

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe
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QUIZ 2 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad (\text{nonrelativistic, source moving})$$

$$z = \frac{v/u}{1 - v/u} \quad (\text{nonrelativistic, observer moving})$$

$$z = \sqrt{\frac{1 + \beta}{1 - \beta}} - 1 \quad (\text{special relativity, with } \beta = v/c)$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:

Trailing clock reads later by an amount $\beta\ell_0/c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right), \quad \vec{p} = \gamma m_0 \vec{v}, \quad E = \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2},$$

$$p^2 \equiv |\vec{p}|^2 - (p^0)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -(m_0 c)^2.$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R} \right)^2 = \frac{8\pi}{3} G \rho - \frac{kc^2}{R^2}, \quad \ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2} \right) R,$$

$$\rho_m(t) = \frac{R^3(t_i)}{R^3(t)} \rho_m(t_i) \text{ (matter)}, \quad \rho_r(t) = \frac{R^4(t_i)}{R^4(t)} \rho_r(t_i) \text{ (radiation)}.$$

$$\dot{\rho} = -3 \frac{\dot{R}}{R} \left(\rho + \frac{p}{c^2} \right), \quad \Omega \equiv \rho / \rho_c, \quad \text{where } \rho_c = \frac{3H^2}{8\pi G}.$$

Flat ($k = 0$):

$$R(t) \propto t^{2/3} \quad \text{(matter-dominated)},$$

$$R(t) \propto t^{1/2} \quad \text{(radiation-dominated)},$$

$$\Omega = 1.$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed ($k > 0$):

$$ct = \alpha(\theta - \sin \theta), \quad \frac{R}{\sqrt{k}} = \alpha(1 - \cos \theta),$$

$$\Omega = \frac{2}{1 + \cos \theta} > 1,$$

where $\alpha \equiv \frac{4\pi G \rho}{3 c^2} \left(\frac{R}{\sqrt{k}} \right)^3$.

Open ($k < 0$):

$$ct = \alpha(\sinh \theta - \theta), \quad \frac{R}{\sqrt{\kappa}} = \alpha(\cosh \theta - 1),$$

$$\Omega = \frac{2}{1 + \cosh \theta} < 1,$$

where $\alpha \equiv \frac{4\pi G \rho}{3 c^2} \left(\frac{R}{\sqrt{\kappa}} \right)^3$,

$$\kappa \equiv -k > 0.$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = - \left(1 - \frac{2GM}{rc^2} \right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2} \right)^{-1} dr^2$$

$$+ r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

GEODESIC EQUATION:

$$\frac{d}{ds} \left\{ g_{ij} \frac{dx^j}{ds} \right\} = \frac{1}{2} (\partial_i g_{k\ell}) \frac{dx^k}{ds} \frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} (\partial_\mu g_{\lambda\sigma}) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \quad (\text{energy density})$$

$$p = \frac{1}{3}u \quad \rho = u/c^2 \quad (\text{pressure, mass density})$$

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \quad (\text{number density})$$

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3}, \quad (\text{entropy density})$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions,} \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots \approx 1.202.$$

$$g_\gamma = g_\gamma^* = 2,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\text{Fermion factor}} \times \underbrace{3}_{\text{3 species } \nu_e, \nu_\mu, \nu_\tau} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4},$$

$$g_\nu^* = \underbrace{\frac{3}{4}}_{\text{Fermion factor}} \times \underbrace{3}_{\text{3 species } \nu_e, \nu_\mu, \nu_\tau} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2},$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\text{Fermion factor}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2},$$

$$g_{e^+e^-}^* = \underbrace{\frac{3}{4}}_{\text{Fermion factor}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} = 3.$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G} \right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106 \text{ MeV} \gg kT \gg m_e = 0.511 \text{ MeV}$, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11} \right)^{1/3} .$$

HORIZON DISTANCE:

$$\begin{aligned} \ell_{p,\text{horizon}}(t) &= R(t) \int_0^t \frac{c}{R(t')} dt' \\ &= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).} \end{cases} \end{aligned}$$

PHYSICAL CONSTANTS:

$$G = 6.673 \times 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$\begin{aligned} k &= \text{Boltzmann's constant} = 1.381 \times 10^{-16} \text{ erg/K} \\ &= 8.617 \times 10^{-5} \text{ eV/K} \end{aligned}$$

$$\begin{aligned} \hbar &= \frac{h}{2\pi} = 1.055 \times 10^{-27} \text{ erg}\cdot\text{s} \\ &= 6.582 \times 10^{-16} \text{ eV}\cdot\text{s} \end{aligned}$$

$$c = 2.998 \times 10^{10} \text{ cm/s}$$

$$1 \text{ yr} = 3.156 \times 10^7 \text{ s}$$

$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg}$$

$$1 \text{ GeV} = 10^9 \text{ eV} = 1.783 \times 10^{-24} \text{ gram (where } c \equiv 1) .$$