MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 6, 2005
Prof. Alan Guth

QUIZ 1 SOLUTIONS

Quiz Date: October 18, 2005

Corrected 11/7/05: Factors of c Added to Problem 3 Solution

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(4 points)* What was the first external galaxy that was shown to be at a distance significantly greater than the most distant known objects in our galaxy? How was the distance estimated?

Ans: (Weinberg, page 20) The first galaxy shown to be at a distance beyond the size of our galaxy was Andromeda, also known by its Messier number, M31. It is the nearest spiral galaxy to our galaxy. The distance was determined (by Hubble) using Cepheid variable stars, for which the absolute luminosity is proportional to the period. A measurement of a particular Cepheid's period determines the star's absolute luminosity, which, compared to the measured luminosity, determines the distance to the star. (Hubble's initial measurement of the distance to Andromeda used a badly-calibrated version of this period-luminosity relationship and consequently underestimated the distance by more than a factor of two; nonetheless, the initial measurement still showed that the Andromeda Nebula was an order of magnitude more distant than the most distant known objects in our own galaxy.)

(b) *(5 points)* What is recombination? Did galaxies begin to form before or after recombination? Why?

Ans: (Weinberg, pages 64 and 73) Recombination refers to the formation of neutral atoms out of charged nuclei and electrons. Galaxies began to form after recombination. Prior to recombination, the strong electromagnetic interactions between photons and matter produced a high pressure which effectively counteracted the gravitational attraction between particles. Once the universe became transparent to radiation, the matter no longer interacted significantly with the photons and consequently began to undergo gravitational collapse into large clumps.

(c) *(4 points)* In Chapter IV of his book, Weinberg develops a "recipe for a hot universe," in which the matter of the universe is described as a gas in thermal equilbrium at a very high temperature, in the vicinity of 10^9 K (several thousand million degrees Kelvin). Such a thermal equilibrium gas is completely described by specifying its temperature and the density of the conserved quantities. Which of the following is on this list of conserved quantities? Circle as many as apply.

(i) baryon number (ii) energy per particle (iii) proton number

(iv) electric charge (v) pressure

Ans: (Weinberg, page 91) The correct answers are (i) and (iv). A third conserved quantity, lepton number, was not included in the multiple-choice options.

(d) *(4 points)* The wavelength corresponding to the mean energy of a CMB (cosmic microwave background) photon today is approximately equal to which of the following quantities? (You may wish to look up the values of various physical constants at the end of the quiz.)

(i) 2 fm (2×10^{-15} m)

(ii) 2 microns (2×10^{-6} m)

(iii) 2 mm (2×10^{-3} m)

(iv) 2 m.

Ans: (Ryden, page 23) The correct answer is (iii).

If you did not remember this number, you could estimate the answer by remembering that the characteristic temperature of the cosmic microwave background is approximately 3 Kelvin. The typical photon energy is then on the order of kT, from which we can find the frequency as $E = h\nu$. The wavelength of the photon is then $\lambda = \nu/c$. This approximation gives $\lambda = 5.3$ mm, which is not equal to the correct answer, but it is much closer to the correct answer than to any of the other choices.

(e) *(4 points)* What is the equivalence principle?

Ans: (Ryden, page 27) In its simplest form, the equivalence principle says that the gravitational mass of an object is identical to its inertial mass. This equality implies the equivalent statement that it is impossible to distinguish (without additional information) between an observer in a reference frame accelerating with acceleration $\vec{a}$ and an observer in an inertial reference frame subject to a gravitational force $-m_{obs}\vec{a}$.

(Actually, what the equivalence principle really says is that the ratio of the gravitational to inertial masses m_g/m_i is universal, that is, independent of the material properties of the object in question. The ratio does not necessarily need to be 1. However, once we know that the two types of masses are proportional, we can simply define the gravitational coupling G to make them equal. To see this, consider a theory of gravity where $m_g/m_i = q$. Then the gravitational force law is

$$m_i a = -\frac{GMm_g}{r^2} ,$$

or

$$a = -\frac{GqM}{r^2} \ .$$

At this point, if we define $G' = Gq$, we have a gravitational theory with gravitational coupling G' and inertial mass equal to gravitational mass.)

(f) *(4 points)* Why is it difficult for Earth-based experiments to look at the small wavelength portion of the graph of CMB energy density per wavelength vs. wavelength?

Ans: (Weinberg, page 67) The Earth's atmosphere is increasingly opaque for wavelength shorter than .3 cm. Therefore, radiation at these wavelengths will be absorbed and rescattered by the Earth's atmosphere; observations of the cosmic microwave background at small wavelengths must be performed above the Earth's atmosphere.

PROBLEM 2: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(30 points)*

This question was Problem 5 on Problem Set 2.

(a) We use the chain rule to write the definition of the Hubble parameter in terms of derivatives with respect to θ:

$$H(\theta) = \frac{1}{R}\frac{dR}{d\theta}\frac{d\theta}{dt} \ .$$

The parametric equations for R and t for an open, matter-dominated universe are given by

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) \ .$$

Recall that the hyperbolic trigonometric functions are differentiated as

$$\frac{d}{d\theta}\sinh\theta = \cosh\theta \ ,$$

$$\frac{d}{d\theta}\cosh\theta = \sinh\theta \ ,$$

so the parametric equations can be differentiated to give

$$\frac{dR}{d\theta} = \alpha\sqrt{\kappa}\sinh\theta \ ,$$

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(\cosh\theta - 1) = \frac{1}{d\theta/dt} \ .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{\kappa}\alpha(\cosh\theta - 1)}\right]\left[\alpha\sqrt{\kappa}\sinh\theta\right]\left[\frac{c}{\alpha(\cosh\theta - 1)}\right]$$

$$= \boxed{\frac{c\sinh\theta}{\alpha(\cosh\theta - 1)^2} \ .}$$

(b) This problem can be attacked by at least three different methods. While you were expected to use only one, we will show all three.

(i) One way to find ρ is to use

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \ .$$

This is usually the safest method to find ρ for a cosmological model, since the above equation is one of the general Friedmann equations. The equation requires that the universe be homogeneous and isotropic, but it is valid for any form of matter. By contrast, the two other methods that will be shown below are valid only for "matter-dominated" universes (i.e., universes that are dominated by nonrelativistic matter, for which the pressure is always negligible). One can rewrite this equation as

$$\frac{8\pi}{3}G\rho = H^2 + \frac{kc^2}{R^2} \ .$$

Recalling that we described open universes by using $\kappa \equiv -k$, this can be rewritten as

$$\frac{8\pi}{3}G\rho = H^2 - \frac{\kappa c^2}{R^2} \ .$$

Replacing H by the answer in part (a) and R by its parametric equation, one finds

$$\frac{8\pi}{3}G\rho = \frac{c^2\sinh^2\theta}{\alpha^2(\cosh\theta - 1)^4} - \frac{\kappa c^2}{\alpha^2\kappa(\cosh\theta - 1)^2}$$

$$= \frac{c^2}{\alpha^2(\cosh\theta - 1)^4}\left[\sinh^2\theta - (\cosh\theta - 1)^2\right] \ .$$

Now make use of the hypertrigonometric identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

to simplify:

$$\sinh^2\theta - (\cosh\theta - 1)^2 = \sinh^2\theta - \cosh^2\theta + 2\cosh\theta - 1$$
$$= 2(\cosh\theta - 1)\ ,$$

so

$$\frac{8\pi}{3}G\rho = \frac{2c^2}{\alpha^2(\cosh\theta - 1)^3}\ .$$

Dividing both sides of the equation by $(8\pi/3)G$, one finds

$$\boxed{\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3}\ .}$$

(ii) The equation for α in the formula sheet,

$$\alpha = \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

can be solved for ρ to give

$$\rho = \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{GR^3}\ .$$

Then substitute the parametric equation for $R(\theta)$:

$$\rho = \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{G}\frac{1}{\alpha^3\kappa^{3/2}(\cosh\theta - 1)^3}$$
$$= \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3}\ .$$

(iii) ρ can also be found from $\ddot{R} = -(4\pi/3)G\rho R$, as long as we know that the universe is matter-dominated. (Be careful, however, about applying this formula in other situations: if the pressure cannot be neglected, then this equation has to be modified.) To evaluate $\ddot{R}$, again use the chain rule. Starting with $\dot{R}$,

$$\dot{R} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \alpha\sqrt{\kappa}\sinh\theta\frac{c}{\alpha(\cosh\theta - 1)} = \frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\ .$$

Then

$$\ddot{R} = \frac{d\dot{R}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left[\frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\right]\frac{c}{\alpha(\cosh\theta - 1)}$$
$$= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)}\left[\frac{\cosh\theta}{\cosh\theta - 1} - \frac{\sinh^2\theta}{(\cosh\theta - 1)^2}\right]$$
$$= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}\left[\cosh\theta(\cosh\theta - 1) - \sinh^2\theta\right]$$
$$= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}(1 - \cosh\theta) = -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2}\ .$$

So

$$\ddot{R} = -\frac{4\pi}{3}G\rho R \quad\Longrightarrow\quad -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} = -\frac{4\pi}{3}G\rho\alpha\sqrt{\kappa}(\cosh\theta - 1)\ ,$$

and

$$\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3}\ .$$

(c) The critical mass density satisfies the cosmological evolution equations for $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c\ .$$

Then

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2}\ .$$

Now replace H by the answer to part (a), and ρ by the answer to part (b):

$$\Omega = \frac{8\pi G}{3}\left[\frac{3}{4\pi}\frac{c^2}{G\alpha^2(\cosh\theta - 1)^3}\right]\left[\frac{\alpha^2(\cosh\theta - 1)^4}{c^2\sinh^2\theta}\right]$$
$$= 2\frac{\cosh\theta - 1}{\sinh^2\theta} = 2\frac{\cosh\theta - 1}{\cosh^2\theta - 1}$$
$$= 2\frac{\cosh\theta - 1}{(\cosh\theta + 1)(\cosh\theta - 1)} = \boxed{\frac{2}{\cosh\theta + 1}\ .}$$

The answer can be written even more compactly, if one wishes, by using a further hypertrigonometric identity:

$$\Omega = \frac{2}{\cosh\theta + 1} = \frac{1}{\cosh^2\frac{1}{2}\theta} = \operatorname{sech}^2\frac{1}{2}\theta\ .$$

(d) The basic formula that determines the physical value of the horizon distance is given by Eq. (5.7) of the lecture notes:

$$\ell_{p,\text{horizon}}(t) = R(t) \int_0^t \frac{c}{R(t')} dt' \ .$$

The complication here is that R is given as a function of θ, rather than t. The problem is handled, however, by a simple change of integration variables. One can change the integral over t' to an integral over θ', provided that one replaces

$$dt' \to \frac{dt'}{d\theta'} d\theta' = \frac{\alpha}{c}(\cosh\theta' - 1) d\theta' \ .$$

One must also re-express the limits of integration in terms of θ. So

$$\begin{aligned}
\ell_{p,\text{horizon}}(\theta) &= R(\theta) \int_0^\theta \frac{c}{R(\theta')} \frac{dt'}{d\theta'} d\theta' \\
&= \alpha\sqrt{\kappa}(\cosh\theta - 1) \int_0^\theta \frac{c}{\alpha\sqrt{\kappa}(\cosh\theta' - 1)} \frac{\alpha}{c} (\cosh\theta' - 1)\, d\theta' \ . \\
&= \alpha(\cosh\theta - 1) \int_0^\theta d\theta' = \boxed{\alpha\,\theta(\cosh\theta - 1) \ .}
\end{aligned}$$

(e) The key to this problem is the use of power series expansions. In general, any sufficiently smooth function $f(x)$ can be expanded about the point x_0 by the series

$$\begin{aligned}
f(x) = f(x_0) &+ \frac{1}{1!} f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 \\
&+ \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \dots \ ,
\end{aligned}$$

where the prime is used to denote a derivative. In particular, the exponential, sinh, and cosh functions can be expanded about $\theta = 0$ by the formulas

$$\begin{aligned}
e^\theta &= 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \dots \\
\sinh\theta &= \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^5}{7!} \dots \\
\cosh\theta &= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \dots \ .
\end{aligned}$$

For this problem, we expand the parametric equations for $R(\theta)$ and $t(\theta)$, keeping the first nonvanishing term in the power series expansions:

$$\begin{aligned}
t &= \frac{\alpha}{c}(\sinh\theta - \theta) = \frac{\alpha}{c}\left(\frac{\theta^3}{3!} + \dots\right) \\
R &= \alpha\sqrt{\kappa}(\cosh\theta - 1) = \alpha\sqrt{\kappa}\left(\frac{\theta^2}{2!} + \dots\right) \ .
\end{aligned}$$

The first expression can be solved for θ, giving

$$\theta \approx \left(\frac{6ct}{\alpha}\right)^{1/3} \ ,$$

which can be substituted into the second expression to give

$$\boxed{R \approx \frac{1}{2}\alpha\sqrt{\kappa}\left(\frac{6ct}{\alpha}\right)^{2/3} \ .}$$

The power series expansions for the sinh and cosh are valid whenever the terms left out are much smaller than the last term kept, which happens when $\theta \ll 1$. Given the above relation between θ and t, this condition is equivalent to

$$t \ll \frac{\alpha}{6c} \ .$$

Thus,

$$\boxed{t^* \approx \frac{\alpha}{6c}, \text{ or } t^* \approx \frac{\alpha}{c} \ .}$$

Since there is no precise meaning to the statement that an approximation is valid, there is no precise value for t^*. It is possible to be more precise by placing criteria on the size of the first omitted term in the series, and using these criteria to derive a more precise value for t^*. These expressions for t^* are always in the form of a dimensionless constant times α/c. This approach is very good, but it was not required to get full credit for this problem.

(f) From part (c), the expression for Ω is given by

$$\Omega = \frac{2}{\cosh\theta + 1} \ .$$

So,

$$1 - \Omega = 1 - \frac{2}{\cosh\theta + 1} = \frac{\cosh\theta - 1}{\cosh\theta + 1} .$$

Expanding numerator and denominator in power series,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots}{2 + \frac{\theta^2}{2!} + \ldots} .$$

Keeping only the leading terms,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2}}{2} = \frac{1}{4}\theta^2 ,$$

so

$$\boxed{1 - \Omega \approx \frac{1}{4}\left(\frac{6ct}{\alpha}\right)^{2/3} .}$$

This result shows that the deviation of Ω from 1 is amplified with time. This fact leads to a conundrum called the "flatness problem", which will be discussed later in the course.

A common mistake (very minor) was to keep extra terms, especially in the denominator. Keeping extra terms allows a higher degree of accuracy, so there is nothing wrong with it. However, one should always be sure to keep ***all*** terms of a given order, since keeping only a subset of terms may or may not increase the accuracy. In this case, an extra term in the denominator can be rewritten as a term in the numerator:

$$\begin{aligned}\frac{\frac{\theta^2}{2!}}{2 + \frac{\theta^2}{2!}} &= \frac{1}{4}\frac{\theta^2}{1 + \frac{\theta^2}{4}} = \frac{1}{4}\theta^2\left(1 - \frac{\theta^2}{4} + \ldots\right) \\ &= \frac{1}{4}\theta^2 - \frac{1}{16}\theta^4 + \ldots ,\end{aligned}$$

where I used the expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots .$$

Thus, the extra term in the denominator is equivalent to a term in the numerator of order θ^4, but other terms proportional to θ^4 have been dropped. So, it is not worthwhile to keep the 2nd term in the expansion of the denominator.

PROBLEM 3: TRACING A LIGHT PULSE THROUGH A RADIATION-DOMINATED UNIVERSE *(25 points)*

(a) The physical horizon distance is given in general by

$$\ell_{p,\text{horizon}} = R(t)\int_0^{t_f} \frac{c}{R(t)}\,dt ,$$

so in this case

$$\ell_{p,\text{horizon}} = bt^{1/2}\int_0^{t_f} \frac{c}{bt^{1/2}}\,dt = \boxed{2ct_f .}$$

(b) If the source is at the horizon distance, it means that a photon leaving the source at $t = 0$ would just be reaching the origin at t_f. So, $\boxed{t_e = 0 .}$

(c) The coordinate distance between the source and the origin is the coordinate horizon distance, given by

$$\ell_{c,\text{horizon}} = \int_0^{t_f} \frac{c}{bt^{1/2}}\,dt = \boxed{\frac{2ct_f^{1/2}}{b} .}$$

(d) The photon starts at coordinate distance $2c\sqrt{t_f}/b$, and by time t it will have traveled a coordinate distance

$$\int_0^t \frac{c}{bt'^{1/2}}\,dt' = \frac{2c\sqrt{t}}{b}$$

toward the origin. Thus the photon will be at coordinate distance

$$\ell_c = \frac{2c}{b}\left(\sqrt{t_f} - \sqrt{t}\right)$$

from the origin, and hence a physical distance

$$\ell_p(t) = R(t)\ell_c = \boxed{2c\left(\sqrt{t\,t_f} - t\right) .}$$

(e) To find the maximum of $\ell_p(t)$, we differentiate it and set the derivative to zero:

$$\frac{d\ell_p}{dt} = \left(\sqrt{\frac{t_f}{t}} - 2\right)c ,$$

so the maximum occurs when

$$\sqrt{\frac{t_f}{t_{\max}}} = 2 \; ,$$

or

$$\boxed{t_{\max} = \frac{1}{4} t_f \; .}$$

PROBLEM 4: TRANSVERSE DOPPLER SHIFTS *(20 points)*

(a) Describing the events in the coordinate system shown, the Xanthu is at rest, so its clocks run at the same speed as the coordinate system time variable, t. The emission of the wavecrests of the radio signal are therefore separated by a time interval equal to the time interval as measured by the source, the Xanthu:

$$\Delta t = \Delta t_s \; .$$

Since the Emmerac is moving perpendicular to the path of the radio waves, at the moment of reception its distance from the Xanthu is at a minimum, and hence its rate of change is zero. Hence successive wavecrests will travel the same distance, as long as $c\Delta t \ll a$. Since the wavecrests travel the same distance, the time separation of their arrival at the Emmerac is Δt, the same as the time separation of their emission. The clocks on the Emmerac, however, and running slowly by a factor of

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \; .$$

The time interval between wave crests as measured by the receiver, on the Emmerac, is therefore smaller by a factor of γ,

$$\Delta t_r = \frac{\Delta t_s}{\gamma} \; .$$

Thus, there is a $\boxed{\text{blueshift.}}$ The redshift parameter z is defined by

$$\frac{\Delta t_r}{\Delta t_s} = 1 + z \; ,$$

so

$$\frac{1}{\gamma} = 1 + z \; ,$$

or

$$\boxed{z = \frac{1-\gamma}{\gamma} \; .}$$

Recall that $\gamma > 1$, so z is negative.

(b) Describing this situation in the coordinate system shown, this time the source on the Xanthu is moving, so the clocks at the source are running slowly. The time between wavecrests, measured in coordinate time t, is therefore larger by a factor of γ than Δt_s, the time as measured by the clock on the source:

$$\Delta t = \gamma \, \Delta t_s \; .$$

Since the radio signal is emitted when the Xanthu is at its minimum separation from the Emmerac, the rate of change of the separation is zero, so each wavecrest travels the same distance (again assuming that $c\Delta t \ll a$). Since the Emmerac is at rest, its clocks run at the same speed as the coordinate time t, and hence the time interval between crests, as measured by the receiver, is

$$\Delta t_r = \Delta t = \gamma \, \Delta t_s \; .$$

Thus the time interval as measured by the receiver is longer than that measured by the source, and hence it is a $\boxed{\text{redshift.}}$ The redshift parameter z is given by

$$1 + z = \frac{\Delta t_r}{\Delta t_s} = \gamma \; ,$$

so

$$\boxed{z = \gamma - 1 \; .}$$

(c) The events described in (a) can be made to look a lot like the events described in (b) by transforming to a frame of reference that is moving to the right at speed v_0 — i.e., by transforming to the rest frame of the Emmerac. In this frame the Emmerac is of course at rest, and the Xanthu is traveling on the trajectory

$$(x{=}-v_0 t, y{=}a, z{=}0) \; ,$$

as in part (b). However, just as the transformation causes the x-component of the velocity of the Xanthu to change from zero to a negative value, so the x-component of the velocity of the radio signal will be transformed from zero to a negative value. Thus in this frame the radio signal will not be traveling along the y-axis, so the events will not match those described in (b). The situations described in (a) and (b) are therefore physically distinct (which they must be if the redshifts are different, as we calculated above).