MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

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Prof. Alan Guth

QUIZ 3 FORMULA SHEET

USEFUL INFORMATION:

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \;, \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \;, \quad \vec{p} = \gamma m_0 \vec{v} \;, \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \;,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \;.$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \;, \quad \ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R \;,$$

$$\rho_m(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho_m(t_i) \;\; \text{(matter)}, \quad \rho_r(t) = \frac{R^4(t_i)}{R^4(t)}\,\rho_r(t_i) \;\; \text{(radiation)}.$$

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \;, \quad \Omega \equiv \rho/\rho_c \;, \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \;.$$

Flat ($k = 0$):

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)} \;,$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)} \;,$$

$$\Omega = 1 \;.$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed ($k > 0$):

$$ct = \alpha(\theta - \sin\theta) \;, \quad \frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \;,$$

$$\Omega = \frac{2}{1 + \cos\theta} > 1 \;,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{k}}\right)^3 \;.$$

Open ($k < 0$):

$$ct = \alpha\left(\sinh\theta - \theta\right) \;, \quad \frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \;,$$

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \;,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{\kappa}}\right)^3 \;,$$

$$\kappa \equiv -k > 0 \;.$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$

$$+ r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \;,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

$$\text{or:} \qquad \frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

$$g_\gamma = g_\gamma^* = 2\ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ ,$$

$$g_\nu^* = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2}\ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2}\ ,$$

$$g_{e^+e^-}^* = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3\ .$$

CHEMICAL EQUILIBRIUM:

Ideal Gas of Classical Nonrelativistic Particles:

$$n_i = g_i \frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} \, e^{(\mu_i - m_i c^2)/kT} \ .$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} \left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \ .$$

HORIZON DISTANCE:

$$\ell_{p,\text{horizon}}(t) = R(t) \int_0^t \frac{c}{R(t')} dt'$$

$$= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).} \end{cases}$$

COSMOLOGICAL CONSTANT:

$$u_{\rm vac} = \rho_{\rm vac} c^2 = \frac{\Lambda c^4}{8\pi G} \ ,$$

$$p_{\rm vac} = -\rho_{\rm vac} c^2 = -\frac{\Lambda c^4}{8\pi G} \ .$$

GENERALIZED COSMOLOGICAL EVOLUTION:

$$x\frac{dx}{dt} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4 + \Omega_{k,0}x^2} \ ,$$

where

$$x \equiv \frac{R(t)}{R(t_0)} \equiv \frac{1}{1+z} \ ,$$

$$\Omega_{k,0} \equiv -\frac{kc^2}{R^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\rm rad,0} - \Omega_{\rm vac,0} \ .$$

Age of universe:

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{x\,dx}{\sqrt{\Omega_{m,0}x + \Omega_{\rm rad,0} + \Omega_{\rm vac,0}x^4 + \Omega_{k,0}x^2}}$$

$$= \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\rm rad,0}(1+z)^4 + \Omega_{\rm vac,0} + \Omega_{k,0}(1+z)^2}} \ .$$

Look-back time:

$$t_{\text{look-back}}(z) =$$

$$\frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\rm rad,0}(1+z')^4 + \Omega_{\rm vac,0} + \Omega_{k,0}(1+z')^2}} \ .$$

ENERGY FLUX FROM A DISTANT SOURCE:

The energy flux received from a source at redshift z_S, for a source that emitted power P isotropically at the time of emission, is given by

$$J = \frac{PH_0^2}{4\pi(1+z_S)^2c^2} \times \begin{cases} \frac{\Omega_{k,0}}{\sinh^2\psi(z_S)} & \text{if } k<0, \\ \frac{1}{I^2} & \text{if } k=0, \\ \frac{|\Omega_{k,0}|}{\sin^2\psi(z_S)} & \text{if } k>0, \end{cases}$$

where

$$I = \int_0^{z_S} \frac{1}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\rm rad,0}(1+z)^4 + \Omega_{\rm vac,0} + \Omega_{k,0}(1+z)^2}}\,dz \ ,$$

and

$$\psi(z_S) = \sqrt{|\Omega_{k,0}|}\, I \ .$$

PHYSICAL CONSTANTS:

$$G = 6.673 \times 10^{-8}\ \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\,\text{erg/K}$$
$$= 8.617 \times 10^{-5}\,\text{eV/K}$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27}\ \text{erg-s}$$
$$= 6.582 \times 10^{-16}\ \text{eV-s}$$

$$c = 2.998 \times 10^{10}\ \text{cm/s}$$

$$\hbar c = 197.3\ \text{MeV-fm}, \quad 1\ \text{fm} = 10^{-15}\ \text{m}$$

$$1\ \text{yr} = 3.156 \times 10^{7}\ \text{s}$$

$$1\ \text{eV} = 1.602 \times 10^{-12}\ \text{erg}$$

$$1\ \text{GeV} = 10^{9}\ \text{eV} = 1.783 \times 10^{-24}\ \text{gram (where } c \equiv 1)\ .$$

Planck Units: The Planck length ℓ_P, the Planck time t_P, the Planck mass m_P, and the Planck energy E_p are given by

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.616 \times 10^{-33}\,\text{cm}\,,$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44}\,\text{s}\,,$$

$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-5}\,\text{g}\,,$$

$$E_P = \sqrt{\frac{\hbar c^5}{G}} = 1.221 \times 10^{19}\,\text{GeV}\,.$$