MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 2, 2005
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 3

QUIZ DATE: Thursday, December 8, 2005

COVERAGE: Lecture Notes 7 and 8; Problem Set 5; Ryden, Chapters 8, 9, and Epilogue; Weinberg, Chapters 6, 7, 8, and the Afterward. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems.**

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, and 2004. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match the coverage of any of the quizzes from previous years.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" at the beginning. For the third quiz, this useful information will be the following:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \quad \Leftarrow \text{or} \Rightarrow \quad H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R \quad \Leftarrow \text{or} \Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \quad \Leftarrow \text{or} \Rightarrow \quad \dot{\varepsilon} = -3\frac{\dot{a}}{a}(\varepsilon + P)$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$R(t) \propto t^{2/3}$ (matter-dominated)

$R(t) \propto t^{1/2}$ (radiation-dominated)

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\,d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\text{vac}} = -\rho_{\text{vac}}c^2 \qquad \rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$G = 6.673 \times 10^{-8}\ \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\,\text{erg/K} = 8.617 \times 10^{-5}\,\text{eV/K}\ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27}\ \text{erg-sec} = 6.582 \times 10^{-16}\ \text{eV-sec}\ ,$$

$$c = 2.998 \times 10^{10}\ \text{cm/sec}$$

$$1\ \text{yr} = 3.156 \times 10^7\ \text{s}$$

$$1\ \text{eV} = 1.602 \times 10^{-12}\ \text{erg}\ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3c^5}{16\pi^3 gG}\right)^{1/4}\ \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t}\ \text{(in sec)}}$$

PROBLEM 1: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

Today the temperature of the cosmic microwave background radiation is 2.7°K. Calculate the number density of photons in this radiation. What is the number density of thermal neutrinos left over from the big bang?

∗ PROBLEM 2: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

∗ PROBLEM 3: A NEW SPECIES OF LEPTON

The following problem was Problem 2, Quiz 3, 1992, worth 25 points.

Suppose the calculations describing the early universe were modified by including an additional, hypothetical lepton, called an 8.286ion. The 8.286ion has roughly the same properties as an electron, except that its mass is given by $mc^2 = 0.750$ MeV.

Parts (a)-(c) of this question require numerical answers, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included. (For example, if you were asked how many meters a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$.)

a) *(5 points)* What would be the number density of 8.286ions, in particles per cubic meter, when the temperature T was given by $kT = 3$ MeV?

b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the value of the mass density at $t = .01$ sec? You may assume that 0.75 MeV $\ll$

$kT \ll 100$ MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, e^+-e^- pairs, and 8.286ion-anti8286ion pairs. Express your answer in the units of gm-cm^{-3}.

c) *(5 points)* Under the same assumptions as in (b), what would be the value of kT, in MeV, at $t = .01$ sec?

d) *(5 points)* When nucleosynthesis calculations are modified to include the effect of the 8.286ion, is the production of helium increased or decreased? Explain your answer in a few sentences.

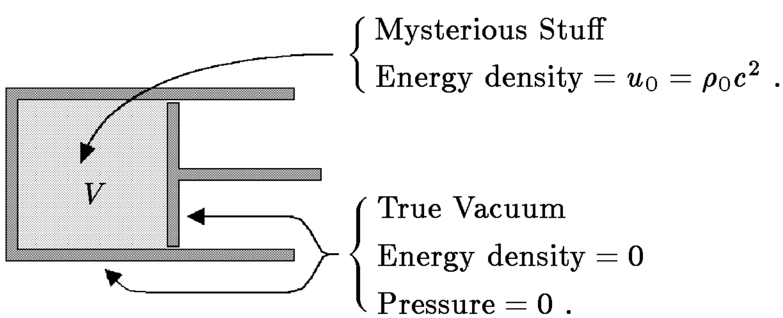
e) *(5 points)* Suppose the neutrinos decouple while $kT \gg 0.75$ MeV. If the 8.286ions are included, what does one predict for the value of T_ν/T_γ today? (Here T_ν denotes the temperature of the neutrinos, and T_γ denotes the temperature of the cosmic background radiation photons.)

∗ PROBLEM 4: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

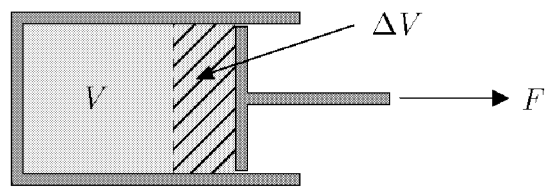
The following problem was Problem 3, Quiz 3, 2002. Although it is couched in the language of Lecture Notes 13, the physics is really the same as the pressure calculations in Lecture Notes 7, so a modified form of this problem would be fair for the coming quiz.

In Lecture Notes 13, a thought experiment involving a piston was used to show that $p = -\rho c^2$ for any substance for which the energy density remains constant under expansion. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

∗ PROBLEM 5: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

Consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and the same ***mysterious stuff*** that was introduced in the previous problem. Since the mass density of mysterious stuff falls off as $1/\sqrt{V}$, where V is the volume, it follows that in an expanding universe the mass density of mysterious stuff falls off as $1/R^{3/2}(t)$.

Suppose that you are given the present value of the Hubble parameter H_0, and also the present values of the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\mathrm{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe t_0 in terms of these quantities.

(a) *(8 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{R(t)}{R(t_0)}$$

for an arbitrary time t. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(b) *(7 points)* Write an integral expression for the age of the universe t_0. The expression should depend only on H_0 and the various contributions to Ω_0 listed above ($\Omega_{m,0}$, $\Omega_{r,0}$, etc.), but it might include x as a variable of integration.

Extra Credit for Super-Sharpies (no partial credit): For 5 points extra credit, you can calculate the angular diameter $\Delta\theta$ of the image of a spherical object at redshift z which had a physical diameter w at the time of emission. You should assume that $\Omega_{\rm tot} < 1$, and also that $\Delta\theta \ll 1$. The calculation is to be carried out for the same model universe described above.

PROBLEM 6: TIME SCALES IN COSMOLOGY

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-37} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the beginning of the processes involved in big bang nucleosynthesis;

(b) the end of the processes involved in big bang nucleosynthesis;

(c) the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{16}$ GeV;

(d) "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

PROBLEM 7: SHORT ANSWERS *(40 points)*

The following problem was Problem 1, Quiz 3, 2002. The questions concerning Steven Weinberg's book are relevant to Quiz 3 of 2005.

(a) *(6 points)* In chapter 6 of *The First Three Minutes*, Steven Weinberg posed the question, "Why was there no systematic search for this [cosmic background] radiation, years before 1965?" In discussing this issue, he contrasted it with the history of two different elementary particles, each of which were predicted approximately 20 years before they were first detected. Name one of these two elementary particles. (If you name them both correctly, you will get 3 points extra credit. However, one right and one wrong will get you 4 points for the question, compared to 6 points for just naming one particle and getting it right.)

Answer: ____________________

2nd Answer (optional): ____________________

(b) *(6 points)* In Chapter 6 of **The First Three Minutes**, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)

(i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.

(ii) There was a breakdown in communication between theorists and experimentalists.

(iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.

(iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

(v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.

(vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

(c) *(8 points)* In Chapter 6 of Rowan-Robinson's *Cosmology*, he discusses the observational evidence which indicates that galaxies and clusters of galaxies contain large amounts of dark matter — matter which is not seen, but which is detected indirectly through its gravitational effects. State whether each of the following statements is true or false. (2 points for each right answer, no penalty for guessing wrong, and remember that a statement that is partly true and partly false counts as false.)

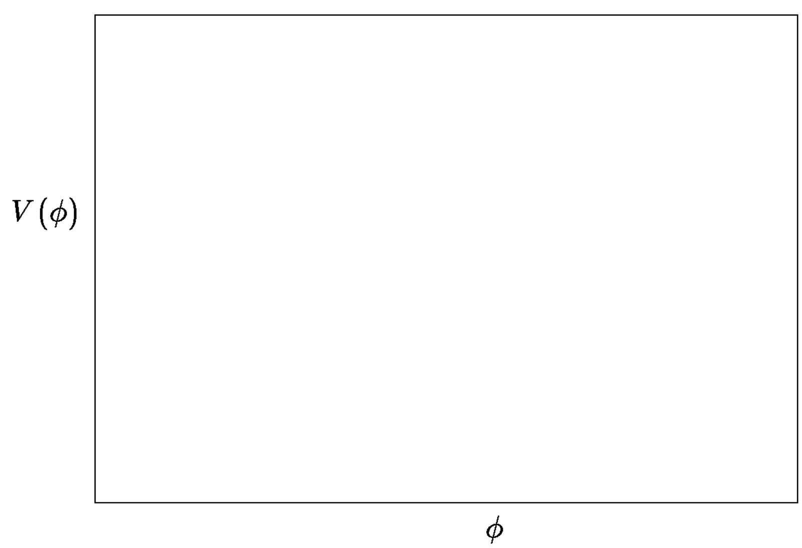
(i) Measurements of the rotation curves (rotational velocity as a function of distance from the center) of spiral galaxies are found to be approximately flat, even when extended to large radii. If, however, the mass were concentrated where the light is seen, these rotation curves would fall off with distance from the center. **T** or **F** .

(ii) For elliptical galaxies, measurements of the Doppler spreading of the emission lines are used to determine the typical speeds of stars and gas, which can be related by the virial theorem to the gravitational potential energy. This Doppler spreading is much smaller than would be expected in the absence of dark matter, indicating that about 90% of the total mass of the galaxy is in the form of a halo of dark matter. **T** or **F** .

(iii) The masses of rich clusters can be estimated by observing the pattern of X-ray emission. **T** or **F** .

(iv) The masses of individual stars can be determined by their spectral characteristics, and these masses can be added to find the mass of the galaxy. **T** or **F** .

(d) *(6 points)* On the graph below, sketch the potential energy function $V(\phi)$ (at zero temperature) that is assumed in the new inflationary universe theory. Label the location of the true vacuum and false vacuum.



(e) *(6 points)* The word "supersymmetry" refers to a symmetry that relates the behavior of one certain class of particles with the behavior of another class.

What are the names of these two classes *(2 points)*? What physical property distinguishes particles of one class from the particles of the other *(4 points)*?

1st Class: ________________ 2nd Class: ________________

Physical distinction: ________________________________

(f) *(8 points)* Grand unified theories unify three of the four known classes of particle interactions. For 2 points each, name these three, and also name the one that is left out.

Included: ________________

Included: ________________

Included: ________________

Excluded: ________________

∗ PROBLEM 8: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

The following problem was Problem 4, Quiz 3, 2004.

On April 13, 2000, the Sloan Digital Sky Survey announced the discovery of what was then the most distant object known in the universe: a quasar at $z = 5.82$. To explain to the public how this object fits into the universe, the SDSS posted on their website an article by Michael Turner and Craig Wiegert titled "How Can An Object We See Today be 27 Billion Light Years Away If the Universe is only 14 Billion Years Old?" Using a model with $H_0 = 65$ km-s^{-1}-Mpc^{-1}, $\Omega_m = 0.35$, and $\Omega_\Lambda = 0.65$, they claimed

(a) that the age of the universe is 13.9 billion years.

(b) that the light that we now see was emitted when the universe was 0.95 billion years old.

(c) that the distance to the quasar, as it would be measured by a ruler today, is 27 billion light-years.

(d) that the distance to the quasar, at the time the light was emitted, was 4.0 billion light-years.

(e) that the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing, is 1.8 times the velocity of light.

The goal of this problem is to check all of these conclusions, although you are of course not expected to actually work out the numbers. Your answers can be expressed in terms of H_0, Ω_m, Ω_Λ, and z. Definite integrals need not be evaluated.

Note that Ω_m represents the present density of nonrelativistic matter, expressed as a fraction of the critical density; and Ω_Λ represents the present density of vacuum energy, expressed as a fraction of the critical density. In answering each of the following questions, you may consider the answer to any previous part — whether you answered it or not — as a given piece of information, which can be used in your answer.

(a) *(15 points)* Write an expression for the age t_0 of this model universe?

(b) *(5 points)* Write an expression for the time t_e at which the light which we now receive from the distant quasar was emitted.

(c) *(10 points)* Write an expression for the present physical distance $\ell_{\text{phys},0}$ to the quasar.

(d) *(5 points)* Write an expression for the physical distance $\ell_{\text{phys,e}}$ between us and the quasar at the time that the light was emitted.

(e) *(5 points)* Write an expression for the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing.

SOLUTIONS

PROBLEM 1: NUMBER DENSITIES IN THE COSMIC BACK-GROUND RADIATION

In general, the number density of a particle in the black-body radiation is given by

$$n = g^* \frac{\xi(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3$$

For photons, one has $g^* = 2$. Then

$$\left.\begin{aligned} k &= 1.381 \times 10^{-16} \text{erg}/^\circ\text{K} \\ T &= 2.7\,^\circ\text{K} \\ \hbar &= 1.055 \times 10^{-27} \text{erg-sec} \\ c &= 2.998 \times 10^{10} \text{cm/sec} \end{aligned}\right\} \implies \left(\frac{kT}{\hbar c}\right)^3 = 1.638 \times 10^3 \text{cm}^{-3} \ .$$

Then using $\xi(3) \simeq 1.202$, one finds

$$\boxed{n_\gamma = 399/\text{cm}^3 \ .}$$

For the neutrinos,

$$g^*_\nu = 2 \times \frac{3}{4} = \frac{3}{2} \quad \text{per species.}$$

The factor of 2 is to account for ν and $\bar{\nu}$, and the factor of $3/4$ arises from the Pauli exclusion principle. So for three species of neutrinos one has

$$g^*_\nu = \frac{9}{2} \ .$$

Using the result

$$T_\nu^3 = \frac{4}{11} T_\gamma^3$$

from Problem 8 of Problem Set 3 (2000), one finds

$$\begin{aligned} n_\nu &= \left(\frac{g^*_\nu}{g^*_\gamma}\right)\left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma \\ &= \left(\frac{9}{4}\right)\left(\frac{4}{11}\right) 399\text{cm}^{-3} \\ \implies &\quad \boxed{n_\nu = 326/\text{cm}^3 \text{ (for all three species combined).}} \end{aligned}$$

PROBLEM 2: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{\pi^4}{30\zeta(3)}\,kT\ .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\,kT\ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{2\pi^4}{45\zeta(3)}\,k\ .}$$

Numerically, this gives $3.602\,k$, where k is the Boltzman constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is $7/8$ times the number of spin states, and g^* is $3/4$ times the number of spin states. So the average energy per particle is

$$
\begin{aligned}
E &= \frac{g \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{g^* \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} \\
&= \frac{\dfrac{7}{8} \dfrac{\pi^2}{30} \dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} \dfrac{(kT)^3}{(\hbar c)^3}} \\
&= \boxed{\frac{7\pi^4}{180\zeta(3)} kT \ .}
\end{aligned}
$$

Numerically, $E = 3.1514\,kT$.

Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8}\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{135\zeta(3)}\,k\ .}$$

Numerically, this gives $S = 4.202\,k$.

PROBLEM 3: A NEW SPECIES OF LEPTON

a) The number density is given by the formula at the start of the exam,

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ .$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$n = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\ \cancel{\text{MeV}}}{6.582\times 10^{-16}\ \cancel{\text{eV-sec}} \times 2.998\times 10^{10}\ \cancel{\text{cm-sec}^{-1}}}\right)^3$$
$$\times \left(\frac{10^6\ \cancel{\text{eV}}}{1\ \cancel{\text{MeV}}}\right)^3 \times \left(\frac{10^2\ \cancel{\text{cm}}}{1\text{ m}}\right)^3$$
$$= 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\times 10^6\times 10^2}{6.582\times 10^{-16}\times 2.998\times 10^{10}}\right)^3 \text{ m}^{-3}\ .$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\times 10^6\times 10^2}{6.582\times 10^{-16}\times 2.998\times 10^{10}}\right)^3\ .}$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho \quad .$$

During the radiation-dominated era $R(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot{R}}{R} = \frac{1}{2t} \quad .$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3} G\rho \quad .$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \quad .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}\ \text{cm}^3\ \text{sec}^{-2}\ \text{g}^{-1}$, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732 \times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9\ \text{g/cm}^3$.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \quad .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion – anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\text{tot}} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2}\frac{1}{g_{\text{tot}}}\hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/R$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\text{rest}} T^3(t) R^3(t)$$

where $g_{\text{rest}} = g_{\text{tot}} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 R^3(t) = g_{\text{rest}} T^3(t) R^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\text{rest}}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 4: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} \ .$$

(The above expression is proportional to $1/\sqrt{V + \Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1 + \frac{\Delta V}{V}}} = \frac{u_0}{1 + \frac{1}{2}\frac{\Delta V}{V}} = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) \quad .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) \ ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V}U_0 .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2}u_0 .}$$

PROBLEM 5: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF

(a) The critical density ρ_c is defined as that density for which $k = 0$, where the Friedmann equation from the front of the exam implies that

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} .$$

Thus the critical density today is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} .$$

The mass density today of any species X is then related to $\Omega_{X,0}$ by

$$\rho_{X,0} = \rho_c\Omega_{X,0} = \frac{3H_0^2\Omega_{X,0}}{8\pi G} .$$

The total mass density today is then expressed in terms of its four components as

$$\rho_0 = \frac{3H_0^2}{8\pi G}\left[\Omega_{m,0} + \Omega_{r,0} + \Omega_{v,0} + \Omega_{\mathrm{ms},0}\right] .$$

But we also know how each of these contributions to the mass density scales with $x(t)$: $\rho_m \propto 1/x^3$, $\rho_r \propto 1/x^4$, $\rho_v \propto 1$, and $\rho_{\mathrm{ms}} \propto 1/\sqrt{V} \propto 1/x^{3/2}$. Inserting these factors,

$$\rho(t) = \frac{3H_0^2}{8\pi G}\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}}\right] .$$

(b) The Friedmann equation then becomes

$$\left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G}{3}\frac{3H_0^2}{8\pi G}\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}}\right] - \frac{kc^2}{R^2} .$$

Defining

$$H_0^2\Omega_{k,0} = -\frac{kc^2}{R^2(t_0)} ,$$

so

$$-\frac{kc^2}{R^2(t)} = -\frac{kc^2}{R^2(t_0)}\frac{1}{x^2} = \frac{H_0^2\Omega_{k,0}}{x^2} ,$$

and then the Friedmann equation becomes

$$\left(\frac{\dot{x}}{x}\right)^2 = H_0^2\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}\right] .$$

Applying this equation today, when $\dot{x}/x = H_0$, one finds that

$$\boxed{\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{v,0} - \Omega_{\mathrm{ms},0} .}$$

Rearranging the equation for $(\dot{x}/x)^2$ above,

$$H_0\,\mathrm{d}t = \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} .$$

The age of the universe is found by integrating over the full range of x, which starts from 0 when the universe is born, and is equal to 1 today. So

$$\boxed{t_0 = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} .}$$

Extra Credit for Super-Sharpies (no partial credit):

Since $\Omega_{\mathrm{tot}} < 1$, we use the Robertson-Walker open universe form

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + R^2(t)\left\{\frac{\mathrm{d}r^2}{1+r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} ,$$

where I have started with the general form from the front of the exam, and replaced k by -1. To discuss the radial transmission of light rays it is useful to define a new radial coordinate

$$r = \sinh\psi \ ,$$

so

$$\mathrm{d}r = \cosh\psi\,\mathrm{d}\psi = \sqrt{1+r^2}\,\mathrm{d}\psi \ ,$$

where I used the identity that $\cosh^2\psi = 1 + \sinh^2\psi$. The metric can then be rewritten as

$$\mathrm{d}s^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \ .$$

Light rays then travel with $\mathrm{d}\tau^2 = 0$, so

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{c}{R(t)} \ .$$

If a light ray leaves the object at time t_e and arrives at Earth today, then it will travel an interval of ψ given by

$$\psi = \int_{t_e}^{t_0} \frac{c}{R(t')}\,\mathrm{d}t' \ .$$

We will need to know ψ, but we don't know either t_e or $R(t)$. So we need to manipulate the right-hand side of the above equation to express it in terms of things that we do know. Changing integration variables from t' to x, where $x = R(t')/R(t_0)$, one finds $\mathrm{d}x = \dot{x}\,\mathrm{d}t'$, and then

$$\psi = \int_{x_e}^{1} \frac{c}{R(t_0)}\frac{1}{x}\frac{\mathrm{d}x}{\dot{x}} \ .$$

Using $H = \dot{x}/x$,

$$\psi = \frac{c}{R(t_0)}\int_{x_e}^{1} \frac{\mathrm{d}x}{x^2 H} \ .$$

Now use the formula for $H = \dot{x}/x$ from part (b), so

$$\psi = \frac{c}{R(t_0)H_0}\int_{x_e}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \ .$$

Here

$$\boxed{x_e = \frac{R(t_e)}{R(t_0)} = \frac{1}{1+z}\ ,}$$

and the coefficient in front of the integral can be evaluated using the Friedman equation for $k = -1$:

$$H_0^2 = \frac{8\pi}{3} G\rho_0 + \frac{c^2}{R^2(t_0)} = H_0^2 \Omega_0 + \frac{c^2}{R^2(t_0)}\ ,$$

so

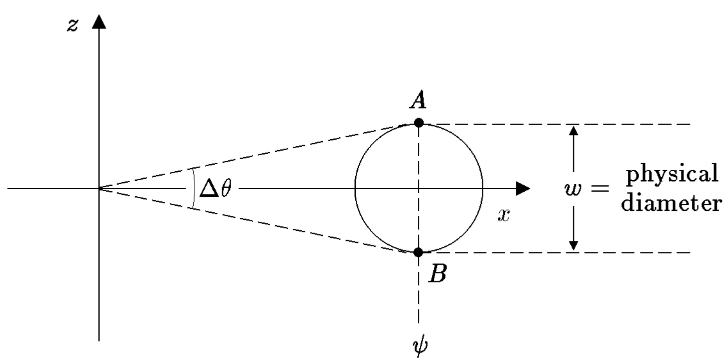
$$\frac{c^2}{R^2(t_0)\, H_0^2} = 1 - \Omega_0 = \Omega_{k,0}\ .$$

Finally, then, the expression for ψ can be written

$$\boxed{\psi = \sqrt{\Omega_{k,0}} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}}\ ,}$$

where x_e is given by the boxed equation above.

Once we know ψ, the rest is straightforward. We draw a picture in comoving coordinates of the light rays leaving the object and arriving at Earth:



In this picture $\Delta\theta$ is the angular size that would be measured. Using the $\mathrm{d}\theta^2$ part of the metric,

$$\mathrm{d}s^2 = R^2(t) \sinh^2 \psi\, \mathrm{d}\theta^2\ ,$$

we can relate w, the physical size of the object at the time of emission, to $\Delta\theta$:

$$w = R(t_e) \sinh \psi\, \Delta\theta\ .$$

To evaluate $R(t_e)$ we can use

$$R(t_e) = x_e R(t_0) = \frac{x_e c}{H_0\sqrt{\Omega_{k,0}}} \ .$$

Finally, then,

$$\boxed{\Delta\theta = \frac{w H_0 \sqrt{\Omega_{k,0}}}{x_e c \sinh\psi} \ ,}$$

where ψ is given by the boxed equation above.

PROBLEM 6: TIME SCALES IN COSMOLOGY

(a) 1 sec. *[This is the time at which the weak interactions begin to "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) 10^{-37} sec. *[We learned in Lecture Notes 7 that kT was about 1 MeV at $t = 1$ sec. Since 1 GeV = 1000 MeV, the value of kT that we want is 10^{19} times higher. In the radiation-dominated era $T \propto R^{-1} \propto t^{-1/2}$, so we get 10^{-38} sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) 10^{-5} sec. *[As in (c), we can use $t \propto T^{-2}$, with $kT \approx 1$ MeV at $t = 1$ sec.]*

PROBLEM 7: SHORT ANSWERS *(40 points)*

(a) The correct answers were the $\boxed{\text{neutrino}}$ and the $\boxed{\text{antiproton.}}$ The neutrino was first hypothesized by Wolfgang Pauli in 1932 (in order to explain the kinematics of beta decay), and first detected in the 1950s. After the positron was discovered in 1932, the antiproton was thought likely to exist, and the Bevatron in Berkeley was built to look for antiprotons. It made the first detection in the 1950s.

(b) The correct answers were $\boxed{\text{(ii), (v) and (vi).}}$ The others were incorrect for the following reasons:

  (i) the earliest prediction of the CMB temperature, by Alpher and Herman in 1948, was 5 degrees, not 0.1 degrees.

(iii) Weinberg quotes his experimental colleagues as saying that the 3° K radiation could have been observed "long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s." To Weinberg, however, the historically interesting question is not when the radiation could have been observed, but why radio astronomers did not know that they ought to try.

(iv) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: "It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe".

(c)

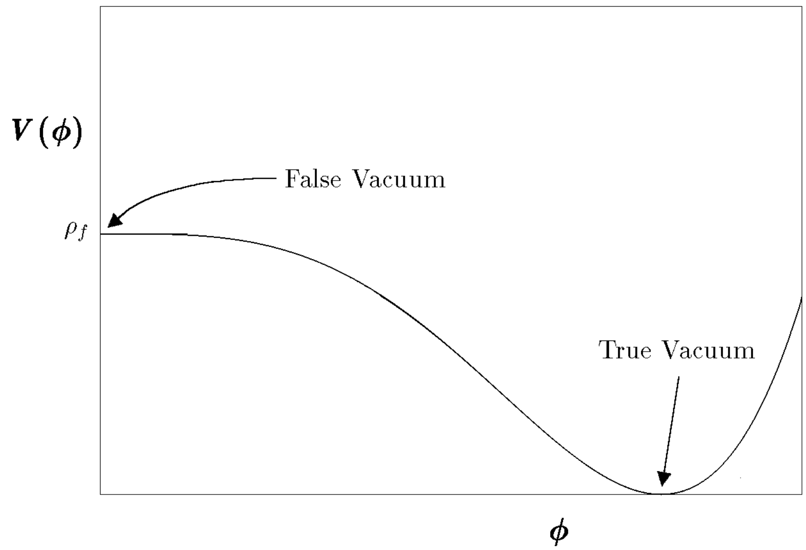
(i) $\boxed{\text{True.}}$ The rotation curves do not drop outside the luminous part of spiral galaxies but approach an approximately constant velocity. This indicates that there is a dark halo of matter well beyond the luminous part of the galaxy.

(ii) $\boxed{\text{False.}}$ The Doppler spreading is much *larger* than would be expected in the absence of dark matter, not smaller as was stated.

(iii) $\boxed{\text{True.}}$ The intensity and spectrum of the X-rays along a particular line of sight are used to estimate the density and temperature of the gas along that line of sight, and this information is then used with the equations of hydrostatic equilibrium to build a model of the cluster. See, for example, Claude Canizares' article in *Astrophysics and Space Science* **267** (1-4): 251-260, 1999, "Dark Matter in Clusters of Galaxies". For grading, however, all students were given 2 points for this question no matter what answer was given, on the grounds that the the Rowan-Robinson textbook was not very clear on this point.

(iv) $\boxed{\text{False.}}$ Most of the mass of the galaxy is in the dark matter, not stars.

(d)



(e) Supersymmetry relates $\boxed{\text{bosons and fermions.}}$

The physical distinction between these is that

- $\boxed{\text{fermions obey the Pauli exclusion principle and bosons do not.}}$

Alternatively, one could say that

- bosons are described in quantum theory by wave functions that are symmetric under the interchange of two identical particles, while fermions are described by wave functions that are antisymmetric under the interchange of two identical particles.

As a third alternative, one could say that

- bosons have integer spin (i.e., 0, 1, 2, ...) and fermions half-integer spin (i.e., 1/2, 3/2, ...).

(These three alternatives are all closely related. The second alternative is really a more detailed version of the first. The third alternative, concerning the spin, has no obvious relation to the first two, but nontheless in relativistic quantum field theory there is a "spin-statistics" theorem which implies that any integer-spin particle must be described by a symmetric wave function, and any half-integer-spin particle must be described by an antisymmetric one.)

(f) Grand unified theories unify the strong, weak, and electromagnetic interactions. Only gravity is excluded.

— Problem and solution written by Jamie Portsmouth, with some editing by AHG.

PROBLEM 8: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

(a) Since $\Omega_m + \Omega_\Lambda = 0.35 + 0.65 = 1$, the universe is flat. It therefore obeys a simple form of the Friedmann equation,

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G(\rho_m + \rho_\Lambda) \ ,$$

where the overdot indicates a derivative with respect to t, and the term proportional to k has been dropped. Using the fact that $\rho_m \propto 1/R^3(t)$ and $\rho_\Lambda =$ const, the energy densities on the right-hand side can be expressed in terms of their present values $\rho_{m,0}$ and $\rho_\Lambda \equiv \rho_{\Lambda,0}$. Defining

$$x(t) \equiv \frac{R(t)}{R(t_0)} \ ,$$

one has

$$\begin{aligned}\left(\frac{\dot{x}}{x}\right)^2 &= \frac{8\pi}{3}G\left(\frac{\rho_{m,0}}{x^3} + \rho_\Lambda\right) \\ &= \frac{8\pi}{3}G\rho_{c,0}\left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \\ &= H_0^2\left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \ .\end{aligned}$$

Here we used the facts that

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}}; \qquad \Omega_{\Lambda,0} \equiv \frac{\rho_\Lambda}{\rho_{c,0}} \ ,$$

and

$$H_0^2 = \frac{8\pi}{3}G\rho_{c,0} \ .$$

The equation above for $(\dot{x}/x)^2$ implies that

$$\dot{x} = H_0\, x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}} \ ,$$

which in turn implies that

$$\mathrm{d}t = \frac{1}{H_0}\frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .$$

Using the fact that x changes from 0 to 1 over the life of the universe, this relation can be integrated to give

$$t_0 = \int_0^{t_0} \mathrm{d}t = \boxed{\frac{1}{H_0} \int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}}} \ .$$

The answer can also be written as

$$\boxed{t_0 = \frac{1}{H_0} \int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}}}$$

or

$$\boxed{t_0 = \frac{1}{H_0} \int_0^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}} \ ,}$$

where in the last answer I changed the variable of integration using

$$x = \frac{1}{1+z} \ ; \qquad \mathrm{d}x = -\frac{\mathrm{d}z}{(1+z)^2} \ .$$

Note that the minus sign in the expression for $\mathrm{d}x$ is canceled by the interchange of the limits of integration: $x = 0$ corresponds to $z = \infty$, and $x = 1$ corresponds to $z = 0$.

Your answer should look like one of the above boxed answers. You were not expected to complete the numerical calculation, but for pedagogical purposes I will continue. The integral can actually be carried out analytically, giving

$$\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}} + \sqrt{\Omega_{\Lambda,0}}}{\sqrt{\Omega_m}}\right) \ .$$

Using

$$\frac{1}{H_0} = \frac{9.778 \times 10^9}{h_0} \ \mathrm{yr} \ ,$$

where $H_0 = 100\,h_0\,\text{km-sec}^{-1}\text{-Mpc}^{-1}$, one finds for $h_0 = 0.65$ that

$$\frac{1}{H_0} = 15.043 \times 10^9 \ \mathrm{yr} \ .$$

Then using $\Omega_m = 0.35$ and $\Omega_{\Lambda,0} = 0.65$, one finds

$$\boxed{t_0 = 13.88 \times 10^9 \text{ yr} .}$$

So the SDSS people were right on target.

(b) Having done part (a), this part is very easy. The dynamics of the universe is of course the same, and the question is only slightly different. In part (a) we found the amount of time that it took for x to change from 0 to 1. The light from the quasar that we now receive was emitted when

$$x = \frac{1}{1+z} ,$$

since the cosmological redshift is given by

$$1 + z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} .$$

Using the expression for dt from part (a), the amount of time that it took the universe to expand from $x = 0$ to $x = 1/(1+z)$ is given by

$$t_e = \int_0^{t_e} \mathrm{d}t = \boxed{\frac{1}{H_0}\int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Again one could write the answer other ways, including

$$\boxed{t_0 = \frac{1}{H_0}\int_z^{\infty} \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0}}} .}$$

Again you were expected to stop with an expression like the one above. Continuing, however, the integral can again be done analytically:

$$\int_0^{x_{\max}} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}x_{\max}^3} + \sqrt{\Omega_{\Lambda,0}}\, x_{\max}^{3/2}}{\sqrt{\Omega_m}}\right) .$$

Using $x_{\max} = 1/(1+5.82) = .1466$ and the other values as before, one finds

$$\boxed{t_e = \frac{0.06321}{H_0} = 0.9509 \times 10^9 \text{ yr} .}$$

So again the SDSS people were right.

(c) To find the physical distance to the quasar, we need to figure out how far light can travel from $z = 5.82$ to the present. Since we want the present distance, we multiply the coordinate distance by $R(t_0)$. For the flat metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + R^2(t)\left\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right\} ,$$

the coordinate velocity of light (in the radial direction) is found by setting $\mathrm{d}s^2 = 0$, giving

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{R(t)} .$$

So the total coordinate distance that light can travel from t_e to t_0 is

$$\ell_c = \int_{t_e}^{t_0} \frac{c}{R(t)}\,\mathrm{d}t .$$

This is not the final answer, however, because we don't explicitly know $R(t)$. We can, however, change variables of integration from t to x, using

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \frac{\mathrm{d}x}{\dot{x}} .$$

So

$$\ell_c = \frac{c}{R(t_0)} \int_{x_e}^{1} \frac{\mathrm{d}x}{x\,\dot{x}} ,$$

where x_e is the value of x at the time of emission, so $x_e = 1/(1+z)$. Using the equation for $\dot{x}$ from part (a), this integral can be rewritten as

$$\ell_c = \frac{c}{H_0 R(t_0)} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .$$

Finally, then

$$\boxed{\ell_{\text{phys},0} = R(t_0)\,\ell_c = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Alternatively, this result can be written as

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}\, x + \Omega_{\Lambda,0}\, x^4}}\ ,}$$

or by changing variables of integration to obtain

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .}$$

Continuing for pedagogical purposes, this time the integral has no analytic form, so far as I know. Integrating numerically,

$$\int_0^{5.82} \frac{\mathrm{d}z'}{\sqrt{0.35\,(1+z')^3 + 0.65}} = 1.8099\ ,$$

and then using the value of $1/H_0$ from part (a),

$$\boxed{\ell_{\text{phys},0} = 27.23 \text{ light-yr}\ .}$$

Right again.

(d) $\ell_{\text{phys},e} = R(t_e)\ell_c$, so

$$\ell_{\text{phys},e} = \frac{R(t_e)}{R(t_0)}\, \ell_{\text{phys},0} = \boxed{\frac{\ell_{\text{phys},0}}{1+z}\ .}$$

Numerically this gives

$$\boxed{\ell_{\text{phys},e} = 3.992 \times 10^9 \text{ light-yr}\ .}$$

The SDSS announcement is still okay.

(e) The speed defined in this way obeys the Hubble law exactly, so

$$v = H_0\, \ell_{\text{phys},0} = c \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .$$

Then

$$\boxed{\frac{v}{c} = \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .}$$

Numerically, we have already found that this integral has the value

$$\boxed{\frac{v}{c} = 1.8099\ .}$$

The SDSS people get an A.