a second after the Big Bang? Indicate any condition that is being relaxed to make up for the missing density.

PROBLEM 18: SPECIAL RELATIVITY DOPPLER SHIFT

The following problem was taken from Problem 2, Quiz 1, 2004, where it counted 20 points.

Consider the Doppler shift of radio waves, for a case in which both the source and the observer are moving. Suppose the source is a spaceship moving with speed \( v_s \) relative to the space station Alpha-7, while the observer is on another spaceship, moving in the opposite direction from Alpha-7.

(a) (10 points) Use the results of part (a) to determine the Doppler shift of the radio waves as received by the observer. (Recall that radio waves are electromagnetic waves, just like light waves.)

(b) (10 points) Calculate the Doppler shift of the radio waves as received by the observer.

(c) (10 points) Calculate the Doppler shift of the radio waves as received by the observer.

(d) (10 points) Calculate the Doppler shift of the radio waves as received by the observer.

The following problem was taken from Problem 2, Quiz 1, 2004, where it counted 15 points.

PROBLEM 18: SPECIAL RELATIVITY DOPPLER SHIFT

The following problem was taken from Problem 2, Quiz 1, 2004, where it counted 15 points.

Consider the Doppler shift of radio waves, for a case in which both the source and the observer are moving. Suppose the source is a spaceship moving with speed \( v_s \) relative to the space station Alpha-7, while the observer is on another spaceship, moving in the opposite direction from Alpha-7.

(a) (10 points) Use the results of part (a) to determine the Doppler shift of the radio waves as received by the observer. (Recall that radio waves are electromagnetic waves, just like light waves.)

(b) (10 points) Calculate the Doppler shift of the radio waves as received by the observer.

(c) (10 points) Calculate the Doppler shift of the radio waves as received by the observer.

(d) (10 points) Calculate the Doppler shift of the radio waves as received by the observer.

The following problem was taken from Problem 2, Quiz 1, 2004, where it counted 15 points.
SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?
(35 points)

a) Doppler predicted the Doppler effect in 1842.

b) Most of the stars of our galaxy, including our sun, lie in a flat disk. We therefore see much more light when we look across the plane of the disk than when we look in any other direction.

c) Hubble's original paper on the expansion of the universe was based on a study of only 18 galaxies. I am not sure why Weinberg refers to 18 galaxies, but it is possible that the text of Hubble's article indicated that 18 of these galaxies were measured with more reliability than the rest.

d) During a time interval in which the linear size of the universe grows by 1%, the horizon distance grows by more than 1%.

To see why, note that the horizon distance is equal to the scale factor times the comoving horizon distance. The scale factor grows by 1% during this time interval, but the comoving horizon distance also grows, since light from the distant galaxies has had more time to reach us.

e) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

f) (i) the average distance between photons:

A photon's distance is proportional to the size of the universe.

Photons are neither created nor destroyed, so the only effect is that the average distance between them is stretched with the expansion of the universe.

(ii) the typical wavelength of the radiation:

A photon's wavelength is also proportional to the size of the universe.

Since the universe expands uniformly, all distances grow by the same factor. Since the universe expands uniformly, all distances grow by the same factor. Since the universe expands uniformly, all distances grow by the same factor.

(iii) the number density of photons in the radiation:

The number density is inversely proportional to the cubic power of the size of the universe.

From (i), the average distance between photons grows in proportion to the size of the universe. Since the volume of a cube is proportional to the cube of its side, the average distance between photons grows as the cube of the size of the universe.

(iv) the energy density of the radiation:

The energy density is inversely proportional to the fourth power of the size of the universe.

The energy of each photon is proportional to its frequency, and hence inversely proportional to its wavelength. So from (ii) the energy of each photon is inversely proportional to its wavelength. The energy of each photon is inversely proportional to the wavelength.

Therefore, the energy density of the radiation is inversely proportional to the fourth power of the size of the universe.

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(v) the temperature of the radiation:
inversely proportional to the size of the universe
(The temperature is directly proportional to the average energy of a photon, which according to (iv) is inversely proportional to the size of the universe.)

PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY

25 points

a) According to Eq. (3.7),
\[ H(t) = \frac{1}{R(t)} \frac{dR}{dt} \]
So in this case
\[ \frac{1}{R(t)} \frac{dR}{dt} = H_0, \]
which can be rewritten as
\[ \frac{dR}{R} = H_0 dt. \]
Integrating,
\[ \ln R = H_0 t + c, \]
where \( c \) is a constant of integration. Exponentiating,
\[ R = e^{H_0 t}, \]
where \( b = e^c \) is an arbitrary constant.

b) Consider a cube of side \( \ell \) in the comoving coordinates system diagram. The physical length of each side is then \( R(t) \ell \), so the physical volume is
\[ V(t) = \frac{4}{3} \pi R^3(t) \ell^3. \]
Since the mass density is fixed at \( \rho = \rho_0 \), the total mass inside this cube at any time is given by
\[ \rho_0 \times \frac{4}{3} \pi R^3(t) \ell^3. \]
In the absence of matter creation the total mass within a comoving volume would not change, so the increase in mass described by the above equation must be attributed to matter creation. The rate of matter creation per unit time per unit volume is then
\[ \frac{dM}{dt} = \frac{3}{H_0} \frac{dR}{dt} \rho_0 = \frac{3}{H_0} H_0 = 3 \rho_0. \]

You were not asked to insert numbers, but it is worthwhile to consider the numerical value after the exam, to see what this answer is telling us. Suppose \( H_0 = 70 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1} \), and take \( \rho_0 \) to be the critical density, \( \rho_c = \frac{3}{8} \pi G H_0^2 \).

Then
\[ \frac{3}{H_0} H_0 = \frac{1}{10} \text{H} \]
which can be rewritten as
\[ \frac{1}{10} H = \frac{M}{\rho} \]
where \( \rho = \rho_c \) is an arbitrary constant.

To put this number into more meaningful terms, note that the mass of a hydrogen atom is \( 1.67 \times 10^{-27} \text{ kg} \), and that 1 year = \( 3.15 \times 10^7 \) s. The rate of matter creation required for the steady-state universe theory can then be expressed as number of hydrogen atoms per cubic meter per billion years! Needless to say, such a rate of matter creation is far too large by any standard. The physical length of each side is then \( R(t) \ell \), so the physical volume

PROBLEM 3: DID YOU DO THE READING?

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b) Kant proposed that the faint nebulae seen in the sky are distant galaxies, similar to the Milky Way.

c) The Milky Way galaxy has a diameter of about 80,000 light-years, and a thickness of 6,000 light-years.

d) The mathematical theory of an expanding universe, in the context of general relativity, was invented by Alexandre Friedmann in 1922. (Actually the 1922 paper discussed only closed universes, but Friedmann published a second paper on open universes in 1924.) Willem de Sitter published his model of the universe in 1917. De Sitter's model was initially believed to be static, but it was later discovered that it appeared static only because it was written in peculiar coordinates—in fact it was also an expanding model. While Friedmann's equations described the general case of a homogeneous isotropic expanding universe, de Sitter's model was more specific: it was a model devoid of matter, with the expansion driven by a positive cosmological constant. The intended answer for this question was Friedmann, but full credit was given for either Friedmann or de Sitter.

e) It was Bernard Burke who told Arno Penzias about the prediction of radio noise from the big bang.

PROBLEM 4: AN EXPONENTIALLY EXPANDING UNIVERSE

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by

$$ H = \frac{\dot{R}}{R} $$

So

$$ H = \frac{\chi R_0 e^{\chi t}}{R_0 e^{\chi t}} = \chi. $$

(b) According to Eq. (3.8), the coordinate velocity of light is given by

$$ \frac{dx}{dt} = c \frac{R_0 e^{-\chi t}}{R(t)} = c R_0 e^{-\chi t}. $$

Integrating,

$$ x(t) = c R_0 \int_0^t e^{-\chi t'} dt' = c R_0 \left[ \frac{-1}{\chi} e^{-\chi t'} \right]_0^t = c \chi R_0 \left( 1 - e^{-\chi t} \right). $$

(c) From Eq. (3.11), or from the front of the quiz, one has

$$ 1 + z = \frac{R(t)}{R(t_e)} $$

Here $t_e = 0$, so

$$ (1) - \chi = 1 + z. $$

(d) The coordinate distance is $x(t_r)$, where $x(t)$ is the function found in part (b), and $t_r$ is the time found in part (c). So

$$ e^{\chi t_r} = 1 + z, $$

and

$$ x(t_r) = c \chi R_0 \left( 1 - e^{-\chi t} \right) = c \chi R_0 \left( 1 - \frac{1}{1 + z} \right). $$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$ x_p(t_r) = R(t_r) x(t_r) = c \chi \frac{R_0}{1 + z} \left( 1 - e^{-\chi t} \right). $$

PROBLEM 5: "DID YOU DO THE READING?"

(a) The distinguishing quantity is $\Omega \equiv \frac{\rho}{\rho c^2}$. The universe is open if $\Omega < 1$, flat if $\Omega = 1$, or closed if $\Omega > 1$. (q) The distinguishing quantity is $\zeta / \chi$, where $\zeta (\chi) = 1$ is the function found in part (p).

(b) The temperature of the microwave background today is about 3 Kelvin. (The best determination to date* was made by the COBE satellite, which measured the cosmic microwave background spectrum from the full COBE FIRAS data. The Cosmic Microwave Background Spectrum from the Full COBE FIRAS Data, D.J. Fixsen, E.S. Cheng, J.C. Patch, J.M. Gales, J.C. Mather, R.A. Shafer, and E.L. Wright, The Astrophysical Journal, vol. 473, p. 576 (1996).)

(c) The intended answer for (b) is $\Omega_L / \Omega = H$.

So

$$ \frac{\Omega_L}{\Omega} = H. $$

(d) The intended answer for (c) is $\Omega_L / \Omega = H$.

(e) From Eq. (3.11), or from the front of the quiz, one has

$$ (1) \cdot \chi = \frac{\Omega}{\Omega_0} \frac{R_0}{R(t_e)} = \frac{\rho}{\rho c^2}. $$

(f) According to Eq. (3.12), the coordinate velocity of light is given by

$$ \frac{dx}{dt} = c \frac{R_0 e^{-\chi t}}{R(t)} = c R_0 e^{-\chi t}. $$

Integrating,

$$ x(t) = c R_0 \int_0^t e^{-\chi t'} dt' = c R_0 \left[ \frac{-1}{\chi} e^{-\chi t'} \right]_0^t = c \chi R_0 \left( 1 - e^{-\chi t} \right). $$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$ x_p(t_r) = R(t_r) x(t_r) = c \chi \frac{R_0}{1 + z} \left( 1 - e^{-\chi t} \right). $$

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The physical separation at $t'$ is given by the scale factor times the coordinate distance at time $t$.

The answer to this latter question is then

$$e^{t'/r} - e^{t'/r} = \frac{\Omega}{\Omega + 1}$$

which fills the universe, but one has not said, for example, how the distance between the particle and origin varies with time. The answer to this latter question is then determined by the evolution of the scale factor,

$$\left(\frac{\Omega}{\Omega + 1}\right)^2 = e^{t'/r} - e^{t'/r}$$

The physical separation at $t'$ is given by the scale factor times the coordinate distance at time $t$.

The coordinate distance at time $t$ is equal to the time $t$ coordinate distance.

Thus, the key to this problem is to work in comoving coordinates.

The absolute luminosity ($L$) of a Cepheid variable is equal to the starting standard, by a factor of $5$ to $10$. From the period one can estimate the absolute luminosity of the star, and then one uses the apparent luminosity and the 1929.

The cosmic microwave background is observed to be highly isotropic. From the period one can estimate the absolute luminosity of the star, and then one uses the apparent luminosity and the 1929.

The cosmic microwave background is observed to be highly isotropic.
PROBLEM 7: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION

(40 points)

(a) (5 points)

The cosmological redshift is given by the usual form,

\( 1 + z = \frac{R(t_0)}{R(te)} \).

For light emitted by an object at time \( te \), the redshift of the received light is

\( 1 + z = \frac{R(t_0)}{R(te)} = \gamma \frac{t_0}{te} \).

So,

\( z = \gamma \frac{t_0}{te} - 1 \).

(b) (5 points)

The coordinates \( t_0 \) and \( te \) are cosmic time coordinates. The "look-back" time as defined in the exam is then the interval \( t_0 - te \). We can write this as

\( t_0 - te = t_0 \left[ 1 - \frac{t_0}{te} \right] \gamma \).

We can use the result of part (a) to eliminate \( te/t_0 \) in favor of \( z \). From (a),

\( \frac{te}{t_0} = (1 + z) - \frac{1}{\gamma} \).

Therefore,

\( t_0 - te = t_0 \left[ 1 - \frac{1}{\gamma} \right] \).

(c) (10 points)

The present value of the physical distance to the object, \( \ell_p(t_0) \), is found from

\( \ell_p(t_0) = \frac{R(t_0)}{c} \int_{te}^{t_0} cR(t) \, dt \).

Calculating this integral gives

\( \ell_p(t_0) = \frac{ct_0}{\gamma} \left[ 1 - \gamma \frac{t_0}{te} \right] \).

Factoring \( t_0/\gamma \) out of the parentheses gives

\( \ell_p(t_0) = \frac{ct_0}{\gamma} \left[ 1 - (1 + z) \gamma \right] \).

This can be written in terms of \( z \) and \( \gamma \) using the result of part (a) as well.

\( \left[ \frac{t_0}{\gamma} \right] \frac{1}{1 - \gamma} = 0 \).

Finally then,

\( \frac{t_0}{\gamma} = \frac{\gamma H(t_0)}{(\gamma H)(t_0)} = \gamma H \).

This can be rewritten in terms of \( z \) and \( \gamma \) using the result of part (a) as well.

\( \left[ \frac{t_0}{\gamma} \right] = \frac{1}{1 - \gamma} = 0 \).

(d) (10 points)

A nearly identical problem was worked through in Problem 8 of Problem Set 1. The energy of the observed photons will be redshifted by a factor of \( (1 + z) \). In addition the rate of arrival of photons will be redshifted relative to the rate of photon emission, reducing the flux by another factor of \( (1 + z) \). Consequently, the observed power will be redshifted by two factors of \( (1 + z) \) to\( P/((1 + z)^2) \).

Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance \( \ell_c \).

\( [z + 1] \frac{z}{z + 1} \frac{1}{z + 1} \).
Problem 8: Did you do the reading?

We introduced the Andromeda Nebula in Lecture Notes 3. In 1609 Galileo used the telescope to observe the Andromeda Nebula. In 1923, Individual stars in the Andromeda Nebula were resolved by Hubble. So Individual stars in the Andromeda Nebula were resolved by Hubble.

1+πℓ/4)

\[\left[\frac{1}{\sqrt{\frac{\zeta}{c}}}\right] \times \frac{\Delta z}{\Delta t} = \zeta + 1\]

\[\left[\frac{1}{\sqrt{\frac{\zeta}{c}}}\right] \times \frac{\Delta z}{\Delta t} = \zeta + 1\]

\[\frac{\Delta z}{\Delta t} = \frac{2 + 1}{2 - 1}\]

\[\zeta + 1 = \frac{2 + 1}{2 - 1}\]

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\[\Delta \zeta = \frac{2 + 1}{2 - 1}\]
At this time it will have traveled a coordinate distance \(\ell\), where \(c\) is the surface of the sun.

\[
\left[ \frac{1}{s^2} \right] \left[ \frac{\ell}{c^2} \right] = \frac{c^2}{H_0^2} = \frac{1}{s^2} \frac{\ell}{c^2} = \frac{\ell}{c^2} 
\]

\[
\left[ \frac{1}{s^2} \right] \left[ \frac{\ell}{c^2} \right] = \frac{c^2}{H_0^2} = \frac{1}{s^2} \frac{\ell}{c^2} = \frac{\ell}{c^2}
\]

\[
\left[ \frac{1}{s^2} \right] \left[ \frac{\ell}{c^2} \right] = \frac{c^2}{H_0^2} = \frac{1}{s^2} \frac{\ell}{c^2} = \frac{\ell}{c^2}
\]

\[a)\] In general, the Hubble constant is given by

\[
H = \frac{\ell}{c^2}
\]

\[b)\] In general, the (physical) horizon distance is given by

\[
\frac{\ell}{c^2} = \frac{1}{s^2} \frac{\ell}{c^2} = \frac{\ell}{c^2}
\]

\[c)\] The coordinate speed of light is

\[
H = \frac{\ell}{c^2}
\]

\[d)\] The physical separation is just the scale factor times the coordinate separation.

\[
\frac{\ell}{c^2} = \frac{1}{s^2} \frac{\ell}{c^2} = \frac{\ell}{c^2}
\]

\[e)\] Let \(t_1\) be the time at which the lightpulse is equidistant from the two galaxies. So the coordinate distance that light

\[
ct = \frac{\ell}{c^2}
\]

\[f)\] Any patch of the night sky would look as bright as the surface of the sun, so the entire sky will appear as bright as the horizon

\[
\frac{\ell}{c^2} = \frac{1}{s^2} \frac{\ell}{c^2} = \frac{\ell}{c^2}
\]

\[g)\] False. [No chemical reactions are seen.]

\[h)\] True. [The microwave radiation can boost the CN molecule from its ground state to a low-lying excited state.]

\[i)\] False. [The CN molecule can rotate about each other. The population of this low-lying excited state is therefore

\[
\frac{\ell}{c^2} = \frac{1}{s^2} \frac{\ell}{c^2} = \frac{\ell}{c^2}
\]

\[j)\] False. [Precise data was not obtained until the COBE satellite, in 1990.

\[k)\] The Hubble constant is determined by the intensity of the microwave radiation. This population

\[
\frac{\ell}{c^2} = \frac{1}{s^2} \frac{\ell}{c^2} = \frac{\ell}{c^2}
\]

\[l)\] True. [The microwave background research in the 1970s, before was no connection was made with cosmology.

\[m)\] The COBE satellite in the 1990s, before was no connection was made with cosmology.

\[n)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[o)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[p)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[q)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[r)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[s)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[t)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[u)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[v)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[w)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[x)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[y)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[z)\] The microwave background research in the 1970s, before was no connection was made with cosmology.

\[\text{PROBLEM 9: A FLAT UNIVERSE WITH } H(0)\]

\[\frac{\ell}{c^2} \propto (H(0))^2\]
apply to the rate of arrival of photons, so the rate of photon arrival at the sphere given by
\[
\frac{A}{R(t)^2} = \frac{A}{R(t)^2}.
\]
Next consider the rate of arrival of the photons at the sphere. In lecture we figured out that if a periodic wave is emitted at time \( t_0 \) and observed at time \( t \), that the rate of arrival of the wave crests will be slower than the rate of emission by a redshift factor \( 1 + z \). The detector therefore occupies a fraction of the sphere given by
\[
\frac{A}{R(t)^2} = \frac{A}{R(t)^2}.
\]
so this fraction of the emitted photons will strike the detector.

The energy from the quasar will radiate uniformly on the sphere. The detector has a physical area \( A \). So in the comoving coordinate picture its area in square
notches would be \( A/R(t)^2 \). The detector therefore occupies a fraction of the sphere given by
\[
\frac{A}{R(t)^2} = \frac{A}{R(t)^2}.
\]

Substituting the expression for \( t \), one has
\[
\frac{t}{t_0} = \frac{c}{R(t)}.
\]

As found in part (d), the physical distance that the light travels between
emission by a redshift factor \( 1 + z \).

According to Hubble's law, the speed is equal to Hubble's constant times the
physical distance. By combining the answers to parts (a) and (d), one has

\[
v = H(t) \ell(t)
\]

which is exactly equal to the horizon distance. It is a general rule that the
horizon distance corresponds to infinite redshift. It is a general rule that the

As \( z \to \infty \), this expression approaches

\[
\lim_{z \to \infty} \frac{t}{t_0} = \frac{2^5}{2^5}.
\]
Using the answer from (c) and integrating the left-hand side,
\[ z \int [1 - v/c(z + 1)] \frac{v}{c(z + 1)} d\frac{v}{c} = t. \]

In the above expression for \( t \), the answer to part (c) is used.

The problem is worded so that \( V_A \) and \( V_B \) are not, yet the answer to part (d) does need to use the above expression for \( t \) to determine the form of any coordinate system, but the underlying physics can sometimes be obscured by a peculiar choice of coordinates. A change of coordinates can not only change the apparent geometry of space, but it can also mix up space and time in ways that make the geometric meaning of coordinates in one coordinate system disappear. The de Sitter model was first written down in coordinates that made it look static, so everyone believed it was. Later Arthur Eddington and Hermann Weyl (independently) calculated the trajectories of test particles, discovering that they flew apart.

PROBLEM 10: DID YOU DO THE READING?

8.286 QUIZ 1 REVIEW PROBLEM SOLUTIONS, FALL 2005 p. 33
Regardless of its name, recombination was crucial for the clumping of matter into galaxies and stars, because the pressure of the photons in the early universe was enormous. When the matter was ionized, the free electrons interacted strongly with the photons, so the pressure of these photons prevented the matter from clumping. After recombination, however, the matter became very transparent to radiation, and the pressure of the radiation became ineffective.

Incidentally, at roughly the same time as recombination (with big uncertainties), the mass density of the universe changed from being dominated by radiation (photons and neutrinos) to being dominated by nonrelativistic matter. There is no known underlying connection between these two events, and it seems to be something of a coincidence that they occurred at about the same time. The transition from radiation-domination to matter-domination also helped to promote the clumping of matter, but the effect was much weaker—because of the very high velocity of photons and neutrinos, their pressure remained a significant force even after their mass density became much smaller than that of matter.

**Problem 11: Another flat universe with**

\[
R(t) \propto t^{3/5}
\]

(a) According to Eq. (3.7) of the Lecture Notes, 

\[
H(t) = \frac{1}{R(t)} \frac{dR}{dt}
\]

For the special case of 

\[
R(t) = bt^{3/5}
\]

this gives 

\[
H(t) = \frac{1}{bt^{3/5}} \frac{3}{5} = \frac{3}{5} t
\]

(b) According to Eq. (3.8) of the Lecture Notes, the coordinate velocity of light (in comoving coordinates) is given by 

\[
\frac{dx}{dt} = \frac{c}{R(t)}
\]

Since galaxies A and B have physical separation \( \ell_0 \) at time \( t_1 \), their coordinate separation is given by 

\[
\frac{\ell}{c} = \frac{\ell_0}{bt^{3/5}}
\]

The radio signal must cover this coordinate distance in the time interval from \( t_1 \) to \( t_2 \), which implies that 

\[
\int_{t_1}^{t_2} \frac{c}{R(t)} dt = \frac{\ell_0}{bt^{3/5}}
\]

Using the expression for \( R(t) \) and integrating, 

\[
\frac{5}{2} c^2 b \left( t_2^{3/5} - t_1^{3/5} \right) = \ell_0
to solve for \( t_2 \), which implies that 

\[
t_2 = \left( 1 + \frac{2\ell_0}{5ct_1} \right)^{5/2} t_1
\]

(c) The method is the same as in part (a). The coordinate distance between the two galaxies is unchanged, but this time the distance must be traversed in the time interval from \( t_2 \) to \( t_3 \). So, 

\[
\int_{t_2}^{t_3} \frac{c}{R(t)} dt = \frac{\ell_0}{bt^{3/5}}
\]

Using the expression for \( R(t) \) and integrating, 

\[
\frac{5}{2} c^2 b \left( t_3^{3/5} - t_2^{3/5} \right) = \ell_0
\]

Solving for \( t_3 \) gives 

\[
t_3 = \left( 1 + \frac{4\ell_0}{5ct_1} \right)^{5/2} t_1
\]
In agreement with the previous answer,

\[ (\partial_t \nabla) \mathcal{O} + i \nabla \left( \frac{e_i}{v_i} \right) = e_i - e_j \]

Putting this back into the Taylor series gives

\[ \left( 1 + \frac{t}{\Delta t} \right) \left( \frac{1}{v_j} \right) = \frac{t}{\Delta t} \left( \frac{1}{v_j} \right) = 0 \rightarrow \nabla \left( \frac{e_i}{v_i} \right) = 0 \]

Using the first boxed answer to part (c) this can be simplified to

\[ \left( \frac{1}{t - \Delta t} \right) \frac{1}{v_j} \left( \frac{1}{t + \Delta t} \right) = \frac{t}{\Delta t} \]

which when simplified to \( t = \nabla \) becomes

\[ \left( \frac{1}{t + i} \right) \left[ \frac{1}{\Delta t} \left( \frac{1}{v_j} \right) + \frac{1}{t} \left( \frac{1}{v_j} \right) \right] = \frac{t}{\Delta t} \]

Evaluating the necessary derivative gives

\[ (\partial_t \nabla) \mathcal{O} + i \nabla \left( \frac{1}{v_j} \right) = \frac{t}{\Delta t} e_i - e_j \]

For those who prefer the Dirac Delta functions to part (d) can be replaced by \( \delta \) where in this order in the line in the numerator could equally well have been

\[ (\partial_t \nabla) \mathcal{O} + i \nabla \left( \frac{1}{v_j} \right) = \frac{t}{\Delta t} e_i - e_j \]

After being Taylor expanded in powers of \( \Delta t \) to first order one has

\[ (\partial_t \nabla) \mathcal{O} + i \nabla \left( \frac{1}{v_j} \right) = \frac{t}{\Delta t} e_i - e_j \]

Since the answer contains an explicit factor of \( \nabla \) the other factors can be

\[ (\partial_t \nabla) \mathcal{O} + i \nabla \left( \frac{1}{v_j} \right) = \frac{t}{\Delta t} e_i - e_j \]

and the interior between the sending of the two signals and the receipt of the responses will be a factor for times longer than the round trip of the radio signal, which travels a coordinate distance

\[ 2 \Delta t \]

Thus, the cosmic time interval between the receipt of the radio signal from a world

\[ \left( \frac{1}{v_j} \right) \frac{1}{\Delta t} \]

will be

\[ \frac{t}{\Delta t} \]

and a signal sent at

\[ \frac{t}{\Delta t} \]

will be

\[ \frac{t}{\Delta t} \]

So, from the formula at the front of the exam,

\[ \frac{t}{\Delta t} \left[ \frac{1}{v_j} \left( \frac{1}{v_j} \right) + \frac{1}{t} \left( \frac{1}{v_j} \right) \right] = \frac{t}{\Delta t} \]

which can be solved for \( \frac{t}{\Delta t} \) to give

\[ \frac{\Delta t}{\Delta t} \left[ \frac{1}{v_j} \left( \frac{1}{v_j} \right) - \frac{t}{\Delta t} \right] \frac{2 \Delta t}{\Delta t} \]

Information flows between the galaxies is still \( \frac{t}{\Delta t} \) the coordinate distance

\[ \left( \frac{1}{v_j} \right) \frac{1}{\Delta t} \]

The response is the same to what is measured on the galaxy B clock, which is \( \nabla \) the cosmic time interval between the receipt of the message and the response. The response is the same as what is measured on the galaxy B clock, which is \( \nabla \) the coordinate distance. The problem was to find the gravitational field inside the galaxy before the signal was received. The problem that becomes relevant to part (e) of the answer is that which involves a coordinate distance.
PROBLEM 12: THE DECELERATION PARAMETER

From the front of the exam, we are reminded that

\[ R = -\frac{4}{3} \pi G \rho R \]

and

\[ \left( \frac{\dot{R}}{R} \right)^2 = 8 \pi \frac{3}{3} G \rho - k c^2 R^2, \]

where a dot denotes a derivative with respect to time \( t \).

The critical mass density \( \rho_c \) is defined to be the mass density that corresponds to a flat \( (k = 0) \) universe, so from the equation above it follows that

\[ \left( \frac{\dot{R}}{R} \right)^2 = 8 \pi \frac{3}{3} G \rho c. \]

Substituting into the definition of \( q \), we find

\[ q = -\frac{\ddot{R}}{R} \left( t \right) \frac{\dot{R}}{R} \left( t \right) \frac{\dot{R}}{R} \left( t \right) = -\frac{\ddot{R}}{R} R \left( R \dot{R} \right)^2 = \left( \frac{4}{3} \pi G \rho \right) \left( \frac{3}{2} \pi G \rho_c \right) = \frac{1}{2} \Omega. \]

PROBLEM 13: A RADIATION-DOMINATED FLAT UNIVERSE

The flatness of the model universe means that \( k = 0 \), so

\[ \left( \frac{\dot{R}}{R} \right)^2 = 8 \pi \frac{3}{3} G \rho. \]

Since \( \rho \left( t \right) \propto \frac{1}{R^4 \left( t \right)} \), it follows that

\[ \frac{dR}{dt} = \text{constant}. \]

Rewriting this as

\[ R \frac{dR}{dt} = \text{constant} dt, \]

the indefinite integral becomes

\[ \frac{1}{2} R^2 = (\text{constant}) t + c', \]

where \( c' \) is a constant of integration. Different choices for \( c' \) correspond to different choices for the definition of \( t = 0 \). We will follow the standard convention of choosing \( c' = 0 \), which sets \( t = 0 \) to be the time when \( R = 0 \). Thus the above equation implies that \( H \propto \frac{1}{R} \), and therefore

\[ H = \text{constant} \frac{1}{R} \]

or

\[ \frac{\dot{R}}{R} \propto \frac{1}{H}. \]

The Hubble constant is then defined to be the mass density that corresponds to a flat universe, so from the equation above it follows that

\[ \frac{\ddot{R}}{R} - \frac{\dot{R}}{R} \frac{d\rho}{dx} = \left( \frac{H}{\dot{R}} \right) \frac{\dot{R}}{R} = \frac{1}{H^2} \frac{\dot{R}}{R} - \frac{d\rho}{dx} = \frac{\rho}{H^2}. \]

The evolution of an open, matter-dominated universe is described by the fol-

PROBLEM 14: EVOLUTION OF AN OPEN UNIVERSE

For a photon-dominated flat universe

\[ \frac{\dot{R}}{R} \propto \frac{1}{H} \]

we have

\[ \frac{\dot{R}}{R} \propto \frac{1}{H} \]

and

\[ \frac{d\rho}{dx} \propto \frac{\rho}{H^2}, \]

where \( \rho \) is a constant of integration. Different choices for \( \rho \) correspond to different

\[ \rho + \frac{\dot{R}}{R} \frac{d\rho}{dx} = \frac{\rho}{H^2}. \]

The indefinite integral becomes

\[ \rho \frac{\dot{R}}{R} + \frac{\dot{R}}{R} \frac{d\rho}{dx} = \frac{\rho}{H^2} \]

or

\[ \frac{\dot{R}}{R} \propto \frac{\rho}{H^2}. \]

From the front of the exam, we are reminded that

The Hubble constant is then defined to be the mass density that corresponds to a flat universe, so from the equation above it follows that

\[ \frac{\ddot{R}}{R} - \frac{\dot{R}}{R} \frac{d\rho}{dx} = \left( \frac{H}{\dot{R}} \right) \frac{\dot{R}}{R} = \frac{1}{H^2} \frac{\dot{R}}{R} - \frac{d\rho}{dx} = \frac{\rho}{H^2}. \]

and

\[ \frac{\rho}{H^2} = \frac{\rho}{H^2}. \]
PROBLEM 15: ANTICIPATING A BIG CRUNCH

The critical density is given by
\[ \rho_c = \frac{3}{H_0^2} \frac{8}{\pi G} , \]
so the mass density is given by
\[ \rho = \Omega \rho_c = 2 \rho_c = \frac{3}{H_0^2} \frac{4}{\pi G} . \]

Substituting this relation into
\[ H^2 = \frac{8}{3} \pi G \rho - \frac{kc^2}{R^2} , \]
we find
\[ H^2 = 2 H_0^2 - \frac{kc^2}{R^2} , \]
from which it follows that
\[ R \sqrt{k} = c H_0 . \]

Now use
\[ \alpha = \frac{4}{3} \pi G \rho R^3 / k^{3/2} c^2 . \]
Substituting the values we have from Eqs. (1) and (2) for \( \rho \) and \( R/\sqrt{k} \), we have
\[ \alpha = c H_0 . \]

To determine the value of the parameter \( \theta \), use
\[ R \sqrt{k} = \alpha (1 - \cos \theta) , \]
which when combined with Eqs. (2) and (3) implies that \( \cos \theta = 0 \).
The equation \( \cos \theta = 0 \) has multiple solutions, but we know that the \( \theta \)-parameter for a closed matter-dominated universe varies between 0 and \( \pi \) during the expansion phase of the universe. Within this range, \( \theta = 0 \) represents the closed universe. Thus, the age of the universe at the time these measurements are made is given by
\[ t = \frac{\alpha c}{1 - \sin \theta} = \frac{1}{H_0} \left( \pi/2 - 1 \right) . \]

The total lifetime of the closed universe corresponds to \( \theta = 2 \pi \), or
\[ t_{\text{final}} = \frac{2 \pi \alpha c}{1 - \sin \theta} = 2 \pi H_0 , \]
so the time remaining before the big crunch is given by
\[ t_{\text{final}} - t = \frac{1}{H_0} \left( \frac{3 \pi}{2} + 1 \right) . \]

PROBLEM 16: A POSSIBLE MODIFICATION OF NEWTON’S LAW

OF GRAVITY (35 points)

(a) Substituting the equation for \( M(r_i) \), given on the quiz, into the differential equation for \( r \), also given on the quiz, one finds:
\[ \ddot{r} = -\frac{4}{3} \pi G r_i \rho_i r^2 + \gamma r n . \]

Dividing both sides of the equation by \( r_i \), one has
\[ \ddot{r} r_i = -\frac{4}{3} \pi G r^2 i \rho_i r^2 + \gamma r n r_i . \]

Substituting \( u = r/r_i \), this becomes
\[ \ddot{u} = -\frac{4}{3} \pi G \rho_i u^2 + \gamma u , \]
which when combined with Eqs. (1) and (2) implies that \( \gamma = 0 \).

(b) The only dependence on \( r_i \) occurs in the last term, which is proportional to \( r_n r_i \), and hence this dependence disappears if \( n = 1 \), since the zeroth power of any positive number is 1.

(c) This is exactly the same as the case discussed in the lecture notes. At \( t = 0 \), the universe is closed, with radius \( r_i \), and the expansion phase of the universe begins. Within this range, \( \theta = 0 \) represents the closed universe. Thus, the age of the universe at the time these measurements are made is given by
\[ t = \frac{\alpha c}{1 - \sin \theta} = \frac{1}{H_0} \left( \pi/2 - 1 \right) . \]
(a) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it?

(b) What is the value of the Newtonian gravitational constant \( G \) in Planck units?

(c) What is the radius of the Planck length? How is it obtained by a process of elimination as long as you remember that the Planck units are the smallest units of length possible in the universe?

(d) What is the Planck length? Is it of the order of \( 10^{-35} \) m, or \( 10^{-15} \) m, or \( 10^{-3} \) m? (Note that this answer could be obtained by a process of elimination as long as you remember that the Planck units are the smallest units of length possible in the universe.)

(e) What is the Olber's paradox? What is the primary resolution of it?
Note that it guarantees that \( |\alpha| \geq |\gamma| \) and \( \gamma \geq |\beta| \) so long as \( \gamma \geq |\beta| \) and \( |\beta| \geq |\alpha| \) which is true by the observer can be written as

\[
\frac{x^2}{\sigma^2} + 1 \geq \frac{x^2}{\sigma^2} + \gamma \geq \frac{x^2}{\sigma^2} \geq |\beta|^2
\]

By the observer can be written as

\[
\frac{x^2}{\sigma^2} + 1 \leq \frac{x^2}{\sigma^2} + \gamma \leq \frac{x^2}{\sigma^2} \leq |\beta|^2
\]

Alpharetta, the wavelength seen by the observer will be the same as if the photons are received by the observer can be written as

\[
\frac{x^2}{\sigma^2} + 1 \leq \frac{x^2}{\sigma^2} + \gamma \leq \frac{x^2}{\sigma^2} \leq |\beta|^2
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