MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 10, 2005
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QUIZ 2

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Your Name

Problem	Maximum	Score
1	25	
2	20	
3	30	
4	35	
TOTAL	110	

USEFUL INFORMATION:

DOPPLER SHIFT:

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot R}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \quad \Leftarrow \text{or} \Rightarrow \quad H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2}$$

$$\ddot R = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R \quad \Leftarrow \text{or} \Rightarrow \quad \frac{\ddot a}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

$$\dot\rho = -3\frac{\dot R}{R}\left(\rho + \frac{p}{c^2}\right) \quad \Leftarrow \text{or} \Rightarrow \quad \dot\varepsilon = -3\frac{\dot a}{a}(\varepsilon + P)$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$R(t) \propto t^{2/3}$ (matter-dominated)

$R(t) \propto t^{1/2}$ (radiation-dominated)

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot R}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot R = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\,(\sinh\theta - \theta)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or: $$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

PHYSICAL CONSTANTS:

$$G = 6.673 \times 10^{-8}\ \mathrm{cm}^3 \cdot \mathrm{g}^{-1} \cdot \mathrm{s}^{-2}$$

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\,\mathrm{erg/K} = 8.617 \times 10^{-5}\,\mathrm{eV/K} \ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27}\ \text{erg-sec} = 6.582 \times 10^{-16}\ \text{eV-sec} \ ,$$

$$c = 2.998 \times 10^{10}\ \mathrm{cm/sec}$$

$$1\ \mathrm{yr} = 3.156 \times 10^7\ \mathrm{s}$$

$$1\ \mathrm{eV} = 1.602 \times 10^{-12}\ \mathrm{erg} \ .$$

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(4 points)* For an open universe with a positive mass density, Ryden shows (in chapter 4) that the radius of curvature $R_{\rm curv} \equiv R(t)/\sqrt{-k}$ and the Hubble length $\ell_{\rm Hubble} \equiv c/H_0$ obey one of the following relations:

(i) $R_{\rm curv} > \ell_{\rm Hubble}$

(ii) $R_{\rm curv} = \ell_{\rm Hubble}$

(iii) $R_{\rm curv} < \ell_{\rm Hubble}$

Which of these relations is true?

(b) *(6 points)* Give a derivation of the relation in part (a).

(c) *(5 points)* As Ryden discusses in chapter 5, the universe today contains not only the photons of the cosmic microwave background (CMB), but also photons that originated as starlight. Including both direct starlight and starlight absorbed and reradiated by dust, the ratio of energy densities $\varepsilon_{\rm Starlight}/\varepsilon_{\rm CMB}$ has approximately which of the following values:

(i) 10^{-10} (ii) 10^{-5} (iii) 10^{-1} (iv) 10^{5} (v) 10^{10}

(d) *(10 points)* For a flat universe that contains only radiation and nonrelativistic matter, Ryden (chapter 6) writes the Friedmann equation as

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \ ,$$

where H_0 refers to the present value of the Hubble parameter H, and $\Omega_{r,0}$ and $\Omega_{m,0}$ refer to the present values of the mass densities in radiation and matter, respectively, compared to the critical density. Ryden rearranges this formula to take the form

$$H_0 \, \mathrm{d}t = \frac{a \, \mathrm{d}a}{A} \left[1 + \frac{a}{a_{rm}} \right]^B \ , \tag{1}$$

where A and B are constants that might depend on the parameters H_0, $\Omega_{r,0}$, and $\Omega_{m,0}$. (Ryden wrote A and B explicitly, but for the purpose of this question I have not.) a_{rm} is the scale factor of radiation-matter equality: i.e., the scale factor when the energy densities of radiation and matter are equal. Recall that $a(t)$ is the notation Ryden uses for the scale factor, which in the Lecture Notes is called $R(t)$.

(i) *(4 points)* Write an expression for a_{rm} in terms of all or some of the parameters H_0, $\Omega_{r,0}$, and $\Omega_{m,0}$.

(ii) *(6 points)* Derive Eq. (1) above, and find the values of A and B.

(e) BONUS QUESTION *(1 point)*: Where does Barbara Ryden suggest writing the Friedmann equation?

PROBLEM 2: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(20 points)*

The following problem was Problem 5 of Problem Set 4.

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{R^5(t)} \; .$$

Assuming that the model universe is flat, calculate

(a) *(5 points)* The behavior of the scale factor, $R(t)$. You should be able to find $R(t)$ up to an arbitrary constant of proportionality.

(b) *(5 points)* The value of the Hubble parameter $H(t)$, as a function of t.

(c) *(5 points)* The physical horizon distance, $\ell_{p,\mathrm{horizon}}(t)$.

(d) *(5 points)* The mass density $\rho(t)$.

PROBLEM 3: AN EXERCISE IN TWO-DIMENSIONAL METRICS *(30 points)*

(a) *(8 points)* Consider first a two-dimensional space with coordinates r and θ. The metric is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \; .$$

Consider the curve described by

$$r(\theta) = (1 + \epsilon \sin\theta)\, r_0 \; ,$$

where ϵ and r_0 are constants, and θ runs from θ_1 to θ_2. Write an expression, in the form of a definite integral, for the length S of this curve.

(b) *(5 points)* Now consider a two-dimensional space with the same two coordinates r and θ, but this time the metric will be

$$\mathrm{d}s^2 = \left(1 + \frac{r^2}{a^2}\right) \mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2 \; ,$$

where a is a constant. θ is a periodic (angular) variable, with a range of 0 to 2π, with 2π identified with 0. What is the length R of the path from the origin ($r = 0$) to the point $r = r_0, \theta = 0$, along the path for which $\theta = 0$ everywhere along the path? You can leave your answer in the form of a definite integral. (Be sure, however, to specify the limits of integration.)

(c) *(7 points)* For the space described in part (b), what is the total area contained within the region $r < r_0$. Again you can leave your answer in the form of a definite integral, making sure to specify the limits of integration.

(d) *(10 points)* Again for the space described in part (b), consider a geodesic described by the usual geodesic equation,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{ g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}s} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell}\right) \frac{\mathrm{d}x^k}{\mathrm{d}s} \, \frac{\mathrm{d}x^\ell}{\mathrm{d}s} \; .$$

The geodesic is described by functions $r(s)$ and $\theta(s)$, where s is the arc length along the curve. Write explicitly both (i.e., for i=1=r and i=2=θ) geodesic equations.

PROBLEM 4: TRAJECTORIES IN AN OPEN UNIVERSE *(35 points)*

Consider the case of an open Robertson-Walker universe. Taking $k = -1$, the spacetime metric can be written in the form

$$c^2\, d\boldsymbol{\tau}^2 = -ds^2 = c^2\, dt^2 - R^2(t) \left\{ \frac{dr^2}{1+r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} \;.$$

We will assume that this metric is given, and that $R(t)$ has been specified. While galaxies are approximately stationary in the comoving coordinate system described by this metric, we can still consider an object that moves in this system. In particular, in this problem we will consider an object that is moving in the θ-direction.

(a) *(7 points)* Express $d\boldsymbol{\tau}/dt$ in terms of $d\theta/dt$, assuming that the object is moving in the θ-direction, with $r = r_0$ and $\phi = \phi_0$.

(b) *(3 points)* For the same situation as in part (a), express $dt/d\boldsymbol{\tau}$ in terms of $d\theta/dt$.

(c) *(10 points)* Suppose the object travels on a trajectory given by the function $\theta_p(t)$ between some time t_1 and some later time t_2, for $r = r_0$ and $\phi = \phi_0$ everywhere on the trajectory. We are not claiming that this trajectory is necessarily a geodesic, so perhaps some kind of rocket engine is needed to cause the object to move on this trajectory. Write an integral which gives the total amount of time that a clock attached to the object would record for this journey.

(d) *(10 points)* Using the formulas at the front of the exam, derive the geodesic equation of motion for the coordinate θ of the object, assuming again that it is moving in the θ-direction. Specifically, you should derive an equation of the form

$$\frac{d}{d\boldsymbol{\tau}} \left[A \frac{d\theta}{d\boldsymbol{\tau}} \right] = B \left(\frac{dt}{d\boldsymbol{\tau}} \right)^2 + C \left(\frac{dr}{d\boldsymbol{\tau}} \right)^2 + D \left(\frac{d\theta}{d\boldsymbol{\tau}} \right)^2 + E \left(\frac{d\phi}{d\boldsymbol{\tau}} \right)^2 \;,$$

where A, B, C, D, and E are functions of the coordinates, and where some of the functions A, B, C, D, and E might be identically zero. (You need not include any terms that are not present for the specified motion, in the θ-direction.)

(e) *(5 points)* Again assuming that the object is moving in the θ-direction, derive the geodesic equation of motion for the coordinate r of the object. If there are no forces acting on the object other than gravity, does this equation allow the object to continue at $r = r_0$, or will the velocity in the r-direction become nonzero?