MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 29, 2005
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QUIZ 2 SOLUTIONS

Quiz Date: November 10, 2005

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(4 points)* For an open universe with a positive mass density, Ryden shows (in chapter 4) that the radius of curvature $R_{\rm curv} \equiv R(t)/\sqrt{-k}$ and the Hubble length $\ell_{\rm Hubble} \equiv c/H_0$ obey one of the following relations:

(i) $R_{\rm curv} > \ell_{\rm Hubble}$

(ii) $R_{\rm curv} = \ell_{\rm Hubble}$

(iii) $R_{\rm curv} < \ell_{\rm Hubble}$

Which of these relations is true?

Ans: (Ryden, page 50) The correct answer is (i).

(b) *(6 points)* Give a derivation of the relation in part (a).

Ans: (Ryden, page 50) The Friedmann equation (in our notation) is

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{c^2}{R^2_{\rm curv}},$$

where we have substituted the definition of the radius of curvature $R_{\rm curv} \equiv R(t)/\sqrt{-k}$. Dividing through by c^2, we have

$$\frac{1}{\ell^2_{\rm Hubble}} = \frac{8\pi G}{3c^2}\rho + \frac{1}{R^2_{\rm curv}}.$$

Since $\rho > 0$, we must have

$$\frac{1}{\ell^2_{\rm Hubble}} > \frac{1}{R^2_{\rm curv}}$$

and therefore

$$R_{\rm curv} > \ell_{\rm Hubble}.$$

(c) *(5 points)* As Ryden discusses in chapter 5, the universe today contains not only the photons of the cosmic microwave background (CMB), but also photons that originated as starlight. Including both direct starlight and starlight absorbed and reradiated by dust, the ratio of energy densities $\varepsilon_{\rm Starlight}/\varepsilon_{\rm CMB}$ has approximately which of the following values:

(i) 10^{-10} (ii) 10^{-5} (iii) 10^{-1} (iv) 10^{5} (v) 10^{10}

Ans: (Ryden, page 66) The correct answer is (iii).

(d) *(10 points)* For a flat universe that contains only radiation and nonrelativistic matter, Ryden (chapter 6) writes the Friedmann equation as

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \ ,$$

where H_0 refers to the present value of the Hubble parameter H, and $\Omega_{r,0}$ and $\Omega_{m,0}$ refer to the present values of the mass densities in radiation and matter, respectively, compared to the critical density. Ryden rearranges this formula to take the form

$$H_0 \, \mathrm{d}t = \frac{a \, \mathrm{d}a}{A} \left[1 + \frac{a}{a_{rm}} \right]^B \ , \tag{1}$$

where A and B are constants that might depend on the parameters H_0, $\Omega_{r,0}$, and $\Omega_{m,0}$. (Ryden wrote A and B explicitly, but for the purpose of this question I have not.) a_{rm} is the scale factor of radiation-matter equality: i.e., the scale factor when the energy densities of radiation and matter are equal. Recall that $a(t)$ is the notation Ryden uses for the scale factor, which in the Lecture Notes is called $R(t)$.

(i) *(4 points)* Write an expression for a_{rm} in terms of all or some of the parameters H_0, $\Omega_{r,0}$, and $\Omega_{m,0}$.

Ans: (Ryden, page 68) By definition, a_{rm} is the value of the scale factor when $\rho_r(a_{rm}) = \rho_m(a_{rm})$. The energy density in radiation at an arbitrary value of the scale factor is related to its present-day value $\rho_{r,0}$ by $\rho_r(a) = \rho_{r,0}/a^4$, while the energy density of matter is given by $\rho_m(a) = \rho_{m,0}/a^3$. (We use Ryden's normalization for the scale factor, $a_0 = a(t_0) = 1$; otherwise, we would need to carry around an arbitrary constant setting the normalization of a notch.) Therefore the statement that the energy densities in radiation and matter were equal at a_{rm} can be written

$$\frac{\rho_{r,0}}{a_{rm}^4} = \frac{\rho_{m,0}}{a_{rm}^3} .$$

Dividing both sides by the critical density gives

$$\frac{\Omega_{r,0}}{a_{rm}^4} = \frac{\Omega_{m,0}}{a_{rm}^3} ,$$

which can be solved to yield

$$\boxed{a_{rm} = \frac{\Omega_{m,0}}{\Omega_{r,0}}.}$$

(Notice that if we had not set $a_0 = 1$, this equation would need to include the arbitrary constant controlling the definition of a notch.)

(ii) *(6 points)* Derive Eq. (1) above, and find the values of A and B.

Ans: (Ryden, page 94) Starting from the form of the Friedmann equation given above, we pull out an overall factor of $\Omega_{r,0}/a^4$ on the right-hand side,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4}\left(1 + \frac{a}{a_{rm}}\right).$$

Taking the square root yields

$$\frac{H}{H_0} = \frac{da}{a\,dt\,H_0} = \frac{\Omega_{r,0}^{1/2}}{a^2}\left(1 + \frac{a}{a_{rm}}\right)^{1/2},$$

which we rearrange to obtain

$$H_0 dt = \frac{a\,da}{\Omega_{r,0}^{1/2}}\left(1 + \frac{a}{a_{rm}}\right)^{-1/2}.$$

This lets us read off the values of A and B,

$$\boxed{A = \Omega_{r,0}^{1/2}\,, \qquad B = -\frac{1}{2}.}$$

(e) BONUS QUESTION *(1 point)*: Where does Barbara Ryden suggest writing the Friedmann equation?

(Ryden, page 49) On your forehead.

PROBLEM 2: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(20 points)*

(a) The Friedmann equation in a flat universe is

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho .$$

Substituting $\rho = \text{const}/R^5$ and taking the square root of both sides gives

$$\frac{\dot{R}}{R} = \alpha R^{-5/2} \ ,$$

for some constant α. Rearranging to a form we can integrate,

$$dR\, R^{3/2} = \alpha dt ,$$

and therefore

$$\frac{2}{5} R^{5/2} = \alpha t .$$

Notice that once again we have eliminated the arbitrary integration constant by choosing the Big Bang boundary conditions $R = 0$ at $t = 0$. Solving for R yields

$$\boxed{R \propto t^{2/5} .}$$

(b) The Hubble parameter is, from its definition,

$$\boxed{H = \frac{\dot{R}}{R} = \frac{2}{5t} ,}$$

where we have used the time dependence of $R(t)$ that we found in part (a). (Notice that we don't need to know the constant of proportionality left undetermined in part (a), as it cancels between numerator and denominator in this calculation.)

(c) Recall that the horizon distance is the physical distance traveled by a light ray since $t = 0$,

$$\ell_{p,\text{horizon}}(t) = R(t) \int_0^t \frac{c\, dt'}{R(t')} .$$

Using $R(t) \propto t^{2/5}$, we find

$$\ell_{p,\text{horizon}}(t) = ct^{2/5} \int_0^t dt'\, t'^{-2/5}$$

or

$$\ell_{p,\text{horizon}}(t) = ct^{2/5} \left(\frac{5}{3} t^{3/5} \right) = \boxed{\frac{5}{3} ct.}$$

(d) Since we know the Hubble parameter, we can find the mass density $\rho(t)$ easily from the Friedmann equation,

$$\rho(t) = \frac{3H^2}{8\pi G}.$$

Using the result from part (b), we find

$$\boxed{\rho(t) = \frac{3}{50\pi G} \frac{1}{t^2}.}$$

As a check on our algebra, since we found in (a) that $R \propto t^{2/5}$, and knew at the beginning of the calculation that $\rho \propto R^{-5}$, we should find, as we do here, that $\rho \propto t^{-2}$. Notice, however, that in this case we do not leave our answer in terms of some undetermined constant of proportionality; the units of ρ are not arbitrary, and therefore we care about its normalization.

PROBLEM 3: AN EXERCISE IN TWO-DIMENSIONAL METRICS

(30 points)

(a) Since

$$r(\theta) = (1 + \epsilon \sin\theta)\, r_0 \ ,$$

as the angular coordinate θ changes by $\mathrm{d}\theta$, r changes by

$$\mathrm{d}r = \epsilon r_0 \cos\theta\, \mathrm{d}\theta \ .$$

$\mathrm{d}s^2$ is then given by

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \\
&= \epsilon^2 r_0^2 \cos^2\theta\, \mathrm{d}\theta^2 + (1 + \epsilon\sin\theta)^2\, r_0^2\, \mathrm{d}\theta^2 \\
&= \left[\epsilon^2 \cos^2\theta + 1 + 2\epsilon \sin\theta + \epsilon^2 \sin^2\theta\right]\, r_0^2\, \mathrm{d}\theta^2 \\
&= \left[1 + \epsilon^2 + 2\epsilon \sin\theta\right]\, r_0^2\, \mathrm{d}\theta^2 \ ,
\end{aligned}$$

so

$$\mathrm{d}s = r_0 \sqrt{1 + \epsilon^2 + 2\epsilon \sin\theta}\, \mathrm{d}\theta \ .$$

Since θ runs from θ_1 to θ_2 as the curve is swept out,

$$\boxed{S = r_0 \int_{\theta_1}^{\theta_2} \sqrt{1 + \epsilon^2 + 2\epsilon \sin\theta}\, d\theta \ .}$$

(b) Since θ does not vary along this path,

$$\mathrm{d}s = \sqrt{1 + \frac{r^2}{a^2}}\, \mathrm{d}r \ ,$$

and so

$$\boxed{R = \int_0^{r_0} \sqrt{1 + \frac{r^2}{a^2}}\, \mathrm{d}r \ .}$$

(c) Since the metric does not contain a term in $\mathrm{d}r\,\mathrm{d}\theta$, the r and θ directions are orthogonal. Thus, if one considers a small region in which r is in the interval r' to $r' + \mathrm{d}r'$, and θ is in the interval θ' to $\theta' + \mathrm{d}\theta'$, then the region can be treated as a rectangle. The side along which r varies has length $\mathrm{d}s_r = \sqrt{1 + (r'^2/a^2)}\, \mathrm{d}r'$, while the side along which θ varies has length $\mathrm{d}s_\theta = r'\, \mathrm{d}\theta'$. The area is then

$$\mathrm{d}A = \mathrm{d}s_r\, \mathrm{d}s_\theta = r' \sqrt{1 + (r'^2/a^2)}\, \mathrm{d}r'\, \mathrm{d}\theta' \ .$$

To cover the area for which $r < r_0$, r' must be integrated from 0 to r_0, and θ' must be integrated from 0 to 2π:

$$A = \int_0^{r_0} \mathrm{d}r' \int_0^{2\pi} \mathrm{d}\theta'\, r' \sqrt{1 + (r'^2/a^2)} \ .$$

But

$$\int_0^{2\pi} \mathrm{d}\theta' = 2\pi \ ,$$

so

$$\boxed{A = 2\pi \int_0^{r_0} \mathrm{d}r'\, r' \sqrt{1 + (r'^2/a^2)} \ .}$$

You were not asked to carry out the integration, but it can be done by using the substitution $u = x^2$, so $du = 2x\,\mathrm{d}x$. The result is

$$A = \frac{2\pi}{3} a^2 \left[\left(1 + \frac{r_0^2}{a^2} \right)^{3/2} - 1 \right] .$$

(d) The nonzero metric coefficients are given by

$$g_{rr} = 1 + \frac{r^2}{a^2} , \qquad g_{\theta\theta} = r^2 ,$$

so the metric is diagonal. For $i = 1 = r$, the geodesic equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}s} \left\{ g_{rr} \frac{\mathrm{d}r}{\mathrm{d}s} \right\} = \frac{1}{2} \frac{\partial g_{rr}}{\partial r} \frac{\mathrm{d}r}{\mathrm{d}s} \frac{\mathrm{d}r}{\mathrm{d}s} + \frac{1}{2} \frac{\partial g_{\theta\theta}}{\partial r} \frac{\mathrm{d}\theta}{\mathrm{d}s} \frac{\mathrm{d}\theta}{\mathrm{d}s} ,$$

so if we substitute the values from above, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \left\{ \left(1 + \frac{r^2}{a^2} \right) \frac{\mathrm{d}r}{\mathrm{d}s} \right\} = \frac{1}{2} \frac{\partial}{\partial r} \left(1 + \frac{r^2}{a^2} \right) \left(\frac{\mathrm{d}r}{\mathrm{d}s} \right)^2 + \frac{1}{2} \frac{\partial r^2}{\partial r} \left(\frac{\mathrm{d}\theta}{\mathrm{d}s} \right)^2 .$$

Simplifying slightly,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s} \left\{ \left(1 + \frac{r^2}{a^2} \right) \frac{\mathrm{d}r}{\mathrm{d}s} \right\} = \frac{r}{a^2} \left(\frac{\mathrm{d}r}{\mathrm{d}s} \right)^2 + r \left(\frac{\mathrm{d}\theta}{\mathrm{d}s} \right)^2 .}$$

The answer above is perfectly acceptable, but one might want to expand the left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}s} \left\{ \left(1 + \frac{r^2}{a^2} \right) \frac{\mathrm{d}r}{\mathrm{d}s} \right\} = \frac{2r}{a^2} \left(\frac{\mathrm{d}r}{\mathrm{d}s} \right)^2 + \left(1 + \frac{r^2}{a^2} \right) \frac{\mathrm{d}^2 r}{\mathrm{d}s^2} .$$

Inserting this expansion into the boxed equation above, the first term can be brought to the right-hand side, giving

$$\boxed{\left(1 + \frac{r^2}{a^2} \right) \frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = -\frac{r}{a^2} \left(\frac{\mathrm{d}r}{\mathrm{d}s} \right)^2 + r \left(\frac{\mathrm{d}\theta}{\mathrm{d}s} \right)^2 .}$$

The $i = 2 = \theta$ equation is simpler, because none of the g_{ij} coefficients depend on θ, so the right-hand side of the geodesic equation vanishes. One has simply

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left\{r^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\right\} = 0\ .}$$

For most purposes this is the best way to write the equation, since it leads immediately to $r^2(\mathrm{d}\theta/\mathrm{d}s) = const$. However, it is possible to expand the derivative, giving the alternative form

$$r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + 2r\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0\ .$$

PROBLEM 4: TRAJECTORIES IN AN OPEN UNIVERSE *(25 points)*

(a) Since r and ϕ are not changing,

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}t^2 - R^2(t)r_0^2\mathrm{d}\theta^2\ ,$$

from which it follows that

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \frac{1}{c^2}R^2(t)r_0^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\ .$$

Taking a square root,

$$\boxed{\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1 - \frac{1}{c^2}R^2(t)r_0^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2}\ .}$$

(b)

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\dfrac{\mathrm{d}\tau}{\mathrm{d}t}}\ ,$$

so

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{1}{c^2}R^2(t)r_0^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2}}\ .}$$

(c) A clock attached to the object would read the proper time τ, so

$$\boxed{\tau = \int_{t_1}^{t_2} \frac{\mathrm{d}\tau}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_1}^{t_2} \mathrm{d}t \sqrt{1 - \frac{1}{c^2} R^2(t) r_0^2 \left(\frac{\mathrm{d}\theta_p}{\mathrm{d}t}\right)^2}\ .}$$

Note that the subscript p on θ_p is necessary, as it indicates that we are using the specific function $\theta_p(t)$ specified in the problem.

(d) We start with the general form for the geodesic equation, as taken from the formula sheet:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ g_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma}\right) \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ .$$

The metric is diagonal, with nonzero entries

$$g_{tt} = c^2 \qquad g_{rr} = -\frac{R^2(t)}{1+r^2} \qquad g_{\theta\theta} = -R^2(t) r^2 \qquad g_{\phi\phi} = -R^2(t) r^2 \sin^2\theta\ .$$

The equation is valid for each value of μ, but to find the θ-equation we consider the case $\mu = \theta$. Then the diagonal property of the metric implies that only $\nu = \theta$ will contribute to the sum over ν. The left-hand side is then

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ g_{\theta\theta} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ -R^2(t) r^2 \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\}\ .$$

In evaluating the right-hand-side, $\mu = \theta$, while (λ, σ) can take on only the values (t,t) and (θ,θ), as the other terms vanish as a consequence of the fact that $\mathrm{d}r/\mathrm{d}t = \mathrm{d}\phi/\mathrm{d}t = 0$. Thus,

$$\begin{aligned} \text{RHS} &= \frac{1}{2}\left(\frac{\partial g_{tt}}{\partial\theta}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial g_{\theta\theta}}{\partial\theta}\right)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 \\ &= \frac{1}{2}\left(\frac{\partial c^2}{\partial\theta}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial[-R^2(t)r^2]}{\partial\theta}\right)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 \\ &= 0\ . \end{aligned}$$

Thus, the equation becomes

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ R^2(t) r^2 \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \right\} = 0\ .}$$

(e) This time we choose $\mu = r$, and then only $\nu = r$ will give a nonzero contribution to the sum over ν. Thus,

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ -\frac{R^2(t)}{1+r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} .$$

For the right-hand-side, we again need only include the terms $(\lambda, \sigma) = (t, t)$ and $(\lambda, \sigma) = (\theta, \theta)$, so

$$\begin{aligned} \text{RHS} &= \frac{1}{2}\left(\frac{\partial g_{tt}}{\partial r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial g_{\theta\theta}}{\partial r}\right)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 \\ &= \frac{1}{2}\left(\frac{\partial c^2}{\partial r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial[-R^2(t)r^2]}{\partial r}\right)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 \\ &= -rR^2(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 . \end{aligned}$$

Finally, then, the geodesic equation is

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ \frac{R^2(t)}{1+r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} = rR^2(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 .}$$

This equation does not allow $r(\tau) = r_0$ as a solution, because this would imply that $\mathrm{d}r/d\tau = 0$; the left-hand side of the geodesic equation would then vanish, while the right-hand side does not.