MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

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Prof. Alan Guth

QUIZ 2 FORMULA SHEET

DOPPLER SHIFT (For motion along a line):

$z = v/u$ (nonrelativistic, source moving)

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \ , \quad \vec{p} = \gamma m_0 \vec{v} \ , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + \left|\vec{p}\right|^2 c^2} \ ,$$

$$p^2 \equiv \left|\vec{p}\right|^2 - \left(p^0\right)^2 = \left|\vec{p}\right|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \ .$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \ , \quad \ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R \ ,$$

$$\rho_m(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho_m(t_i) \ \text{(matter)}, \quad \rho_r(t) = \frac{R^4(t_i)}{R^4(t)}\,\rho_r(t_i) \ \text{(radiation)}.$$

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \ \ \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat ($k = 0$): $R(t) \propto t^{2/3}$ (matter-dominated) ,

$R(t) \propto t^{1/2}$ (radiation-dominated) ,

$\Omega = 1$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed ($k > 0$): $ct = \alpha(\theta - \sin\theta)$, $\dfrac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)$,

$$\Omega = \frac{2}{1+\cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$): $ct = \alpha\left(\sinh\theta - \theta\right)$, $\dfrac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)$,

$$\Omega = \frac{2}{1+\cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2$$
$$+ r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

or: $$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions ,} \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

$$g_\gamma = g^*_\gamma = 2\ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\substack{\text{Spin}\\ \text{states}}} = \frac{21}{4}\ ,$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\substack{\text{Spin}\\ \text{states}}} = \frac{9}{2}\ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\substack{\text{Spin}\\ \text{states}}} = \frac{7}{2}\ ,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\substack{\text{Spin}\\ \text{states}}} = 3\ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

HORIZON DISTANCE:

$$\ell_{p,\text{horizon}}(t) = R(t) \int_0^t \frac{c}{R(t')} dt'$$

$$= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).} \end{cases}$$

PHYSICAL CONSTANTS:

$G = 6.673 \times 10^{-8}$ cm$^3 \cdot$ g$^{-1} \cdot$ s^{-2}

$k =$ Boltzmann's constant $= 1.381 \times 10^{-16}$ erg/K

$= 8.617 \times 10^{-5}$ eV/K

$\hbar = \dfrac{h}{2\pi} = 1.055 \times 10^{-27}$ erg-s

$= 6.582 \times 10^{-16}$ eV-s

$c = 2.998 \times 10^{10}$ cm/s

1 yr $= 3.156 \times 10^7$ s

1 eV $= 1.602 \times 10^{-12}$ erg

1 GeV $= 10^9$ eV $= 1.783 \times 10^{-24}$ gram (where $c \equiv 1$) .