MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe October 6, 2005
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 1

QUIZ DATE: Tuesday, October 18, 2005

NEW READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapters 4 and 5.

QUIZ COVERAGE: Lecture Notes 1 (sections on the Doppler shift only); Lecture Notes 3, 4, and 5; Problem Sets 1 and 2; Weinberg, Chapters 1-5, Ryden, Chapters 1-3. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems.** The starred problems are the ones that I recommend that you review most carefully: Problems 2, 4, 7, 12, 14, 15, and 18. The starred problems do not include any reading questions, but parts of the reading questions in these Review Problems may also recur on the upcoming quiz.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. Except for a few parts which are clearly marked, they are all problems that I would consider fair for the coming quiz. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, and 2004. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. Since the schedule and the number of quizzes has varied over the years, the coverage of this quiz will not necessarily be the same as Quiz 1 from previous years. In fact, the first quiz this year covers the same material as the first two quizzes in 2000, and the same as Quiz 1 from 2002 and 2004.

REVIEW SESSION: To help you study for the quiz, Jessie Shelton will hold a review session on Sunday, October 16, at 6:00 p.m. The location will be announced on the course website.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" at the beginning. For the first quiz, this useful information will be the following:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Flat ($\Omega \equiv \rho/\rho_c = 1$): $R(t) \propto t^{2/3}$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta) ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2} ,$$

$$\kappa \equiv -k .$$

PROBLEM 1: DID YOU DO THE READING? *(35 points)*

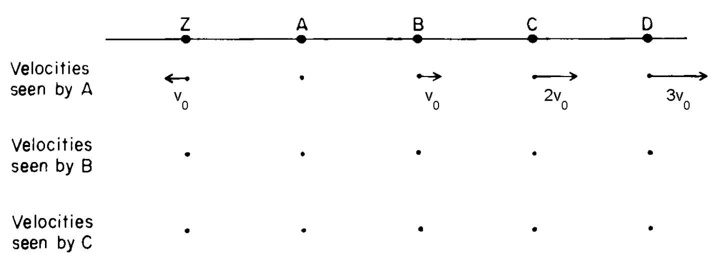
The following problem was Problem 1, Quiz 1, 2000. The parts were each worth 5 points.

a) The Doppler effect for both sound and light waves is named for Johann Christian Doppler, a professor of mathematics at the Realschule in Prague. He predicted the effect for both types of waves in xx42. What are the two digits xx?

b) When the sky is very clear (as it almost never is in Boston), one can see a band of light across the night sky that has been known since ancient times as the Milky Way. Explain in a sentence or two how this band of light is related to the shape of the galaxy in which we live, which is also called the Milky Way.

c) The statement that the distant galaxies are on average receding from us with a speed proportional to their distance was first published by Edwin Hubble in 1929, and has become known as Hubble's law. Was Hubble's original paper based on the study of 2, 18, 180, or 1,800 galaxies?

d) The following diagram, labeled *Homogeneity and the Hubble Law,* was used by Weinberg to explain how Hubble's law is consistent with the homogeneity of the universe:



The arrows and labels from the "Velocities seen by B" and the "Velocities seen by C" rows have been deleted from this copy of the figure, and it is your job to sketch the figure in your exam book with these arrows and labels included. (Actually, in Weinberg's diagram these arrows were not labeled, but the labels are required here so that the grader does not have to judge the precise length of hand-drawn arrows.)

e) The horizon is the present distance of the most distant objects from which light has had time to reach us since the beginning of the universe. The horizon changes with time, but of course so does the size of the universe as a whole. During a time interval in which the linear size of the universe grows by 1%, does the horizon distance

(i) grow by more than 1%, or

(ii) grow by less than 1%, or

(iii) grow by the same 1%?

f) Name the two men who in 1964 discovered the cosmic background radiation. With what institution were they affiliated?

g) At a temperature of 3000 K, the nuclei and electrons that filled the universe combined to form neutral atoms, which interact very weakly with the photons of the background radiation. After this process, known as "recombination," the background radiation expanded freely. Since recombination, how have each of the following quantities varied as the size of the universe has changed? (Your answers should resemble statements such as "proportional to the size of the universe," or "inversely proportional to the square of the size of the universe". The word "size" will be interpreted to mean linear size, not volume.)

(i) the average distance between photons

(ii) the typical wavelength of the radiation

(iii) the number density of photons in the radiation

(iv) the energy density of the radiation

(v) the temperature of the radiation

***PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY** *(25 points)*

The following problem was Problem 2, Quiz 1, 2000.

The steady-state theory of the universe was proposed in the late 1940s by Hermann Bondi, Thomas Gold, and Fred Hoyle, and was considered a viable model for the universe until the cosmic background radiation was discovered and its properties were confirmed. As the name suggests, this theory is based on the hypothesis that the large-scale properties of the universe do not change with time. The expansion of the universe was an established fact when the steady-state theory was invented, but the steady-state theory reconciles the expansion with a steady-state density of matter by proposing that new matter is created as the universe expands, so that the matter density does not fall. Like the conventional theory, the steady-state theory describes a homogeneous, isotropic, expanding universe, so the same comoving coordinate formulation can be used.

a) *(10 points)* The steady-state theory proposes that the Hubble constant, like other cosmological parameters, does not change with time, so $H(t) = H_0$. Find the most general form for the scale factor function $R(t)$ which is consistent with this hypothesis.

b) *(15 points)* Suppose that the mass density of the universe is ρ_0, which of course does not change with time. In terms of the general form for $R(t)$ that you found in part (a), calculate the rate at which new matter must be created for ρ_0 to remain constant as the universe expands. Your answer should have the units of mass per unit volume per unit time. *[If you failed to answer part (a), you will still receive full credit here if you correctly answer the question for an arbitrary scale factor function $R(t)$.]*

PROBLEM 3: DID YOU DO THE READING? *(25 points)*

The following problem was Problem 1, Quiz 1, 1994:

The following questions are worth 5 points each:

a) In 1750 the English instrument maker Thomas Wright published *Original Theory or New Hypothesis of the Universe*. In this book Wright described an astronomical object that is known today as the Crab Nebula, the solar system, the Milky Way, or the local supercluster?

b) In 1755 Immanuel Kant published his *Universal Natural History and Theory of the Heavens*. What new hypothesis was put forward in this book?

c) Estimate the diameter and thickness of the disk of the Milky Way galaxy. Any numbers within a factor of 2 of those given in Weinberg's book will be accepted.

d) The mathematical theory of an expanding universe was first published in 1922 by the Russian mathematician Alexandre Friedmann, the Dutch Astronomer Willem de Sitter, the American astronomer Edwin Hubble, or the Belgian cleric Georges Lemaître?

e) After discovering an inexplicable hiss coming from their radio telescope, Arno Penzias and Robert Wilson of Bell Laboratories learned that P.J.E. Peebles, a Princeton theorist, had calculated that the big bang would produce a background of cosmic radiation with a temperature today of 10° K. What MIT radio astronomer informed them of Peebles' work?

∗PROBLEM 4: AN EXPONENTIALLY EXPANDING UNIVERSE *(20 points)*

The following problem was Problem 2, Quiz 2, 1994, and had also appeared on the 1994 Review Problems. As is the case this year, it was announced that one of the problems on the quiz would come from either the homework or the Review Problems.

Consider a flat (i.e., a $k = 0$, or a Euclidean) universe with scale factor given by

$$R(t) = R_0 e^{\chi t} ,$$

where R_0 and χ are constants.

(a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

(b) *(5 points)* Let (x, y, z, t) be the coordinates of a comoving coordinate system. Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive x-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) *(5 points)* Suppose that we are living on a galaxy along the positive x-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift z. Express the time t_r at which we receive the pulse in terms of z, χ, and any relevant physical constants.

(d) *(5 points)* At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of z, χ, and any relevant physical constants.

PROBLEM 5: "DID YOU DO THE READING?"

(a) The assumptions of homogeneity and isotropy greatly simplify the description of our universe. We find that there are three possibilities for a homogeneous and isotropic universe: an open universe, a flat universe, and a closed universe. What quantity or condition distinguishes between these three cases: the temperature of the microwave background, the value of $\Omega = \rho/\rho_c$, matter vs. radiation domination, or redshift?

(b) What is the temperature, in Kelvin, of the cosmic microwave background today?

(c) Which of the following supports the hypothesis that the universe is isotropic: the distances to nearby clusters, observations of the cosmic microwave background, clustering of galaxies on large scales, or the age and distribution of globular clusters?

(d) Is the distance to the Andromeda Nebula (roughly) 10 kpc, 5 billion light years, 2 million light years, or 3 light years?

(e) Did Hubble discover the law which bears his name in 1862, 1880, 1906, 1929, or 1948?

(f) When Hubble measured the value of his constant, he found $H^{-1} \approx 100$ million years, 2 billion years, 10 billion years, or 20 billion years?

(g) Cepheid variables are important to cosmology because they can be used to estimate the distances to the nearby galaxies. What property of Cepheid variables makes them useful for this purpose, and how are they used?

(h) Cepheid variable stars can be used as estimators of distance for distances up to about 100 light-years, 10^4 light-years, 10^7 light years, or 10^{10} light-years?

[Note for 2004: this question was based on the reading from Joseph Silk's **The Big Bang**, *and therefore would be not be a fair question for this year.]*

(i) Name the two men who in 1964 discovered the cosmic background radiation. With what institution were they affiliated?

(j) At the time of the discovery of the cosmic microwave background, an active but independent effort was taking place elsewhere. P.J.E. Peebles had estimated that the universe must contain background radiation with a temperature of at least 10°K, and Robert H. Dicke, P.G. Roll, and D.T. Wilkinson had mounted an experiment to look for it. At what institution were these people working?

PROBLEM 6: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The following problem was Problem 3, Quiz 2, 1988:

Consider a flat universe filled with a new and peculiar form of matter, with a Robertson–Walker scale factor that behaves as

$$R(t) = bt^{1/3} .$$

Here b denotes a constant.

(a) If a light pulse is emitted at time t_e and observed at time t_o, find the physical separation $\ell_p(t_o)$ between the emitter and the observer, at the time of observation.

(b) Again assuming that t_e and t_o are given, find the observed redshift z.

(c) Find the physical distance $\ell_p(t_o)$ which separates the emitter and observer at the time of observation, expressed in terms of c, t_o, and z (i.e., without t_e appearing).

(d) At an arbitrary time t in the interval $t_e < t < t_o$, find the physical distance $\ell_p(t)$ between the light pulse and the observer. Express your answer in terms of c, t, and t_o.

∗PROBLEM 7: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(40 points)*

The following problem was Problem 3, Quiz 1, 2000.

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{\gamma} ,$$

where b and γ are constants. *[This universe differs from the matter-dominated universe described in the lecture notes in that ρ is not proportional to $1/R^3(t)$. Such behavior is possible when pressures are large, because a gas expanding under pressure can lose energy (and hence mass) during the expansion.]* For the following questions, any of the answers may depend on γ, whether it is mentioned explicitly or not.

a) *(5 points)* Let t_0 denote the present time, and let t_e denote the time at which the light that we are currently receiving was emitted by a distant object. In terms of these quantities, find the value of the redshift parameter z with which the light is received.

b) *(5 points)* Find the "look-back" time as a function of z and t_0. The look-back time is defined as the length of the interval in cosmic time between the emission and observation of the light.

c) *(10 points)* Express the present value of the physical distance to the object as a function of H_0, z, and γ.

d) *(10 points)* At the time of emission, the distant object had a power output P (measured, say, in ergs/sec) which was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux J from this object at the earth today? Express your answer in terms of P, H_0, z, and γ. *[Energy flux (which might be measured in erg-cm^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow.]*

e) *(10 points)* Suppose that the distant object is a galaxy, moving with the Hubble expansion. Within the galaxy a supernova explosion has hurled a jet of material directly towards Earth with a speed v, measured relative to the galaxy, which is comparable to the speed of light c. Assume, however, that the distance the jet has traveled from the galaxy is so small that it can be neglected. With what redshift z_J would we observe the light coming from this jet? Express your answer in terms of all or some of the variables v, z (the redshift of the galaxy), t_0, H_0, and γ, and the constant c.

PROBLEM 8: DID YOU DO THE READING? *(25 points)*

The following problem was Problem 1, Quiz 1, 1996:

The following questions are worth 5 points each.

a) In 1814-1815, the Munich optician Joseph Frauenhofer allowed light from the sun to pass through a slit and then through a glass prism. The light was spread into a spectrum of colors, showing lines that could be identified with known elements — sodium, iron, magnesium, calcium, and chromium. Were these lines dark, or bright *(2 points)*? Why *(3 points)*?

b) The Andromeda Nebula was shown conclusively to lie outside our own galaxy when astronomers acquired telescopes powerful enough to resolve the individual stars of Andromeda. Was this feat accomplished by Galileo in 1609, by Immanuel Kant in 1755, by Henrietta Swan Leavitt in 1912, by Edwin Hubble in 1923, or by Walter Baade and Allan Sandage in the 1950s?

c) Some of the earliest measurements of the cosmic background radiation were made indirectly, by observing interstellar clouds of a molecule called cyanogen (CN). State whether each of the following statements is true or false *(1 point each)*:

(i) The first measurements of the temperature of the interstellar cyanogen were made over twenty years before the cosmic background radiation was directly observed.

(ii) Cyanogen helps to measure the cosmic background radiation by reflecting it toward the earth, so that it can be received with microwave detectors.

(iii) One reason why the cyanogen observations were important was that they gave the first measurements of the equivalent temperature of the cosmic background radiation at wavelengths shorter than the peak of the black-body spectrum.

(iv) By measuring the spectrum of visible starlight that passes through the cyanogen clouds, astronomers can infer the intensity of the microwave radiation that bathes the clouds.

(v) By observing chemical reactions in the cyanogen clouds, astronomers can infer the temperature of the microwave radiation in which they are bathed.

d) In about 280 B.C., a Greek philosopher proposed that the Earth and the other planets revolve around the sun. What was the name of this person? *[Note for 2004: this question was based on readings from Joseph Silk's* **The Big Bang**, *and therefore is not appropriate for Quiz 1 of this year.]*

e) In 1832 Heinrich Wilhelm Olbers presented what we now know as "Olbers' Paradox," although a similar argument had been discussed as early as 1610 by Johannes Kepler. Olbers argued that if the universe were transparent, static, infinitely old, and was populated by a uniform density of stars similar to our sun, then one of the following consequences would result:

(i) The brightness of the night sky would be infinite.

(ii) Any patch of the night sky would look as bright as the surface of the sun.

(iii) The total energy flux from the night sky would be about equal to the total energy flux from the sun.

(iv) Any patch of the night sky would look as bright as the surface of the moon.

Which one of these statements is the correct statement of Olbers' paradox? *[Note for 2004: this question was based on readings from Joseph Silk's* **The Big Bang**, *and therefore is not appropriate for Quiz 1 of this year.]*

PROBLEM 9: A FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$

The following problem was Problem 3, Quiz 1, 1996:

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{3/5} ,$$

where b is a constant.

a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

b) *(5 points)* What is the physical horizon distance at time t?

c) *(5 points)* Suppose a light pulse leaves galaxy A at time t_A and arrives at galaxy B at time t_B. What is the coordinate distance between these two galaxies?

d) *(5 points)* What is the physical separation between galaxy A and galaxy B at time t_A? At time t_B?

e) *(5 points)* At what time is the light pulse equidistant from the two galaxies?

f) *(5 points)* What is the speed of B relative to A at the time t_A? (By "speed," I mean the rate of change of the physical distance with respect to cosmic time, $d\ell_p/dt$.)

g) *(5 points)* For observations made at time t, what is the present value of the physical distance as a function of the redshift z (and the time t)? What physical distance corresponds to $z = \infty$? How does this compare with the horizon distance? (Note that this question does not refer to the galaxies A and B discussed in the earlier parts. In particular, you should not assume that the light pulse left its source at time t_A.)

h) *(5 points)* Returning to the discussion of the galaxies A and B which were considered in parts (c)-(f), suppose the radiation from galaxy A is emitted with total power P. What is the power per area received at galaxy B?

i) *(5 points)* When the light pulse is received by galaxy B, a pulse is immediately sent back toward galaxy A. At what time does this second pulse arrive at galaxy A?

PROBLEM 10: DID YOU DO THE READING? *(20 points)*

The following questions were taken from Problem 1, Quiz 1, 1998:

The following questions are worth 5 points each.

a) In 1917, Einstein introduced a model of the universe which was based on his newly developed general relativity, but which contained an extra term in the equations which he called the "cosmological term." (The coefficient of this term is called the "cosmological constant.") What was Einstein's motivation for introducing this term?

b) When the redshift of distant galaxies was first discovered, the earliest observations were analyzed according to a cosmological model invented by the Dutch astronomer W. de Sitter in 1917. At the time of its discovery, was this model thought to be static or expanding? From the modern perspective, is the model thought to be static or expanding?

c) The early universe is believed to have been filled with thermal, or black-body, radiation. For such radiation the number density of photons and the energy density are each proportional to powers of the absolute temperature T. Say

$$\text{Number density} \propto T^{n_1}$$

$$\text{Energy density} \propto T^{n_2}$$

Give the values of the exponents n_1 and n_2.

d) At about 3,000 K the matter in the universe underwent a certain chemical change in its form, a change that was necessary to allow the differentiation of matter into galaxies and stars. What was the nature of this change?

PROBLEM 11: ANOTHER FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$ *(40 points)*

The following was Problem 3, Quiz 1, 1998:

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{3/5} ,$$

where b is a constant.

a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

b) *(10 points)* Suppose a message is transmitted by radio signal (traveling at the speed of light c) from galaxy A to galaxy B. The message is sent at cosmic time t_1, when the physical distance between the galaxies is ℓ_0. At what cosmic time t_2 is the message received at galaxy B? (Express your answer in terms of ℓ_0, t_1, and c.)

c) *(5 points)* Upon receipt of the message, the creatures on galaxy B immediately send back an acknowledgement, by radio signal, that the message has been received. At what cosmic time t_3 is the acknowledgment received on galaxy A? (Express your answer in terms of ℓ_0, t_1, t_2, and c.)

d) *(10 points)* The creatures on galaxy B spend some time trying to decode the message, finally deciding that it is an advertisement for Kellogg's Corn Flakes (whatever that is). At a time Δt after the receipt of the message, as measured on their clocks, they send back a response, requesting further explanation. At what cosmic time t_4 is the response received on galaxy A? In answering this part, you should not assume that Δt is necessarily small. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, Δt, and c.)

e) *(5 points)* When the response is received by galaxy A, the radio waves will be redshifted by a factor $1 + z$. Give an expression for z. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, t_4, Δt, and c.)

f) *(5 points; No partial credit)* If the time Δt introduced in part (d) is small, the time difference $t_4 - t_3$ can be expanded to first order in Δt. Calculate $t_4 - t_3$ to first order accuracy in Δt. (Express your answer in terms of ℓ_0, t_1, t_2, t_3, t_4, Δt, and c.) [Hint: while this part can be answered by using brute force to expand the answer in part (d), there is an easier way.]

***PROBLEM 12: THE DECELERATION PARAMETER**

The following problem was Problem 2, Quiz 2, 1992, where it counted 10 points out of 100.

Many standard references in cosmology define a quantity called the ***deceleration parameter*** q, which is a direct measure of the slowing down of the cosmic expansion. The parameter is defined by

$$q \equiv -\ddot{R}(t)\frac{R(t)}{\dot{R}^2(t)} .$$

Find the relationship between q and Ω for a matter-dominated universe. *[In case you have forgotten, Ω is defined by*

$$\Omega = \rho/\rho_c ,$$

where ρ is the mass density and ρ_c is the critical mass density (i.e., that mass density which corresponds to $k = 0$).]

PROBLEM 13: A RADIATION-DOMINATED FLAT UNIVERSE

We have learned that a matter-dominated homogeneous and isotropic universe can be described by a scale factor $R(t)$ obeying the equation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \; .$$

This equation in fact applies to any form of mass density, so we can apply it to a universe in which the mass density is dominated by the energy of photons. Recall that the mass density of nonrelativistic matter falls off as $1/R^3(t)$ as the universe expands; the mass of each particle remains constant, and the density of particles falls off as $1/R^3(t)$ because the volume increases as $R^3(t)$. For the photon-dominated universe, the density of photons falls of as $1/R^3(t)$, but in addition the frequency (and hence the energy) of each photon redshifts in proportion to $1/R(t)$. Since mass and energy are equivalent, the mass density of the gas of photons falls off as $1/R^4(t)$.

For a flat (i.e., $k = 0$) matter-dominated universe we learned that the scale factor $R(t)$ is proportional to $t^{2/3}$. How does $R(t)$ behave for a photon-dominated universe?

*PROBLEM 14: EVOLUTION OF AN OPEN UNIVERSE

The following problem was taken from Quiz 2, 1990, where it counted 10 points out of 100.

Consider an open, matter-dominated universe, as described by the evolution equations on the front of the quiz. Find the time t at which $R/\sqrt{\kappa} = 2\alpha$.

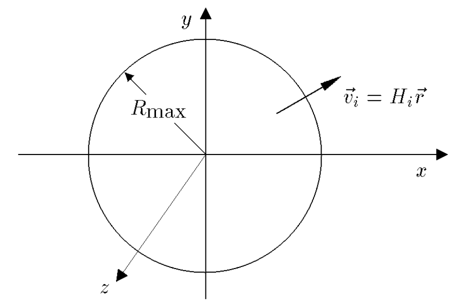
*PROBLEM 15: ANTICIPATING A BIG CRUNCH

Suppose that we lived in a closed, matter-dominated universe, as described by the equations on the front of the quiz. Suppose further that we measured the mass density parameter Ω to be $\Omega_0 = 2$, and we measured the Hubble "constant" to have some value H_0. How much time would we have before our universe ended in a big crunch, at which time the scale factor $R(t)$ would collapse to 0?

PROBLEM 16: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(35 points)*

The following problem was Problem 2, Quiz 2, 2000.

In Lecture Notes 4 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density ρ_i and an initial pattern of velocities corresponding to Hubble expansion: $\vec{v}_i = H_i\vec{r}$:



We denoted the radius at time t of a particle which started at radius r_i by the function $r(r_i, t)$. Assuming Newton's law of gravity, we concluded that each particle would experience an acceleration given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i,t)}\,\hat{r} \; ,$$

where $M(r_i)$ denotes the total mass contained initially in the region $r < r_i$, given by

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i \; .$$

Suppose that the law of gravity is modified to contain a new, repulsive term, producing an acceleration which grows as the nth power of the distance, with a strength that is independent of the mass. That is, suppose $\vec{g}$ is given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i,t)}\,\hat{r} + \gamma r^n(r_i,t)\,\hat{r} \; ,$$

where γ is a constant. The function $r(r_i, t)$ then obeys the differential equation

$$\ddot{r} = -\frac{GM(r_i)}{r^2(r_i,t)} + \gamma r^n(r_i,t) \; .$$

a) *(6 points)* As done in the lecture notes, we define

$$u(r_i, t) \equiv r(r_i, t)/r_i \ .$$

Write the differential equation obeyed by u.

b) *(6 points)* For what value of the power n is the differential equation found in part (a) independent of r_i?

c) *(8 points)* Write the initial conditions for u which, when combined with the differential equation found in (a), uniquely determine the function u.

d) *(15 points)* If all is going well, then you have learned that for a certain value of n, the function $u(r_i, t)$ will in fact not depend on r_i, so we can define

$$R(t) \equiv u(r_i, t) \ .$$

Show that the differential equation for R can be integrated once to obtain an equation related to the conservation of energy. The desired equation should include terms depending on R and $\dot{R}$, but not $\ddot{R}$ or any higher derivatives.

If all has not gone so well, you may not know what differential equation $R(t)$ obeys. If that is the case, then for a maximum of 12 points you can consider the generic equation

$$\ddot{R} + \frac{A}{R^p} + BR^q = 0 \ ,$$

where A and B are arbitrary constants, and p and q are positive integers, with $p > 1$. Show that this equation can be integrated once, as described in the previous paragraph.

PROBLEM 17: DID YOU DO THE READING?

The following problem was taken from Problem 1, Quiz 1, 2004, where each part counted 5 points, for a total of 25 points. The reading assignment included the first three chapters of Ryden, **Introduction to Cosmology**, *and the first three chapters of Weinberg,* **The First Three Minutes**.

(a) In 1826, the astronomer Heinrich Olber wrote a paper on a paradox regarding the night sky. What is Olber's paradox? What is the primary resolution of it?

(b) What is the value of the Newtonian gravitational constant G in Planck units? The Planck length is of the order of 10^{-35} m, 10^{-15} m, 10^{15} m, or 10^{35} m?

(c) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it? (For the latter question, a simple "yes" or "no" will suffice.)

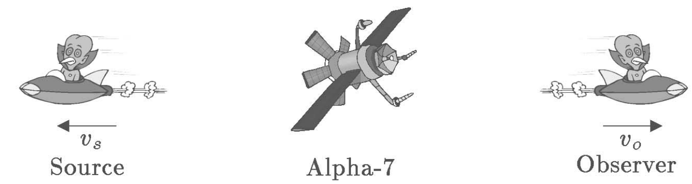
(d) In the "Standard Model" of the universe, when the universe cooled to about 3×10^a K, it became transparent to photons, and today we observe these as the Cosmic Microwave Background (CMB) at a temperature of about 3×10^b K. What are the integers a and b?

(e) What did the universe primarily consist of at about 1/100th of a second after the Big Bang? Include any constituent that is believed to have made up more than 1% of the mass density of the universe.

∗ PROBLEM 18: SPECIAL RELATIVITY DOPPLER SHIFT

The following problem was taken from Problem 2, Quiz 1, 2004, where it counted 20 points.

Consider the Doppler shift of radio waves, for a case in which both the source and the observer are moving. Suppose the source is a spaceship moving with a speed v_s relative to the space station Alpha-7, while the observer is on another spaceship, moving in the opposite direction from Alpha-7 with speed v_o relative to Alpha-7.



(a) *(10 points)* Calculate the Doppler shift z of the radio wave as received by the observer. (Recall that radio waves are electromagnetic waves, just like light except that the wavelength is longer.)

(b) *(10 points)* Use the results of part (a) to determine v_{tot}, the velocity of the source spaceship as it would be measured by the observer spaceship. *(8 points will be given for the basic idea, whether or not you have the right answer for part (a), and 2 points will be given for the algebra.)*

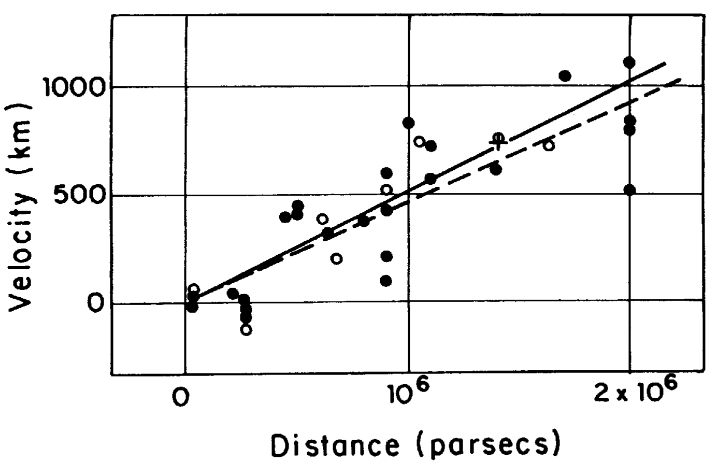
SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(35 points)*

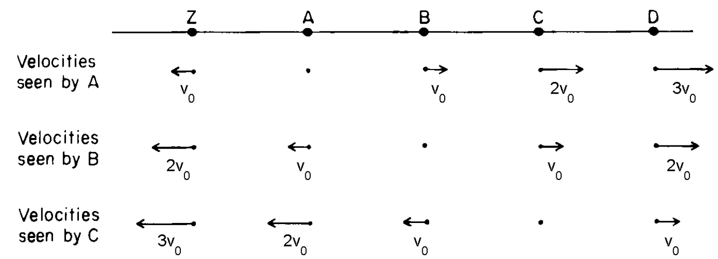
a) Doppler predicted the Doppler effect in **1842.**

b) Most of the stars of our galaxy, including our sun, lie in a flat disk. We therefore see much more light when we look out from earth along the plane of the disk than when we look in any other direction.

c) Hubble's original paper on the expansion of the universe was based on a study of only **18 galaxies.** Well, at least Weinberg's book says 18 galaxies. For my own book I made a copy of Hubble's original graph, which seems to show 24 black dots, each of which represents a galaxy, as reproduced below. The vertical axis shows the recession velocity, in kilometers per second. The solid line shows the best fit to the black dots, each of which represents a galaxy. Each open circle represents a group of the galaxies shown as black dots, selected by their proximity in direction and distance; the broken line is the best fit to these points. The cross shows a statistical analysis of 22 galaxies for which individual distance measurements were not available. I am not sure why Weinberg refers to 18 galaxies, but it is possible that the text of Hubble's article indicated that 18 of these galaxies were measured with more reliability than the rest.



d)



e) During a time interval in which the linear size of the universe grows by 1%, the horizon distance grows by **more than 1%.** To see why, note that the horizon distance is equal to the scale factor times the comoving horizon distance. The scale factor grows by 1% during this time interval, but the comoving horizon distance also grows, since light from the distant galaxies has had more time to reach us.

f) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

g) (i) the average distance between photons: **proportional to the size of the universe** (Photons are neither created nor destroyed, so the only effect is that the average distance between them is stretched with the expansion. Since the universe expands uniformly, all distances grow by the same factor.)

(ii) the typical wavelength of the radiation: **proportional to the size of the universe** (See Lecture Notes 3.)

(iii) the number density of photons in the radiation: **inversely proportional to the cube of the size of the universe** (From (i), the average distance between photons grows in proportion to the size of the universe. Since the volume of a cube is proportional to the cube of the length of a side, the average volume occupied by a photon grows as the cube of the size of the universe. The number density is the inverse of the average volume occupied by a photon.)

(iv) the energy density of the radiation: **inversely proportional to the fourth power of the size of the universe** (The energy of each photon is proportional to its frequency, and hence inversely proportional to its wavelength. So from (ii) the energy of each photon is inversely proportional to the size of the universe, and from (iii) the number density is inversely proportional to the cube of the size.)

(v) the temperature of the radiation: $\boxed{\text{inversely proportional to the size of the universe}}$ (The temperature is directly proportional to the average energy of a photon, which according to (iv) is inversely proportional to the size of the universe.)

PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY *(25 points)*

a) *(10 points)* According to Eq. (3.7),

$$H(t) = \frac{1}{R(t)}\,\frac{dR}{dt}\ .$$

So in this case

$$\frac{1}{R(t)}\,\frac{dR}{dt} = H_0\ ,$$

which can be rewritten as

$$\frac{dR}{R} = H_0\,dt\ .$$

Integrating,

$$\ln R = H_0\,t + c\ ,$$

where c is a constant of integration. Exponentiating,

$$\boxed{R = be^{H_0\,t}}\ ,$$

where $b = e^c$ is an arbitrary constant.

b) *(15 points)* Consider a cube of side ℓ_c drawn on the comoving coordinate system diagram. The physical length of each side is then $R(t)\,\ell_c$, so the physical volume is

$$V(t) = R^3(t)\,\ell_c^3\ .$$

Since the mass density is fixed at $\rho = \rho_0$, the total mass inside this cube at any given time is given by

$$M(t) = R^3(t)\,\ell_c^3\,\rho_0\ .$$

In the absence of matter creation the total mass within a comoving volume would not change, so the increase in mass described by the above equation

must be attributed to matter creation. The rate of matter creation per unit time per unit volume is then given by

$$\begin{aligned}\text{Rate} &= \frac{1}{V(t)}\,\frac{dM}{dt}\\ &= \frac{1}{R^3(t)\,\ell_c^3}\,3R^2(t)\frac{dR}{dt}\,\ell_c^3\,\rho_0\\ &= \frac{3}{R}\,\frac{dR}{dt}\,\rho_0\\ &= \boxed{3H_0\,\rho_0\ .}\end{aligned}$$

You were not asked to insert numbers, but it is worthwhile to consider the numerical value after the exam, to see what this answer is telling us. Suppose we take $H_0 = 70$ km-sec^{-1}-Mpc^{-1}, and take ρ_0 to be the critical density, $\rho_c = 3H_0^2/8\pi G$. Then

$$\begin{aligned}\text{Rate} &= \frac{9H_0^3}{8\pi G}\\ &= \frac{9\times(70\ \text{km-s}^{-1}\text{-Mpc}^{-1})^3}{8\pi\times 6.67260\times10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2}}\\ &= \frac{9\times(70\ \cancel{\text{km}}\text{-s}^{-1}\text{-}\cancel{\text{Mpc}}^{-1})^3}{8\pi\times 6.67260\times10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2}}\\ &\quad\times\left(\frac{1\ \cancel{\text{Mpc}}}{3.086\times10^{22}\ \cancel{\text{m}}}\right)^3\times\left(\frac{10^3\ \cancel{\text{m}}}{\cancel{\text{km}}}\right)^3\\ &= 6.26\times10^{-44}\ \text{kg-m}^{-3}\text{-s}^{-1}\ .\end{aligned}$$

To put this number into more meaningful terms, note that the mass of a hydrogen atom is 1.67×10^{-27} kg, and that 1 year $= 3.156\times10^7$ s. The rate of matter production required for the steady-state universe theory can then be expressed as roughly one hydrogen atom per cubic meter per billion years! Needless to say, such a rate of matter production is totally undetectable, so the steady-state theory cannot be ruled out by the failure to detect matter production.

PROBLEM 3: DID YOU DO THE READING?

a) Wright's book described the disk structure of the Milky Way.

b) Kant proposed that the faint nebulae seen in the sky are distant galaxies, similar to the Milky Way.

c) The Milky Way galaxy has a diameter of about 80,000 light-years, and a thickness of 6,000 light-years.

d) The mathematical theory of an expanding universe, in the context of general relativity, was invented by Alexandre Friedmann in 1922. (Actually the 1922 paper discussed only closed universes, but Friedmann published a second paper on open universes in 1924.) Willem de Sitter published his model of the universe in 1917. De Sitter's model was initially believed to be static, but it was later discovered that it appeared static only because it was written in peculiar coordinates— in fact it was also an expanding model. While Friedmann's equations described the general case of a homogeneous isotropic expanding universe, de Sitter's model was more specific: it was a model devoid of matter, with the expansion driven by a positive cosmological constant. The intended answer for this question was Friedmann, but full credit was given for either Friedmann or de Sitter.

e) It was Bernard Burke who told Arno Penzias about the prediction of radio noise from the big bang.

PROBLEM 4: AN EXPONENTIALLY EXPANDING UNIVERSE

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by

$$H = \dot{R}/R\ .$$

So

$$H = \frac{\chi R_0 e^{\chi t}}{R_0 e^{\chi t}} = \boxed{\chi\ .}$$

(b) According to Eq. (3.8), the coordinate velocity of light is given by

$$\frac{dx}{dt} = \frac{c}{R(t)} = \frac{c}{R_0} e^{-\chi t}\ .$$

Integrating,

$$\begin{aligned} x(t) &= \frac{c}{R_0}\int_0^t e^{-\chi t'}\,dt' \\ &= \frac{c}{R_0}\left[-\frac{1}{\chi}e^{-\chi t'}\right]_0^t \\ &= \boxed{\frac{c}{\chi R_0}\left[1 - e^{-\chi t}\right]\ .} \end{aligned}$$

(c) From Eq. (3.11), or from the front of the quiz, one has

$$1 + z = \frac{R(t_r)}{R(t_e)}\ .$$

Here $t_e = 0$, so

$$\begin{aligned} 1 + z &= \frac{R_0 e^{\chi t_r}}{R_0} \\ \Longrightarrow\quad e^{\chi t_r} &= 1 + z \\ \Longrightarrow\quad & \boxed{t_r = \frac{1}{\chi}\ln(1+z)\ .} \end{aligned}$$

(d) The coordinate distance is $x(t_r)$, where $x(t)$ is the function found in part (b), and t_r is the time found in part (c). So

$$e^{\chi t_r} = 1 + z\ ,$$

and

$$\begin{aligned} x(t_r) &= \frac{c}{\chi R_0}\left[1 - e^{-\chi t_r}\right] \\ &= \frac{c}{\chi R_0}\left[1 - \frac{1}{1+z}\right] \\ &= \frac{cZ}{\chi R_0(1+z)}\ . \end{aligned}$$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$\ell_p(t_r) = R(t_r)x(t_r) = \frac{cze^{\chi t_r}}{\chi(1+z)} = \boxed{\frac{cz}{\chi}\ .}$$

PROBLEM 5: "DID YOU DO THE READING?"

(a) The distinguishing quantity is $\Omega \equiv \rho/\rho_c$. The universe is open if $\Omega < 1$, flat if $\Omega = 1$, or closed if $\Omega > 1$.

(b) The temperature of the microwave background today is about 3 Kelvin. (The best determination to date* was made by the COBE satellite, which measured

* Astrophysical Journal, vol. **473**, p. 576 (1996): *The Cosmic Microwave Background Spectrum from the Full COBE FIRAS Data Sets*, D.J. Fixsen, E.S. Cheng, J.M. Gales, J.C. Mather, R.A. Shafer, and E.L. Wright.

the temperature as 2.728 ± 0.004 Kelvin. The error here is quoted with a 95% confidence limit, which means that the experimenters believe that the probability that the true value lies outside this range is only 5%.)

(c) The cosmic microwave background is observed to be highly isotropic.

(d) The distance to the Andromeda nebula is roughly 2 million light years.

(e) 1929.

(f) 2 billion years. Hubble's value for Hubble's constant was high by modern standards, by a factor of 5 to 10.

(g) The absolute luminosity (*i.e.*, the total light output) of a Cepheid variable star appears to be highly correlated with the period of its pulsations. This correlation can be used to estimate the distance to the Cepheid, by measuring the period and the apparent luminosity. From the period one can estimate the absolute luminosity of the star, and then one uses the apparent luminosity and the $1/r^2$ law for the intensity of a point source to determine the distance r.

(h) 10^7 light-years.

(i) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

(j) Princeton University.

PROBLEM 6: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

The key to this problem is to work in comoving coordinates.

[Some students have asked me why one cannot use "physical" coordinates, for which the coordinates really measure the physical distances. In principle one can use any coordinate system on likes, but the comoving coordinates are the simplest. In any other system it is difficult to write down the trajectory of either a particle or a light-beam. In comoving coordinates it is easy to write the trajectory of either a light beam, or a particle which is moving with the expansion of the universe (and hence standing still in the comoving coordinates). Note, by the way, that when one says that a particle is standing still in comoving coordinates, one has not really said very much about it's trajectory. One has said that it is moving with the matter which fills the universe, but one has not said, for example, how the distance between the particle and origin varies with time. The answer to this latter question is then determined by the evolution of the scale factor, $R(t)$.]

(a) The physical separation at t_o is given by the scale factor times the coordinate distance. The coordinate distance is found by integrating the coordinate velocity, so

$$\ell_p(t_o) = R(t_o)\int_{t_e}^{t_o} \frac{c\,dt'}{R(t')} = bt_o^{1/3}\int_{t_e}^{t_o}\frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2}ct_o^{1/3}\left[t_o^{2/3} - t_e^{2/3}\right]$$

$$\boxed{= \frac{3}{2}ct_o\left[1 - (t_e/t_o)^{2/3}\right]} \ .$$

(b) From the front of the exam,

$$1 + z = \frac{R(t_o)}{R(t_e)} = \left(\frac{t_o}{t_e}\right)^{1/3}$$

$$\Longrightarrow \quad \boxed{z = \left(\frac{t_o}{t_e}\right)^{1/3} - 1} \ .$$

(c) By combining the answers to (a) and (b), one has

$$\boxed{\ell_p(t_o) = \frac{3}{2}ct_o\left[1 - \frac{1}{(1+z)^2}\right]} \ .$$

(d) The physical distance of the light pulse at time t is equal to $R(t)$ times the coordinate distance. The coordinate distance at time t is equal to the starting coordinate distance, $\ell_c(t_e)$, minus the coordinate distance that the light pulse travels between time t_e and time t. Thus,

$$\begin{aligned}
\ell_p(t) &= R(t)\left[\ell_c(t_e) - \int_{t_e}^{t}\frac{c\,dt'}{R(t')}\right] \\
&= R(t)\left[\int_{t_e}^{t_o}\frac{c\,dt'}{R(t')} - \int_{t_e}^{t}\frac{c\,dt'}{R(t')}\right] \\
&= R(t)\int_{t}^{t_o}\frac{c\,dt'}{R(t')} \\
&= bt^{1/3}\int_{t}^{t_o}\frac{c\,dt'}{bt'^{1/3}} = \frac{3}{2}ct^{1/3}\left[t_o^{2/3} - t^{2/3}\right] \\
&= \boxed{\frac{3}{2}ct\left[\left(\frac{t_o}{t}\right)^{2/3} - 1\right]} \ .
\end{aligned}$$

PROBLEM 7: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(40 points)*

a) *(5 points)* The cosmological redshift is given by the usual form,

$$1+z=\frac{R(t_0)}{R(t_e)} \ .$$

For light emitted by an object at time t_e, the redshift of the received light is

$$1+z=\frac{R(t_0)}{R(t_e)}=\left(\frac{t_0}{t_e}\right)^{\gamma} \ .$$

So,

$$\boxed{z=\left(\frac{t_0}{t_e}\right)^{\gamma}-1 \ .}$$

b) *(5 points)* The coordinates t_0 and t_e are cosmic time coordinates. The "look-back" time as defined in the exam is then the interval t_0-t_e. We can write this as

$$t_0-t_e=t_0\left(1-\frac{t_e}{t_0}\right) \ .$$

We can use the result of part (a) to eliminate t_e/t_0 in favor of z. From (a),

$$\frac{t_e}{t_0}=(1+z)^{-1/\gamma} \ .$$

Therefore,

$$\boxed{t_0-t_e=t_0\left[1-(1+z)^{-1/\gamma}\right] \ .}$$

c) *(10 points)* The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0)=R(t_0)\int_{t_e}^{t_0}\frac{c}{R(t)}dt \ .$$

Calculating this integral gives

$$\ell_p(t_0)=\frac{ct_0^{\gamma}}{1-\gamma}\left[\frac{1}{t_0^{\gamma-1}}-\frac{1}{t_e^{\gamma-1}}\right] \ .$$

Factoring $t_0^{\gamma-1}$ out of the parentheses gives

$$\ell_p(t_0)=\frac{ct_0}{1-\gamma}\left[1-\left(\frac{t_0}{t_e}\right)^{\gamma-1}\right] \ .$$

This can be rewritten in terms of z and H_0 using the result of part (a) as well as,

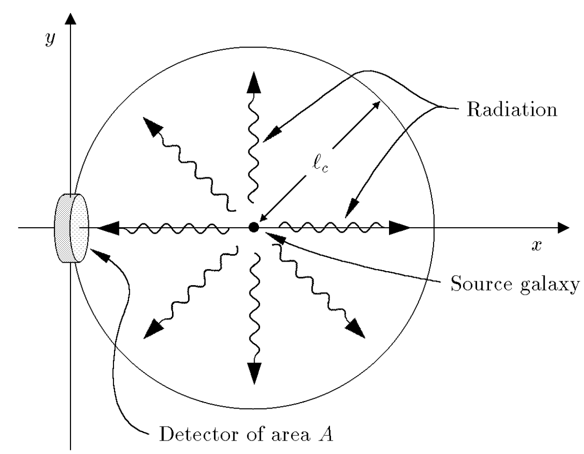
$$H_0=\frac{\dot{R}(t_0)}{R(t_0)}=\frac{\gamma}{t_0} \ .$$

Finally then,

$$\boxed{\ell_p(t_0)=cH_0^{-1}\frac{\gamma}{1-\gamma}\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right] \ .}$$

d) *(10 points)* A nearly identical problem was worked through in Problem 8 of Problem Set 1.

The energy of the observed photons will be redshifted by a factor of $(1+z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of photon emmission, reducing the flux by another factor of $(1+z)$. Consequently, the observed power will be redshifted by two factors of $(1+z)$ to $P/(1+z)^2$.



Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance ℓ_c.

Now consider the photons passing through a patch of the sphere with physical area A. In comoving coordinates the present area of the patch is $A/R(t_0)^2$. Since the object radiates uniformly in all directions, the patch will intercept a fraction $(A/R(t_0)^2)/(4\pi\ell_c^2)$ of the photons passing through the sphere. Thus the power hitting the area A is

$$\frac{(A/R(t_0)^2)}{4\pi\ell_c^2}\frac{P}{(1+z)^2} \ .$$

The radiation energy flux J, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4\pi\ell_p(t_0)^2}\frac{P}{(1+z)^2}$$

where we used $\ell_p(t_0) = R(t_0)\ell_c$. Using the result of part (c) to write J in terms of P, H_0, z, and γ gives,

$$J = \frac{H_0^2}{4\pi c^2}\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{P}{(1+z)^2\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right]^2} \ .$$

e) *(10 points)* Following the solution of Problem 1 of Problem Set 1, we can introduce a fictitious relay station that is at rest relative to the galaxy, but located just next to the jet, between the jet and Earth. As in the previous solution, the relay station simply rebroadcasts the signal it receives from the source, at exactly the instant that it receives it. The relay station therefore has no effect on the signal received by the observer, but allows us to divide the problem into two simple parts.

The distance between the jet and the relay station is very short compared to cosmological scales, so the effect of the expansion of the universe is negligible. For this part of the problem we can use special relativity, which says that the period with which the relay station measures the received radiation is given by

$$\Delta t_{\text{relay station}} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \times \Delta t_{\text{source}} \ .$$

Note that I have used the formula from the front of the exam, but I have changed the size of v, since the source in this case is moving toward the relay station, so the light is blue-shifted. To observers on Earth, the relay station is just a source at rest in the comoving coordinate system, so

$$\Delta t_{\text{observed}} = (1+z)\Delta t_{\text{relay station}} \ .$$

Thus,

$$\begin{aligned} 1+z_J \equiv \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{source}}} &= \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{relay station}}}\frac{\Delta t_{\text{relay station}}}{\Delta t_{\text{source}}} \\ &= (1+z)\big|_{\text{cosmological}} \ \times \ (1+z)\big|_{\text{special relativity}} \\ &= (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \ . \end{aligned}$$

Thus,

$$z_J = (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} - 1 \ .$$

Note added: In looking over the solutions to this problem, I found that a substantial number of students wrote solutions based on the incorrect assumption that the Doppler shift could be treated as if it were entirely due to motion. These students used the special relativity Doppler shift formula to convert the redshift z of the galaxy to a velocity of recession, then subtracted from this the speed v of the jet, and then again used the special relativity Doppler shift formula to find the Doppler shift corresponding to this composite velocity. However, as discussed at the end of Lecture Notes 3, the cosmological Doppler shift is given by

$$1+z \equiv \frac{\Delta t_o}{\Delta t_e} = \frac{R(t_o)}{R(t_e)} \ , \tag{3.11}$$

and is not purely an effect caused by motion. It is really the combined effect of the motion of the distant galaxies and the gravitational field that exists between the galaxies, so the special relativity formula relating z to v does not apply.

PROBLEM 8: DID YOU DO THE READING?

a) The lines were dark, caused by absorption of the radiation in the cooler, outer layers of the sun.

b) Individual stars in the Andromeda Nebula were resolved by Hubble in 1923.

[The other names and dates are not without significance. In 1609 Galileo built his first telescope; during 1609-10 he resolved the individual stars of the Milky Way, and also discovered that the surface of the moon is irregular, that Jupiter has moons of its own, that Saturn has handles (later recognized as rings), that the sun has spots, and that Venus has phases. In 1755 Immanuel Kant published his *Universal Natural History and Theory of the Heavens*, in which he suggested that at least some of the nebulae are galaxies like our own. In 1912 Henrietta Leavitt discovered the relationship between the period and luminosity of Cepheid variable stars. In the 1950s Walter Baade and Allan Sandage recalibrated the extra-galactic distance scale, reducing the accepted value of the Hubble constant by about a factor of 10.]

c)

(i) True. [In 1941, A. McKellar discovered that cyanogen clouds behave as if they are bathed in microwave radiation at a temperature of about 2.3° K, but no connection was made with cosmology.]

(ii) False. [Any radiation reflected by the clouds is far too weak to be detected. It is the bright starlight shining through the cloud that is detectable.]

(iii) True. [Electromagnetic waves at these wavelengths are mostly blocked by the Earth's atmosphere, so they could not be detected directly until high altitude balloons and rockets were introduced into cosmic background radiation research in the 1970s. Precise data was not obtained until the COBE satellite, in 1990.]

(iv) True. [The microwave radiation can boost the CN molecule from its ground state to a low-lying excited state, a state in which the C and N atoms rotate about each other. The population of this low-lying state is therefore determined by the intensity of the microwave radiation. This population is measured by observing the absorption of starlight passing through the clouds, since there are absorption lines in the visible spectrum caused by transitions between the low-lying state and higher energy excited states.]

(v) False. [No chemical reactions are seen.]

d) Aristarchus. [The heliocentric picture was never accepted by other Greek philosophers, however, and was not revived until the publication of *De Revolutionibus Orbium Coelestium* (*On the Revolutions of the Celestial Spheres*) by Copernicus in 1543.]

e) (ii) Any patch of the night sky would look as bright as the surface of the sun. [Explanation: The crux of the argument is that the brightness of an object, measured for example by the power per area (i.e., flux) hitting the retina of your eye, does not change as the object is moved further away. The power falls off with the square of the distance, but so does the area of the image on your retina — so the power per area is independent of distance. Under the assumptions stated, your line of sight will eventually hit a star no matter what direction you are looking. The energy flux on your retina will therefore be the same as in the image of the sun, so the entire sky will appear as bright as the surface of the sun.]

PROBLEM 9: A FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$

a) In general, the Hubble constant is given by $H = \dot{R}/R$, where the overdot denotes a derivative with respect to cosmic time t. In this case

$$H = \frac{1}{bt^{3/5}}\,\frac{3}{5}\,bt^{-2/5} = \boxed{\frac{3}{5t}\;.}$$

b) In general, the (physical) horizon distance is given by

$$\ell_{p,\text{horizon}}(t) = R(t)\int_0^t \frac{c}{R(t')}dt'\;.$$

In this case one has

$$\ell_{p,\text{horizon}}(t) = bt^{3/5}\int_0^t \frac{c}{bt'^{3/5}}dt' = ct^{3/5}\,\frac{5}{2}\left[t^{2/5} - 0^{2/5}\right] = \boxed{\frac{5}{2}\,ct\;.}$$

c) The coordinate speed of light is $c/R(t)$, so the coordinate distance that light travels between t_A and t_B is given by

$$\ell_c = \int_{t_A}^{t_B} \frac{c}{R(t')}\,dt' = \int_{t_A}^{t_B} \frac{c}{bt'^{3/5}}\,dt' = \boxed{\frac{5c}{2b}\left(t_B^{2/5} - t_A^{2/5}\right)\;.}$$

d) The physical separation is just the scale factor times the coordinate separation, so

$$\ell_p(t_A) = R(t_A)\,\ell_c = \boxed{\frac{5}{2}ct_A\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right]\;.}$$

$$\ell_p(t_B) = R(t_B)\,\ell_c = \boxed{\frac{5}{2}ct_B\left[1 - \left(\frac{t_A}{t_B}\right)^{2/5}\right]\;.}$$

e) Let t_{eq} be the time at which the light pulse is equidistant from the two galaxies. At this time it will have traveled a coordinate distance $\ell_c/2$, where ℓ_c is the answer to part (c). Since the coordinate speed is $c/R(t)$, the time $t_{\rm eq}$ can be found from:

$$\int_{t_A}^{t_{\rm eq}} \frac{c}{R(t')}\,dt' = \frac{1}{2}\ell_c$$

$$\frac{5c}{2b}\left(t_{\rm eq}^{2/5} - t_A^{2/5}\right) = \frac{5c}{4b}\left(t_B^{2/5} - t_A^{2/5}\right)$$

Solving for $t_{\rm eq}$,

$$\boxed{t_{\rm eq} = \left[\frac{t_A^{2/5} + t_B^{2/5}}{2}\right]^{5/2} .}$$

f) According to Hubble's law, the speed is equal to Hubble's constant times the physical distance. By combining the answers to parts (a) and (d), one has

$$v = H(t_A)\,\ell_p(t_A)$$

$$= \frac{3}{5t_A}\,\frac{5}{2}ct_A\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right] = \boxed{\frac{3}{2}\,c\,\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right] .}$$

g) The redshift for radiation observed at time t can be written as

$$1 + z = \frac{R(t)}{R(t_e)} ,$$

where t_e is the time that the radiation was emitted. Solving for t_e,

$$t_e = \frac{t}{(1+z)^{5/3}} .$$

As found in part (d), the physical distance that the light travels between t_e and t, as measured at time t, is given by

$$\ell_p(t) = R(t)\int_{t_e}^{t} \frac{c}{R(t')}\,dt' = \frac{5}{2}ct\left[1 - \left(\frac{t_e}{t}\right)^{2/5}\right] .$$

Substituting the expression for t_e, one has

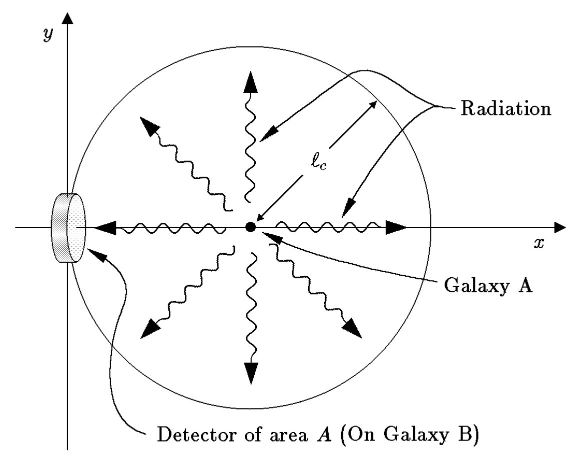
$$\boxed{\ell_p(t) = \frac{5}{2}\,ct\left[1 - \frac{1}{(1+z)^{2/3}}\right] .}$$

As $z \to \infty$, this expression approaches

$$\boxed{\lim_{z\to\infty} \ell_p(t) = \frac{5}{2}\,ct ,}$$

which is exactly equal to the horizon distance. It is a general rule that the horizon distance corresponds to infinite redshift z.

h) Again we will view the problem in comoving coordinates. Put galaxy B at the origin, and galaxy A at a coordinate distance ℓ_c along the x-axis. Draw a sphere of radius ℓ_c, centered galaxy A. Also draw a detector on galaxy B, with physical area A (measured at the present time).



The energy from the quasar will radiate uniformly on the sphere. The detector has a physical area A, so in the comoving coordinate picture its area in square notches would be $A/R(t_B)^2$. The detector therefore occupies a fraction of the sphere given by

$$\frac{[A/R(t_B)^2]}{4\pi\ell_c^2} = \frac{A}{4\pi\ell_p(t_B)^2} ,$$

so this fraction of the emitted photons will strike the detector.

Next consider the rate of arrival of the photons at the sphere. In lecture we figured out that if a periodic wave is emitted at time t_A and observed at time t_B, then the rate of arrival of the wave crests will be slower than the rate of emission by a redshift factor $1 + z = R(t_B)/R(t_A)$. The same argument will apply to the rate of arrival of photons, so the rate of photon arrival at the sphere will be slower than the rate of emission by the factor $1+z$, reducing the energy flux by this factor. In addition, each photon is redshifted in frequency by $1+z$. Since the energy of each photon is proportional to its frequency, the energy flux is reduced by an additional factor of $1+z$. Thus, the rate at which energy reaches the detector is

$$\text{Power hitting detector} = \frac{A}{4\pi\ell_p(t_B)^2}\frac{P}{(1+z)^2} .$$

The red shift z of the light pulse received at galaxy B is given by

$$1+z = \frac{R(t_B)}{R(t_A)} = \left(\frac{t_B}{t_A}\right)^{3/5} .$$

Using once more the expression for $\ell_P(t_B)$ from part (d), one has

$$\boxed{J = \frac{\text{Power hitting detector}}{A} = \frac{P(t_A/t_B)^{6/5}}{25\pi\, c^2\, t_B^2 \left[1-\left(\frac{t_A}{t_B}\right)^{2/5}\right]^2} .}$$

The problem is worded so that t_A, and not z, is the given variable that determines how far galaxy A is from galaxy B. In practice, however, it is usually more useful to express the answer in terms of the redshift z of the received radiation. One can do this by using the above expression for $1+z$ to eliminate t_A in favor of z, finding

$$J = \frac{P}{25\pi\, c^2\, t_B^2(1+z)^{2/3}\left[(1+z)^{2/3}-1\right]^2} .$$

i) Let t'_A be the time at which the light pulse arrives back at galaxy A. The pulse must therefore travel a coordinate distance ℓ_c (the answer to part (c)) between time t_B and t'_A, so

$$\int_{t_B}^{t'_A} \frac{c}{R(t')}\, dt' = \ell_c .$$

Using the answer from (c) and integrating the left-hand side,

$$\frac{5c}{2b}\left(t_A'^{2/5} - t_B^{2/5}\right) = \frac{5c}{2b}\left(t_B^{2/5} - t_A^{2/5}\right) .$$

Solving for t'_A,

$$\boxed{t'_A = \left(2t_B^{2/5} - t_A^{2/5}\right)^{5/2} .}$$

PROBLEM 10: DID YOU DO THE READING?

a) Einstein believed that the universe was static, and the cosmological term was necessary to prevent a static universe from collapsing under the attractive force of normal gravity. [The repulsive effect of a cosmological constant grows linearly with distance, so if the coefficient is small it is important only when the separations are very large. Such a term can be important cosmologically while still being too small to be detected by observations of the solar system or even the galaxy. Recent measurements of distant supernovas ($z \approx 1$), which you may have read about in the newspapers, make it look like maybe there is a cosmological constant after all! Since the cosmological constant is the hot issue in cosmology this season, we will want to look at it more carefully. The best time will be after Lecture Notes 7.]

b) At the time of its discovery, de Sitter's model was thought to be static [although it was known that the model predicted a redshift which, at least for nearby galaxies, was proportional to the distance]. From a modern perspective the model is thought to be expanding.

[It seems strange that physicists in 1917 could not correctly determine if the theory described a universe that was static or expanding, but the mathematical formalism of general relativity can be rather confusing. The basic problem is that when space is not Euclidean there is no simple way to assign coordinates to it. The mathematics of general relativity is designed to be valid for any coordinate system, but the underlying physics can sometimes be obscured by a peculiar choice of coordinates. A change of coordinates can not only distort the apparent geometry of space, but it can also mix up space and time. The de Sitter model was first written down in coordinates that made it look static, so everyone believed it was. Later Arthur Eddington and Hermann Weyl (independently) calculated the trajectories of test particles, discovering that they flew apart.]

c) $n_1 = 3$, and $n_2 = 4$.

d) Above 3,000 K the universe was so hot that the atoms were ionized, dissociated into nuclei and free electrons. At about this temperature, however, the universe was cool enough so that the nuclei and electrons combined to form neutral atoms.

[This process is usually called "recombination," although the prefix "re-" is totally inaccurate, since in the big bang theory these constituents had never been previously combined. As far as I know the word was first used in this context by P.J.E. Peebles, so I once asked him why the prefix was used. He replied that this word is standard terminology in plasma physics, and was carried over into cosmology.]

[Regardless of its name, recombination was crucial for the clumping of matter into galaxies and stars, because the pressure of the photons in the early universe was enormous. When the matter was ionized, the free electrons interacted strongly with the photons, so the pressure of these photons prevented the matter from clumping. After recombination, however, the matter became very transparent to radiation, and the pressure of the radiation became ineffective.]

[Incidentally, at roughly the same time as recombination (with big uncertainties), the mass density of the universe changed from being dominated by radiation (photons and neutrinos) to being dominated by nonrelativistic matter. There is no known underlying connection between these two events, and it seems to be something of a coincidence that they occurred at about the same time. The transition from radiation-domination to matter-domination also helped to promote the clumping of matter, but the effect was much weaker than the effect of recombination— because of the very high velocity of photons and neutrinos, their pressure remained a significant force even after their mass density became much smaller than that of matter.]

PROBLEM 11: ANOTHER FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$

a) According to Eq. (3.7) of the Lecture Notes,

$$H(t) = \frac{1}{R(t)} \frac{dR}{dt} .$$

For the special case of $R(t) = bt^{3/5}$, this gives

$$H(t) = \frac{1}{bt^{3/5}} \frac{3}{5} bt^{-2/5} = \boxed{\frac{3}{5t}} .$$

b) According to Eq. (3.8) of the Lecture Notes, the coordinate velocity of light (in comoving coordinates) is given by

$$\frac{dx}{dt} = \frac{c}{R(t)} .$$

Since galaxies A and B have physical separation ℓ_0 at time t_1, their coordinate separation is given by

$$\ell_c = \frac{\ell_0}{bt_1^{3/5}} .$$

The radio signal must cover this coordinate distance in the time interval from t_1 to t_2, which implies that

$$\int_{t_1}^{t_2} \frac{c}{R(t)}\, dt = \frac{\ell_0}{bt_1^{3/5}} .$$

Using the expression for $R(t)$ and integrating,

$$\frac{5c}{2b}\left(t_2^{2/5} - t_1^{2/5}\right) = \frac{\ell_0}{bt_1^{3/5}} ,$$

which can be solved for t_2 to give

$$\boxed{t_2 = \left(1 + \frac{2\ell_0}{5ct_1}\right)^{5/2} t_1 .}$$

c) The method is the same as in part (b). The coordinate distance between the two galaxies is unchanged, but this time the distance must be traversed in the time interval from t_2 to t_3. So,

$$\int_{t_2}^{t_3} \frac{c}{R(t)}\, dt = \frac{\ell_0}{bt_1^{3/5}} ,$$

which leads to

$$\frac{5c}{2b}\left(t_3^{2/5} - t_2^{2/5}\right) = \frac{\ell_0}{bt_1^{3/5}} .$$

Solving for t_3 gives

$$\boxed{t_3 = \left[\left(\frac{t_2}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{5/2} t_1 .}$$

The above answer is perfectly acceptable, but one could also replace t_2 by using the answer to part (b), which gives

$$\boxed{t_3 = \left(1 + \frac{4\ell_0}{5ct_1}\right)^{5/2} t_1 .}$$

[Alternatively, one could have begun the problem by considering the full round trip of the radio signal, which travels a coordinate distance $2\ell_c$ during the time interval from t_1 to t_3. The problem then becomes identical to part (b), except that the coordinate distance ℓ_c is replaced by $2\ell_c$, and t_2 is replaced by t_3. One is led immediately to the answer in the form of the previous equation.]

d) Cosmic time is defined by the reading of suitably synchronized clocks which are each at rest with respect to the matter of the universe at the same location. (For this problem we will not need to think about the method of synchronization.) Thus, the cosmic time interval between the receipt of the message and the response is the same as what is measured on the galaxy B clocks, which is Δt. The response is therefore sent at cosmic time $t_2+\Delta t$. The coordinate distance between the galaxies is still $\ell_0/R(t_1)$, so

$$\int_{t_2+\Delta t}^{t_4} \frac{c}{R(t)}\,dt = \frac{\ell_0}{b t_1^{3/5}} \,.$$

Integration gives

$$\frac{5c}{2b}\left[t_4^{2/5} - (t_2+\Delta t)^{2/5}\right] = \frac{\ell_0}{b t_1^{3/5}} \,,$$

which can be solved for t_4 to give

$$\boxed{t_4 = \left[\left(\frac{t_2+\Delta t}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{5/2} t_1 \,.}$$

e) From the formula at the front of the exam,

$$1+z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} = \frac{R(t_4)}{R(t_2+\Delta t)} = \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} \,.$$

So,

$$\boxed{z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} = \frac{R(t_4)}{R(t_2+\Delta t)} = \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} - 1 \,.}$$

f) If Δt is small compared to the time that it takes $R(t)$ to change significantly, then the interval between a signal sent at t_3 and a signal sent at $t_3+\Delta t$ will be received with a redshift identical to that observed between two successive crests of a wave. Thus, the separation between the receipt of the acknowledgement

and the receipt of the response will be a factor $(1+z)$ times longer than the time interval between the sending of the two signals, and therefore

$$\begin{aligned} t_4 - t_3 &= (1+z)\Delta t + \mathcal{O}(\Delta t^2) \\ &= \left(\frac{t_4}{t_2+\Delta t}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) \,. \end{aligned}$$

Since the answer contains an explicit factor of Δt, the other factors can be evaluated to zeroth order in Δt:

$$\boxed{t_4 - t_3 = \left(\frac{t_4}{t_2}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) \,,}$$

where to first order in Δt the t_4 in the numerator could equally well have been replaced by t_3.

For those who prefer the brute force approach, the answer to part (d) can be Taylor expanded in powers of Δt. To first order one has

$$t_4 = t_3 + \left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} \Delta t + \mathcal{O}(\Delta t^2) \,.$$

Evaluating the necessary derivative gives

$$\frac{\partial t_4}{\partial \Delta t} = \left[\left(\frac{t_2+\Delta t}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{3/2} \left(\frac{t_2+\Delta t}{t_1}\right)^{-3/5} \,,$$

which when specialized to $\Delta t = 0$ becomes

$$\left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} = \left[\left(\frac{t_2}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{3/2} \left(\frac{t_2}{t_1}\right)^{-3/5} \,.$$

Using the first boxed answer to part (c), this can be simplified to

$$\begin{aligned} \left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t=0} &= \left(\frac{t_3}{t_1}\right)^{3/5} \left(\frac{t_2}{t_1}\right)^{-3/5} \\ &= \left(\frac{t_3}{t_2}\right)^{3/5} \,. \end{aligned}$$

Putting this back into the Taylor series gives

$$t_4 - t_3 = \left(\frac{t_3}{t_2}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) \,,$$

in agreement with the previous answer.

PROBLEM 12: THE DECELERATION PARAMETER

From the front of the exam, we are reminded that

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

and

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} ,$$

where a dot denotes a derivative with respect to time t. The critical mass density ρ_c is defined to be the mass density that corresponds to a flat ($k = 0$) universe, so from the equation above it follows that

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho_c .$$

Substituting into the definition of q, we find

$$q = -\ddot{R}(t)\frac{R(t)}{\dot{R}^2(t)} = -\frac{\ddot{R}}{R}\left(\frac{R}{\dot{R}}\right)^2$$

$$= \left(\frac{4\pi}{3}G\rho\right)\left(\frac{3}{8\pi G\rho_c}\right) = \frac{1}{2}\frac{\rho}{\rho_c} = \boxed{\frac{1}{2}\Omega .}$$

PROBLEM 13: A RADIATION-DOMINATED FLAT UNIVERSE

The flatness of the model universe means that $k = 0$, so

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho .$$

Since

$$\rho(t) \propto \frac{1}{R^4(t)} ,$$

it follows that

$$\frac{dR}{dt} = \frac{\text{const}}{R} .$$

Rewriting this as

$$R\,dR = \text{const}\,dt ,$$

the indefinite integral becomes

$$\frac{1}{2}R^2 = (\text{const})t + c' ,$$

where c' is a constant of integration. Different choices for c' correspond to different choices for the definition of $t = 0$. We will follow the standard convention of choosing $c' = 0$, which sets $t = 0$ to be the time when $R = 0$. Thus the above equation implies that $R^2 \propto t$, and therefore

$$\boxed{R(t) \propto t^{1/2}}$$

for a photon-dominated flat universe.

PROBLEM 14: EVOLUTION OF AN OPEN UNIVERSE

The evolution of an open, matter-dominated universe is described by the following parametric equations:

$$ct = \alpha(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1) .$$

Evaluating the second of these equations at $R/\sqrt{\kappa} = 2\alpha$ yields a solution for θ:

$$2\alpha = \alpha(\cosh\theta - 1) \quad\Longrightarrow\quad \cosh\theta = 3 \quad\Longrightarrow\quad \theta = \cosh^{-1}(3) .$$

We can use these results in the first equation to solve for t. Noting that

$$\sinh\theta = \sqrt{\cosh^2\theta - 1} = \sqrt{8} = 2\sqrt{2} ,$$

we have

$$\boxed{t = \frac{\alpha}{c}\left[2\sqrt{2} - \cosh^{-1}(3)\right] .}$$

Numerically, $t \approx 1.06567\,\alpha/c$.

PROBLEM 15: ANTICIPATING A BIG CRUNCH

The critical density is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} ,$$

so the mass density is given by

$$\rho = \Omega_0 \rho_c = 2\rho_c = \frac{3H_0^2}{4\pi G} . \tag{1}$$

Substituting this relation into

$$H_0^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} ,$$

we find

$$H_0^2 = 2H_0^2 - \frac{kc^2}{R^2} ,$$

from which it follows that

$$\frac{R}{\sqrt{k}} = \frac{c}{H_0} . \tag{2}$$

Now use

$$\alpha = \frac{4\pi}{3} \frac{G\rho R^3}{k^{3/2} c^2} .$$

Substituting the values we have from Eqs. (1) and (2) for ρ and $R/\sqrt{k}$, we have

$$\alpha = \frac{c}{H_0} . \tag{3}$$

To determine the value of the parameter θ, use

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) ,$$

which when combined with Eqs. (2) and (3) implies that $\cos\theta = 0$. The equation $\cos\theta = 0$ has multiple solutions, but we know that the θ-parameter for a closed matter-dominated universe varies between 0 and π during the expansion phase of the universe. Within this range, $\cos\theta = 0$ implies that $\theta = \pi/2$. Thus, the age of the universe at the time these measurements are made is given by

$$\begin{aligned} t &= \frac{\alpha}{c}(\theta - \sin\theta) \\ &= \frac{1}{H_0}\left(\frac{\pi}{2} - 1\right) . \end{aligned}$$

The total lifetime of the closed universe corresponds to $\theta = 2\pi$, or

$$t_{\text{final}} = \frac{2\pi\alpha}{c} = \frac{2\pi}{H_0} ,$$

so the time remaining before the big crunch is given by

$$t_{\text{final}} - t = \frac{1}{H_0}\left[2\pi - \left(\frac{\pi}{2} - 1\right)\right] = \boxed{\left(\frac{3\pi}{2} + 1\right)\frac{1}{H_0}} .$$

PROBLEM 16: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(35 points)*

(a) Substituting the equation for $M(r_i)$, given on the quiz, into the differential equation for r, also given on the quiz, one finds:

$$\ddot{r} = -\frac{4\pi}{3}\frac{G r_i^3 \rho_i}{r^2} + \gamma r^n .$$

Dividing both sides of the equation by r_i, one has

$$\frac{\ddot{r}}{r_i} = -\frac{4\pi}{3}\frac{G r_i^2 \rho_i}{r^2} + \gamma \frac{r^n}{r_i} .$$

Substituting $u = r/r_i$, this becomes

$$\boxed{\ddot{u} = -\frac{4\pi}{3}\frac{G\rho_i}{u^2} + \gamma u^n r_i^{n-1} .}$$

(b) The only dependence on r_i occurs in the last term, which is proportional to r_i^{n-1}. This dependence disappears if $\boxed{n = 1}$, since the zeroth power of any positive number is 1.

(c) This is exactly the same as the case discussed in the lecture notes, since the initial conditions do not depend on the differential equation. At $t = 0$,

$$\begin{aligned} r(r_i, 0) &= r_i \text{ (definition of } r_i) \\ \dot{r}(r_i, 0) &= H_i r_i \text{ (since } \vec{v}_i = H_i \vec{r}). \end{aligned}$$

Dividing these equations by r_i one has the intial conditions

$$\boxed{\begin{aligned} u &= 1 \\ \dot{u} &= H_i \ . \end{aligned}}$$

(d) $R(t)$ should obey the differential equation obtained in part (a) for u, for the value of n that was obtained in part (b): $n = 1$. So,

$$\ddot{R} = -\frac{4\pi}{3}\frac{G\rho_i}{R^2} + \gamma R \ .$$

Multiplying the equation by $\dot{R} \equiv dR/dt$, one finds

$$\frac{d^2R}{dt^2}\frac{dR}{dt} = \left\{ -\frac{4\pi}{3}\frac{G\rho_i}{R^2} + \gamma R \right\} \frac{dR}{dt} \ ,$$

which can be rewritten as

$$\frac{d}{dt}\left\{ \frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{R} - \frac{1}{2}\gamma R^2 \right\} = 0 \ .$$

Thus the quantity inside the curly brackets must be constant. Following the lecture notes, I will call this constant E:

$$\boxed{\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{R} - \frac{1}{2}\gamma R^2 = E \ .}$$

Or one can use the standardly defined quantity

$$k = -\frac{2E}{c^2} \ ,$$

in which case the equation can be written

$$-\frac{2}{c^2}\left\{ \frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{4\pi}{3}\frac{G\rho_i}{R} - \frac{1}{2}\gamma R^2 \right\} = k \ .$$

One can then rewrite the equation in the more standard form

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}\frac{G\rho_i}{R^3} + \gamma - \frac{kc^2}{R^2} \ .}$$

Additional Note: Historically, the constant γ corresponds to the "cosmological constant" which was introduced by Albert Einstein in 1917 in an effort to build a static model of the universe. The cosmological constant Λ is related to γ by

$$\gamma = \frac{1}{3}\Lambda c^2 \ .$$

Since the time-dependent mass density $\rho(t) = \rho_i/R^3$, the differential equation can be rewritten as

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - \frac{kc^2}{R^2} \ .$$

Since the cosmological constant contributes like a constant addition to the mass density, modern physicists interpret it as the mass density of the vacuum. The mass density of the vacuum is then related to Einstein's cosmological constant Λ by

$$\rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G} \ .$$

PROBLEM 17: DID YOU DO THE READING? *(25 points)*

(a) In 1826, the astronomer Heinrich Olber wrote a paper on a paradox regarding the night sky. What is Olber's paradox? What is the primary resolution of it?

(Ryden, Chapter 2, Pages 6-8)

Ans: Olber's paradox is that the night sky appears to be dark, instead of being uniformly bright. The primary resolution is that the universe has a finite age, and so the light from stars beyond the horizon distance has not reached us yet. (However, even in the steady-state model of the universe, the paradox is resolved because the light from distant stars will be red-shifted beyond the visible spectrum).

(b) What is the value of the Newtonian gravitational constant G in Planck units? The Planck length is of the order of 10^{-35} m, 10^{-15} m, 10^{15} m, or 10^{35} m?

(Ryden, Chapter 1, Page 3)

Ans: $G = 1$ in Planck units, by definition.

The Planck length is of the order of 10^{-35} m. (Note that this answer could be obtained by a process of elimination as long as you remember that the Planck length is much smaller than 10^{-15} m, which is the typical size of a nucleus).

(c) What is the Cosmological Principle? Is the Hubble expansion of the universe consistent with it?

(Weinberg, Chapter 2, Pages 21-23; Ryden, Chapter 2, Page 11)

Ans: The Cosmological Principle states that there is nothing special about our location in the universe, i.e. the universe is homogeneous and isotropic.

Yes, the Hubble expansion is consistent with it (since there is no center of expansion).

(d) In the "Standard Model" of the universe, when the universe cooled to about 3×10^a K, it became transparent to photons, and today we observe these as the Cosmic Microwave Background (CMB) at a temperature of about 3×10^b K. What are the integers a and b?

(Weinberg, Chapter 3; Ryden, Chapter 2, Page 22)

$a = 3$, $b = 0$.

(e) What did the universe primarily consist of at about 1/100th of a second after the Big Bang? Include any constituent that is believed to have made up more than 1% of the mass density of the universe.

(Weinberg, Chapter 1, Page 5)

Ans: Electrons, positrons, neutrinos, and photons.

PROBLEM 18: SPECIAL RELATIVITY DOPPLER SHIFT *(20 points)*

(a) The easiest way to solve this problem is by a double application of the standard special-relativity Doppler shift formula, which was given on the front of the exam:

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 , \tag{18.1}$$

where $\beta = v/c$. Remembering that the wavelength is stretched by a factor $1+z$, we find immediately that the wavelength of the radio wave received at Alpha-7 is given by

$$\lambda_{\rm Alpha-7} = \sqrt{\frac{1+v_s/c}{1-v_s/c}}\,\lambda_{\rm emitted} . \tag{18.2}$$

The photons that are received by the observer are in fact never received by Alpha-7, but the wavelength found by the observer will be the same as if Alpha-7 acted as a relay station, receiving the photons and retransmitting them at the received wavelength. So, applying Eq. (18.1) again, the wavelength seen by the observer can be written as

$$\lambda_{\rm observed} = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\,\lambda_{\rm Alpha-7} . \tag{18.3}$$

Combining Eqs. (18.2) and (18.3),

$$\lambda_{\rm observed} = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\sqrt{\frac{1+v_s/c}{1-v_s/c}}\,\lambda_{\rm emitted} , \tag{18.4}$$

so finally

$$\boxed{z = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\sqrt{\frac{1+v_s/c}{1-v_s/c}} - 1 .} \tag{18.5}$$

(b) Although we used the presence of Alpha-7 in determining the redshift z of Eq. (18.5), the redshift is not actually affected by the space station. So the special-relativity Doppler shift formula, Eq. (18.1), must directly describe the redshift resulting from the relative motion of the source and the observer. Thus

$$\sqrt{\frac{1+v_{\rm tot}/c}{1-v_{\rm tot}/c}} - 1 = \sqrt{\frac{1+v_o/c}{1-v_o/c}}\sqrt{\frac{1+v_s/c}{1-v_s/c}} - 1 . \tag{18.6}$$

The equation above determines $v_{\rm tot}$ in terms of v_o and v_s, so the rest is just algebra. To simplify the notation, let $\beta_{\rm tot} \equiv v_{\rm tot}/c$, $\beta_o \equiv v_o/c$, and $\beta_s \equiv v_s/c$. Then

$$1+\beta_{\rm tot} = \frac{1+\beta_o}{1-\beta_o}\frac{1+\beta_s}{1-\beta_s}(1-\beta_{\rm tot})$$

$$\beta_{\rm tot}\left[1+\frac{1+\beta_o}{1-\beta_o}\frac{1+\beta_s}{1-\beta_s}\right] = \frac{1+\beta_o}{1-\beta_o}\frac{1+\beta_s}{1-\beta_s} - 1$$

$$\beta_{\rm tot}\left[\frac{(1-\beta_o-\beta_s+\beta_o\beta_s)+(1+\beta_o+\beta_s+\beta_o\beta_s)}{(1-\beta_o)(1-\beta_s)}\right] = \frac{(1+\beta_o+\beta_s+\beta_o\beta_s)-(1-\beta_o-\beta_s+\beta_o\beta_s)}{(1-\beta_o)(1-\beta_s)}$$

$$\beta_{\rm tot}[2(1+\beta_o\beta_s)] = 2(\beta_o+\beta_s)$$

$$\beta_{\rm tot} = \frac{\beta_o+\beta_s}{1+\beta_o\beta_s}$$

$$\boxed{v_{\rm tot} = \frac{v_o+v_s}{1+\dfrac{v_o v_s}{c^2}} .} \tag{18.7}$$

The final formula is the relativistic expression for the addition of velocities. Note that it guarantees that $|v_{\rm tot}| \le c$ as long as $|v_o| \le c$ and $|v_s| \le c$.