MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 3, 2007
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 2

QUIZ DATE: Tuesday, November 6, 2007, during the normal class time.

COVERAGE: Lecture Notes 6 and Lecture Notes 7; Problem Sets 4, 5, and 6; Weinberg, *The First Three Minutes,* Chapters 4–8; Ryden, *Introduction to Cosmology*, Chapter 5. However, Ryden's Chapter 5 is intended to help you understand the lecture material, so there will be no quiz questions based specifically on this material. Chapters 4 and 5 of Weinberg's book are packed with numbers; you need not memorize these numbers, but you should be familiar with their orders of magnitude. **One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems.** The starred problems are the ones that I recommend that you review most carefully: Problems 1, 2, 4, 6, 9, 10, 12, 13, 14, 15, and 16. There is only one reading question, Problem 17.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, and 2005. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match the coverage of any of the quizzes from previous years. Quiz 2 of this year covers more than Quiz 2 did in 2004 or 2005.

REVIEW SESSION AND OFFICE HOUR: To help you study for the quiz, Barton Zwiebach will hold a review session on Monday, November 5, at 7:00 pm, in Room 2-105. Note that this is a room that we have not used before. Yi Mao will hold an office hour on Monday, November 5, from 4:00 - 5:00 pm, in Room 8-310. Note that this is also a room that we have not used before.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" for your reference. For the second quiz, this useful information will be the following:

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \;, \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \;, \quad \vec{p} = \gamma m_0 \vec{v} \;, \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} \;,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 \;.$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \;, \quad \ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R \;,$$

$$\rho_m(t) = \frac{R^3(t_i)}{R^3(t)} \rho_m(t_i) \text{ (matter)}, \quad \rho_r(t) = \frac{R^4(t_i)}{R^4(t)} \rho_r(t_i) \text{ (radiation)}.$$

$$\dot{\rho} = -3 \frac{\dot{R}}{R} \left(\rho + \frac{p}{c^2}\right) \;, \quad \Omega \equiv \rho/\rho_c \;, \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \;.$$

$$\text{Flat } (k=0): \quad R(t) \propto t^{2/3} \quad \text{(matter-dominated)} \; ,$$
$$R(t) \propto t^{1/2} \quad \text{(radiation-dominated)} \; ,$$
$$\Omega = 1 \; .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\text{Closed } (k>0): \quad ct = \alpha(\theta - \sin\theta) \; , \quad \frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \; ,$$
$$\Omega = \frac{2}{1+\cos\theta} > 1 \; ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{k}}\right)^3 \; .$$

$$\text{Open } (k<0): \quad ct = \alpha\left(\sinh\theta - \theta\right) \; , \quad \frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \; ,$$
$$\Omega = \frac{2}{1+\cosh\theta} < 1 \; ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{\kappa}}\right)^3 \; ,$$
$$\kappa \equiv -k > 0 \; .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \; ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

$$\text{or:} \quad \frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions ,} \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

$$g_\gamma = g^*_\gamma = 2\ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ ,$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2}\ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2}\ ,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3\ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

PHYSICAL CONSTANTS:

$$G = 6.673 \times 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$\begin{aligned} k &= \text{Boltzmann's constant} = 1.381 \times 10^{-16} \, \text{erg/K} \\ &= 8.617 \times 10^{-5} \, \text{eV/K} \ , \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055 \times 10^{-27} \text{ erg-s} \\ &= 6.582 \times 10^{-16} \text{ eV-s} \ , \end{aligned}$$

$$c = 2.998 \times 10^{10} \text{ cm/s}$$

$$1 \text{ yr} = 3.156 \times 10^7 \text{ s}$$

$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg}$$

$$1 \text{ GeV} = 10^9 \text{ eV} = 1.783 \text{ g } (c \equiv 1) \ .$$

∗ PROBLEM 1: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(30 points)*

The following problem was Problem 3, Quiz 2, 1998.

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

(a) *(7 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi =$ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(8 points)* Write an expression for the physical horizon distance ℓ_{phys} at time t. You should leave your answer in the form of a definite integral.

The form of $R(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.,* dominated by nonrelativistic matter), then $R(t)$ is described by the parametric equations

$$ct = \alpha(\theta - \sin\theta) ,$$

$$R = \alpha(1 - \cos\theta) ,$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{c^2} .$$

These equations are identical to those on the front of the exam, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where θ is the parameter used to describe the evolution.

(d) *(5 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

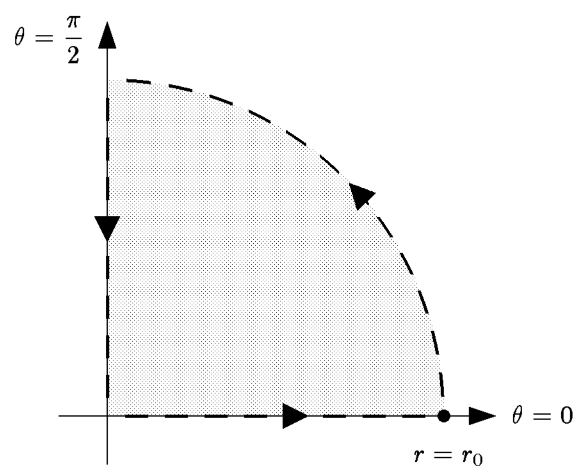
∗ PROBLEM 2: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC *(25 points)*

The following problem was Problem 3, Quiz 2, 1994:

Suppose a two dimensional space, described in polar coordinates (r, θ), has a metric given by

$$ds^2 = (1 + ar)^2\, dr^2 + r^2(1 + br)^2\, d\theta^2 \ ,$$

where a and b are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the $\theta = 0$ line to $r = r_0$, then moving at fixed r to $\theta = \pi/2$, and then moving back to the origin at fixed θ. The path is shown below:



a) *(10 points)* Find the total length of this path.

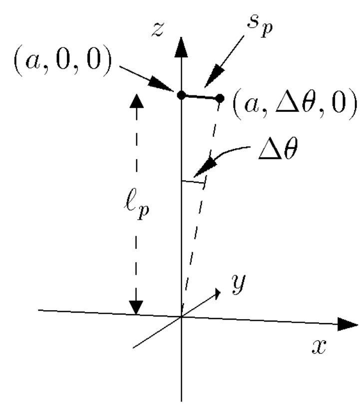
b) *(15 points)* Find the area enclosed by this path.

PROBLEM 3: GEOMETRY IN A CLOSED UNIVERSE *(25 points)*

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with $k = 1$. The questions below all pertain to some fixed time t, so the scale factor can be written simply as R, dropping its explicit t-dependence.

A small rod has one end at the point $(r = a,\ \theta = 0,\ \phi = 0)$ and the other end at the point $(r = a,\ \theta = \Delta\theta,\ \phi = 0)$. Assume that $\Delta\theta \ll 1$.



(a) Find the physical distance ℓ_p from the origin ($r = 0$) to the first end $(a, 0, 0)$ of the rod. You may find one of the following integrals useful:

$$\int \frac{dr}{\sqrt{1 - r^2}} = \sin^{-1} r$$

$$\int \frac{dr}{1 - r^2} = \frac{1}{2} \ln \left(\frac{1 + r}{1 - r} \right) .$$

(b) Find the physical length s_p of the rod. Express your answer in terms of the scale factor R, and the coordinates a and $\Delta\theta$.

(c) Note that $\Delta\theta$ is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance ℓ_p, the physical length s_p, and the scale factor R.

∗ PROBLEM 4: THE GENERAL SPHERICALLY SYMMETRIC METRIC *(20 points)*

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three dimensional space which is spherically symmetric about a particular point, coordinates can be found so that the metric has the form

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

for some function $\rho(r)$. The coordinates θ and ϕ have their usual ranges: θ varies between 0 and π, and ϕ varies from 0 to 2π, where $\phi = 0$ and $\phi = 2\pi$ are identified. Given this metric, consider the sphere whose outer boundary is defined by $r = r_0$.

(a) Find the physical radius a of the sphere. (By "radius", I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)

(b) Find the physical area of the surface of the sphere.

(c) Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.

(d) Suppose a new radial coordinate σ is introduced, where σ is related to r by

$$\sigma = r^2\ .$$

Express the metric in terms of this new variable.

PROBLEM 5: VOLUMES IN A ROBERTSON-WALKER UNIVERSE *(20 points)*

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}\quad .$$

Calculate the volume $V(r_{\max})$ of the sphere described by

$$r \le r_{\max}\ .$$

You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

∗ PROBLEM 6: THE SCHWARZSCHILD METRIC *(25 points)*

The follow problem was Problem 4, Quiz 3, 1992:

The space outside a spherically symmetric mass M is described by the Schwarzschild metric, given at the front of the exam. Two observers, designated A and B, are located along the same radial line, with values of the coordinate r given by r_A and r_B, respectively, with $r_A < r_B$. You should assume that both observers lie outside the Schwarzschild horizon.

a) *(5 points)* Write down the expression for the Schwarzschild horizon radius $R_{\rm Sch}$, expressed in terms of M and fundamental constants.

b) *(5 points)* What is the proper distance between A and B? It is okay to leave the answer to this part in the form of an integral that you do not evaluate—but be sure to clearly indicate the limits of integration.

c) *(5 points)* Observer A has a clock that emits an evenly spaced sequence of ticks, with proper time separation $\Delta\tau_A$. What will be the coordinate time separation Δt_A between these ticks?

d) *(5 points)* At each tick of A's clock, a light pulse is transmitted. Observer B receives these pulses, and measures the time separation on his own clock. What is the time interval $\Delta\tau_B$ measured by B.

e) *(5 points)* Suppose that the object creating the gravitational field is a static black hole, so the Schwarzschild metric is valid for all r. Now suppose that one considers the case in which observer A lies on the Schwarzschild horizon, so $r_A \equiv R_{\rm Sch}$. Is the proper distance between A and B finite for this case? Does the time interval of the pulses received by B, $\Delta\tau_B$, diverge in this case?

PROBLEM 7: GEODESICS *(20 points)*

The following problem was Problem 4, Quiz 2, 1986:

Ordinary Euclidean two-dimensional space can be described in polar coordinates by the metric

$$ds^2 = dr^2 + r^2\, d\theta^2 \ .$$

(a) Suppose that $r(\lambda)$ and $\theta(\lambda)$ describe a geodesic in this space, where the parameter λ is the arc length measured along the curve. Use the general formula on the front of the exam to obtain explicit differential equations which $r(\lambda)$ and $\theta(\lambda)$ must obey.

(b) Now introduce the usual Cartesian coordinates, defined by

$$x = r\cos\theta \ ,$$

$$y = r\sin\theta \ .$$

Use your answer to (a) to show that the line $y = 1$ is a geodesic curve.

PROBLEM 8: GEODESICS ON THE SURFACE OF A SPHERE

In this problem we will test the geodesic equation by computing the geodesic curves on the surface of a sphere. We will describe the sphere as in Lecture Notes 6, with metric given by

$$ds^2 = a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \quad .$$

(a) Clearly one geodesic on the sphere is the equator, which can be parametrized by $\theta = \pi/2$ and $\phi = \psi$, where ψ is a parameter which runs from 0 to 2π. Show that if the equator is rotated by an angle α about the x-axis, then the equations become:

$$\cos\theta = \sin\psi \sin\alpha$$

$$\tan\phi = \tan\psi \cos\alpha \quad .$$

(b) Using the generic form of the geodesic equation on the front of the exam, derive the differential equation which describes geodesics in this space.

(c) Show that the expressions in (a) satisfy the differential equation for the geodesic. Hint: The algebra on this can be messy, but I found things were reasonably simple if I wrote the derivatives in the following way:

$$\frac{d\theta}{d\psi} = -\frac{\cos\psi \sin\alpha}{\sqrt{1 - \sin^2\psi \sin^2\alpha}} \quad , \qquad \frac{d\phi}{d\psi} = \frac{\cos\alpha}{1 - \sin^2\psi \sin^2\alpha} \quad .$$

∗ PROBLEM 9: GEODESICS IN A CLOSED UNIVERSE

The following problem was Problem 3, Quiz 3, 2000, where it was worth 40 points plus 5 points extra credit.

Consider the case of closed Robertson-Walker universe. Taking $k = 1$, the spacetime metric can be written in the form

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t) \left\{ \frac{dr^2}{1 - r^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\} \quad .$$

We will assume that this metric is given, and that $R(t)$ has been specified. While galaxies are approximately stationary in the comoving coordinate system described by this metric, we can still consider an object that moves in this system. In particular, in this problem we will consider an object that is moving in the radial direction (r-direction), under the influence of no forces other than gravity. Hence the object will travel on a geodesic.

(a) *(7 points)* Express $d\tau/dt$ in terms of dr/dt.

(b) *(3 points)* Express $dt/d\tau$ in terms of dr/dt.

(c) *(10 points)* If the object travels on a trajectory given by the function $r_p(t)$ between some time t_1 and some later time t_2, write an integral which gives the total amount of time that a clock attached to the object would record for this journey.

(d) *(10 points)* During a time interval dt, the object will move a coordinate distance

$$dr = \frac{dr}{dt} dt \ .$$

Let $d\ell$ denote the physical distance that the object moves during this time. By "physical distance," I mean the distance that would be measured by a comoving observer (an observer stationary with respect to the coordinate system) who is located at the same point. The quantity $d\ell/dt$ can be regarded as the physical speed $v_{\rm phys}$ of the object, since it is the speed that would be measured by a comoving observer. Write an expression for $v_{\rm phys}$ as a function of dr/dt and r.

(e) *(10 points)* Using the formulas at the front of the exam, derive the geodesic equation of motion for the coordinate r of the object. Specifically, you should derive an equation of the form

$$\frac{d}{d\tau}\left[A\frac{dr}{d\tau}\right] = B\left(\frac{dt}{d\tau}\right)^2 + C\left(\frac{dr}{d\tau}\right)^2 + D\left(\frac{d\theta}{d\tau}\right)^2 + E\left(\frac{d\phi}{d\tau}\right)^2 ,$$

where A, B, C, D, and E are functions of the coordinates, some of which might be zero.

(f) *(5 points EXTRA CREDIT)* On Problem 4 of Problem Set 3 we learned that in a flat Robertson-Walker metric, the relativistically defined momentum of a particle,

$$p = \frac{m v_{\rm phys}}{\sqrt{1 - \frac{v_{\rm phys}^2}{c^2}}} ,$$

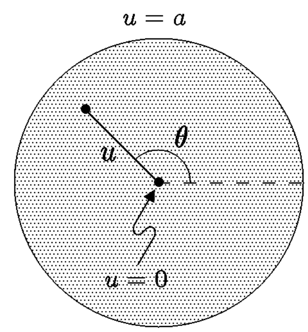
falls off as $1/R(t)$. Use the geodesic equation derived in part (e) to show that the same is true in a closed universe.

∗ PROBLEM 10: A TWO-DIMENSIONAL CURVED SPACE *(40 points)*

The following problem was Problem 3, Quiz 2, 2002.

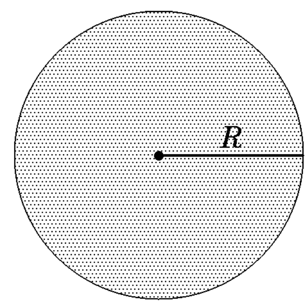
Consider a two-dimensional curved space described by polar coordinates u and θ, where $0 \le u \le a$ and $0 \le \theta \le 2\pi$, and $\theta = 2\pi$ is as usual identified with $\theta = 0$. The metric is given by

$$ds^2 = \frac{a\,du^2}{4u(a-u)} + u\,d\theta^2 \ .$$

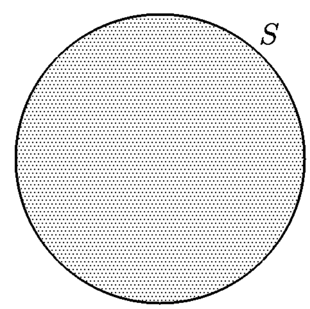


A diagram of the space is shown at the right, but you should of course keep in mind that the diagram does not accurately reflect the distances defined by the metric.

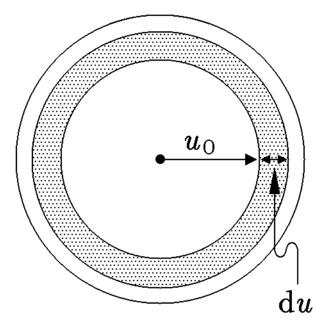
(a) *(6 points)* Find the radius R of the space, defined as the length of a radial (i.e., $\theta = constant$) line. You may express your answer as a definite integral, which you need not evaluate. Be sure, however, to specify the limits of integration.



(b) *(6 points)* Find the circumference S of the space, defined as the length of the boundary of the space at $u = a$.



(c) *(7 points)* Consider an annular region as shown, consisting of all points with a u-coordinate in the range $u_0 \le u \le u_0 + du$. Find the physical area dA of this region, to first order in du.



(d) *(3 points)* Using your answer to part (c), write an expression for the total area of the space.

(e) *(10 points)* Consider a geodesic curve in this space, described by the functions $u(s)$ and $\theta(s)$, where the parameter s is chosen to be the arc length along the curve. Find the geodesic equation for $u(s)$, which should have the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[F(u,\theta)\,\frac{\mathrm{d}u}{\mathrm{d}s}\right] = \ldots\ ,$$

where $F(u,\theta)$ is a function that you will find. (Note that by writing F as a function of u and θ, we are saying that it *could* depend on either or both of them, but we are not saying that it *necessarily* depends on them.) You need not simplify the left-hand side of the equation.

(f) *(8 points)* Similarly, find the geodesic equation for $\theta(s)$, which should have the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[G(u,\theta)\,\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = \ldots\ ,$$

where $G(u,\theta)$ is a function that you will find. Again, you need not simplify the left-hand side of the equation.

PROBLEM 11: EVOLUTION OF MODEL UNIVERSES *(30 points)*

The following problem was Problem 1, Quiz 2, 2004.

This problem is based on Chapter 5 of Ryden. Since her notation is a little different from mine, I am presenting the problem in both notations, and you can answer it in the notation of your choice.

The evolution of a homogeneous, isotropic universe is governed by the following three independent equations:

The Friedmann equation,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \quad \Leftarrow \text{or} \Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2}\ , \tag{1}$$

the fluid equation,

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \quad \Leftarrow \text{or} \Rightarrow \quad \dot{\varepsilon} = -3\frac{\dot{a}}{a}(\varepsilon + P)\ , \tag{2}$$

and the equation of state,

$$p = w\rho c^2 \quad \Leftarrow \text{or} \Rightarrow \quad P = w\varepsilon\ . \tag{3}$$

In Eq. (3) we assume that w is a constant. In this problem we will examine the time evolution of the scale factor, $R(t)$ [or $a(t)$], for different assumptions about the nature of the matter and its equation of state.

(a) *(8 points)* First consider an empty universe ($\rho = \varepsilon = 0$). What are the possible forms for the function $R(t)$ [or $a(t)$], and is the universe open, closed, or flat in each case?

For the rest of the problem we consider a ***flat*** universe, made up of "stuff" that has some constant w relating the pressure and the mass density (according to the equation of state above).

(b) *(8 points)* What value of w corresponds to

(i) nonrelativistic matter?

(ii) relativistic matter (i.e., radiation)?

and

(iii) the cosmological constant?

For this part you may simply state the answers without doing any calculations.

(c) *(6 points)* In such a universe, $\rho \propto R^{-b}$ [or $\varepsilon \propto a^{-b}$], where b is a constant that depends only on w. Find b. For full credit, your answer should show how to derive the expression for b using only mathematics and Eqs. (1), (2), and (3) above.

(d) *(6 points)* Using $\rho \propto R^{-b}$ [or $\varepsilon \propto a^{-b}$] , determine $R(t)$ [or $a(t)$] for both $b = 0$ and $b \neq 0$. For $b \neq 0$ you should express $R(t)$ in terms of t and b. For the case $b = 0$ you should express your answer in terms of the present value of the Hubble constant, H_0. In both cases your answer can contain a "proportional to" sign ($\propto$), or you can introduce an arbitrary constant of proportionality. Again, to obtain full credit you must show how to derive the answer from Eqs. (1), (2), and (3) above.

∗ PROBLEM 12: ROTATING FRAMES OF REFERENCE *(35 points)*

The following problem was Problem 3, Quiz 2, 2004.

In this problem we will use the formalism of general relativity and geodesics to derive the relativistic description of a rotating frame of reference.

The problem will concern the consequences of the metric

$$ds^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,(\mathrm{d}\phi + \omega\,\mathrm{d}t)^2 + \mathrm{d}z^2\right] , \qquad (1)$$

which corresponds to a coordinate system rotating about the z-axis, where ϕ is the azimuthal angle around the z-axis. The coordinates have the usual range for cylindrical coordinates: $-\infty < t < \infty$, $0 \leq r < \infty$, $-\infty < z < \infty$, and $0 \leq \phi < 2\pi$, where $\phi = 2\pi$ is identified with $\phi = 0$.

> *EXTRA INFORMATION*
>
> *To work the problem, you do not need to know anything about where this metric came from. However, it might (or might not!) help your intuition to know that Eq. (1) was obtained by starting with a Minkowski metric in cylindrical coordinates* $\bar{t}$, $\bar{r}$, $\bar{\phi}$, *and* $\bar{z}$,
>
> $$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}\bar{t}^{\,2} - \left[\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\bar{\phi}^2 + \mathrm{d}\bar{z}^2\right] ,$$
>
> *and then introducing new coordinates* t, r, ϕ, *and* z *that are related by*
>
> $$\bar{t} = t, \qquad \bar{r} = r, \quad \bar{\phi} = \phi + \omega t, \quad \bar{z} = z ,$$
>
> *so* $\mathrm{d}\bar{t} = \mathrm{d}t$, $\mathrm{d}\bar{r} = \mathrm{d}r$, $\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \omega\,\mathrm{d}t$, *and* $\mathrm{d}\bar{z} = \mathrm{d}z$.

(a) *(8 points)* The metric can be written in matrix form by using the standard definition

$$ds^2 = -c^2\,\mathrm{d}\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu ,$$

where $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \phi$, and $x^3 \equiv z$. Then, for example, g_{11} (which can also be called g_{rr}) is equal to 1. Find explicit expressions to complete the list of the nonzero entries in the matrix $g_{\mu\nu}$:

$$\begin{aligned}
&g_{11} \equiv g_{rr} = 1 \\
&g_{00} \equiv g_{tt} = ? \\
&g_{20} \equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = ? \qquad (2)\\
&g_{22} \equiv g_{\phi\phi} = ? \\
&g_{33} \equiv g_{zz} = ?
\end{aligned}$$

If you cannot answer part (a), you can introduce unspecified functions $f_1(r)$, $f_2(r)$, $f_3(r)$, and $f_4(r)$, with

$$\begin{aligned} g_{11} &\equiv g_{rr} = 1 \\ g_{00} &\equiv g_{tt} = f_1(r) \\ g_{20} &\equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = f_1(r) \\ g_{22} &\equiv g_{\phi\phi} = f_3(r) \\ g_{33} &\equiv g_{zz} = f_4(r) \ , \end{aligned} \tag{3}$$

and you can then express your answers to the subsequent parts in terms of these unspecified functions.

(b) *(10 points)* Using the geodesic equations from the front of the quiz,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ ,$$

explicitly write the equation that results when the free index μ is equal to 1, corresponding to the coordinate r.

(c) *(7 points)* Explicitly write the equation that results when the free index μ is equal to 2, corresponding to the coordinate ϕ.

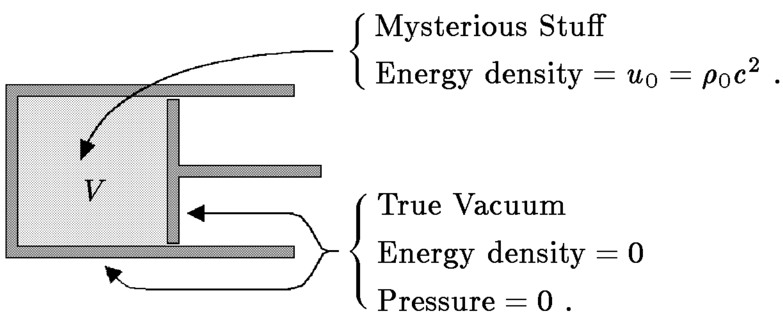
(d) *(10 points)* Use the metric to find an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of $\mathrm{d}r/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$, and $\mathrm{d}z/\mathrm{d}t$. The expression may also depend on the constants c and ω. Be sure to note that your answer should depend on the derivatives of t, ϕ, and z with respect to t, not τ. *(Hint: first find an expression for* $\mathrm{d}\tau/\mathrm{d}t$*, in terms of the quantities indicated, and then ask yourself how this result can be used to find* $\mathrm{d}t/\mathrm{d}\tau$*.)*

∗ PROBLEM 13: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

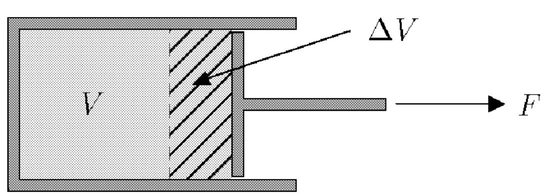
The following problem was Problem 3, Quiz 3, 2002. Although it is couched in the language of Lecture Notes 13, the physics is really the same as the pressure calculations in Lecture Notes 7, so a modified form of this problem would be fair for the coming quiz.

In Lecture Notes 13, a thought experiment involving a piston was used to show that $p = -\rho c^2$ for any substance for which the energy density remains constant under expansion. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

∗ PROBLEM 14: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

Today the temperature of the cosmic microwave background radiation is 2.7°K. Calculate the number density of photons in this radiation. What is the number density of thermal neutrinos left over from the big bang?

∗ PROBLEM 15: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

∗ PROBLEM 16: A NEW SPECIES OF LEPTON

The following problem was Problem 2, Quiz 3, 1992, worth 25 points.

Suppose the calculations describing the early universe were modified by including an additional, hypothetical lepton, called an 8.286ion. The 8.286ion has roughly the same properties as an electron, except that its mass is given by $mc^2 = 0.750$ MeV.

Parts (a)-(c) of this question require numerical answers, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included. (For example, if you were asked how many meters a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$.)

a) *(5 points)* What would be the number density of 8.286ions, in particles per cubic meter, when the temperature T was given by $kT = 3$ MeV?

b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the value of the mass density at $t = .01$ sec? You may assume that 0.75 MeV $\ll$ $kT \ll 100$ MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, e^+-e^- pairs, and 8.286ion-anti8286ion pairs. Express your answer in the units of gm-cm^{-3}.

c) *(5 points)* Under the same assumptions as in (b), what would be the value of kT, in MeV, at $t = .01$ sec?

d) *(5 points)* When nucleosynthesis calculations are modified to include the effect of the 8.286ion, is the production of helium increased or decreased? Explain your answer in a few sentences.

e) *(5 points)* Suppose the neutrinos decouple while $kT \gg 0.75$ MeV. If the 8.286ions are included, what does one predict for the value of T_ν/T_γ today? (Here T_ν denotes the temperature of the neutrinos, and T_γ denotes the temperature of the cosmic background radiation photons.)

PROBLEM 17: DID YOU DO THE READING?

(a) *(5 points)* By what factor does the lepton number per comoving volume of the universe change between temperatures of $kT = 10$ MeV and $kT = 0.1$ MeV? You should assume the existence of the normal three species of neutrinos for your answer.

(b) *(5 points)* Measurements of the primordial deuterium abundance would give good constraints on the baryon density of the universe. However, this abundance is hard to measure accurately. Which of the following is NOT a reason why this is hard to do?

(i) The neutron in a deuterium nucleus decays on the time scale of 15 minutes, so almost none of the primordial deuterium produced in the Big Bang is still present.

(ii) The deuterium abundance in the Earth's oceans is biased because, being heavier, less deuterium than hydrogen would have escaped from the Earth's surface.

(iii) The deuterium abundance in the Sun is biased because nuclear reactions tend to destroy it by converting it into helium-3.

(iv) The spectral lines of deuterium are almost identical with those of hydrogen, so deuterium signatures tend to get washed out in spectra of primordial gas clouds.

(v) The deuterium abundance is so small (a few parts per million) that it can be easily changed by astrophysical processes other than primordial nucleosynthesis.

(c) *(5 points)* Give three examples of hadrons.

(d) *(6 points)* In chapter 6 of *The First Three Minutes*, Steven Weinberg posed the question, "Why was there no systematic search for this [cosmic background] radiation, years before 1965?" In discussing this issue, he contrasted it with the history of two different elementary particles, each of which were predicted approximately 20 years before they were first detected. Name one of these two elementary particles. (If you name them both correctly, you will get 3 points extra credit. However, one right and one wrong will get you 4 points for the question, compared to 6 points for just naming one particle and getting it right.)

Answer: ________________________

2nd Answer (optional): ________________________

(e) *(6 points)* In Chapter 6 of *The First Three Minutes*, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list. (2 points for each right answer, circle at most 3.)

(i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.

(ii) There was a breakdown in communication between theorists and experimentalists.

(iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.

(iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

(v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.

(vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

SOLUTIONS

PROBLEM 1: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\, dt^2 + R^2(t) d\psi^2 \ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{R^2(t)} \ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm \frac{c}{R(t)} \ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\rm hor} = \int_0^t \frac{c}{R(t')} dt' \ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\rm hor}$, which according to the metric is given by

$$\ell_{\rm phys}(t) = \int_{\psi=0}^{\psi=\psi_{\rm hor}} ds = \int_0^{\psi_{\rm hor}} R(t)\, d\psi = \boxed{R(t) \int_0^t \frac{c}{R(t')} dt' \ .}$$

(c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{R(t)} \ .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta) \ .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1-\cos\theta)}{R(t)} \ .$$

Then using $R = \alpha(1 - \cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1 \ .}$$

(d) This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ must also change by 2π. From $R = \alpha(1 - \cos\theta)$, one can see that R returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1 \ .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 6. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

$$x^2 + y^2 + z^2 + w^2 = a^2 \ ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0, 0, 0, 1)$ can be called the north pole. Each point (x, y, z, w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x, y, z, w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0, 0, 0, -1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch.

Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\text{Crunch}}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\text{Crunch}} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\text{Crunch}}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\text{Crunch}} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost 1/2, and it would approach 1/2 as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 2: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC

a) Along the first segment $d\theta = 0$, so $ds^2 = (1 + ar)^2\, dr^2$, or $ds = (1 + ar)\, dr$. Integrating, the length of the first segment is found to be

$$S_1 = \int_0^{r_0} (1 + ar)\, dr = r_0 + \frac{1}{2} a r_0^2 \; .$$

Along the second segment $dr = 0$, so $ds = r(1 + br)\, d\theta$, where $r = r_0$. So the length of the second segment is

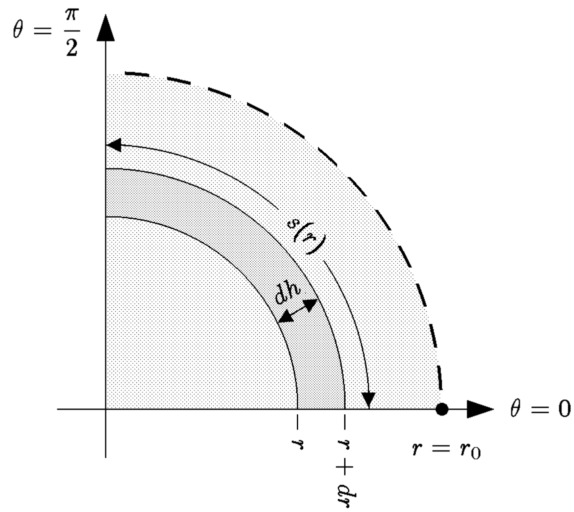
$$S_2 = \int_0^{\pi/2} r_0(1 + br_0)\, d\theta = \frac{\pi}{2} r_0(1 + br_0) \; .$$

Finally, the third segment is identical to the first, so $S_3 = S_1$. The total length is then

$$S = 2S_1 + S_2 = 2\left(r_0 + \frac{1}{2} a r_0^2\right) + \frac{\pi}{2} r_0(1 + br_0)$$

$$= \boxed{\left(2 + \frac{\pi}{2}\right) r_0 + \frac{1}{2}(2a + \pi b) r_0^2 \; .}$$

b) To find the area, it is best to divide the region into concentric strips as shown:



Note that the strip has a coordinate width of dr, but the distance across the width of the strip is determined by the metric to be

$$dh = (1 + ar)\, dr \ .$$

The length of the strip is calculated the same way as S_2 in part (a):

$$s(r) = \frac{\pi}{2} r(1 + br) \ .$$

The area is then

$$dA = s(r)\, dh \ ,$$

so

$$\begin{aligned} A &= \int_0^{r_0} s(r)\, dh \\ &= \int_0^{r_0} \frac{\pi}{2} r(1 + br)(1 + ar)\, dr \\ &= \frac{\pi}{2} \int_0^{r_0} [r + (a + b)r^2 + abr^3]\, dr \\ &= \boxed{\frac{\pi}{2} \left[\frac{1}{2} r_0^2 + \frac{1}{3}(a + b) r_0^3 + \frac{1}{4} abr_0^4\right]} \end{aligned}$$

PROBLEM 3: GEOMETRY IN A CLOSED UNIVERSE

(a) As one moves along a line from the origin to $(a, 0, 0)$, there is no variation in θ or ϕ. So $d\theta = d\phi = 0$, and

$$ds = \frac{R\,dr}{\sqrt{1-r^2}} \ .$$

So

$$\boxed{\ell_p = \int_0^a \frac{R\,dr}{\sqrt{1-r^2}} = R\sin^{-1} a \ .}$$

(b) In this case it is only θ that varies, so $dr = d\phi = 0$. So

$$ds = Rr\,d\theta \ ,$$

so

$$\boxed{s_p = Ra\,\Delta\theta \ .}$$

(c) From part (a), one has

$$a = \sin(\ell_p/R) \ .$$

Inserting this expression into the answer to (b), and then solving for $\Delta\theta$, one has

$$\boxed{\Delta\theta = \frac{s_p}{R\sin(\ell_p/R)} \ .}$$

Note that as $R \to \infty$, this approaches the Euclidean result, $\Delta\theta = s_p/\ell_p$.

PROBLEM 4: THE GENERAL SPHERICALLY SYMMETRIC METRIC

(a) The metric is given by

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\,d\phi^2\right] \ .$$

The radius a is defined as the physical length of a radial line which extends from the center to the boundary of the sphere. The length of a path is just the integral of ds, so

$$a = \int_{\substack{\text{radial path from}\\ \text{origin to } r_0}} ds \ .$$

The radial path is at a constant value of θ and ϕ, so $d\theta = d\phi = 0$, and then $ds = dr$. So

$$a = \int_0^{r_0} dr = \boxed{r_0 \ .}$$

(b) On the surface $r = r_0$, so $dr \equiv 0$. Then

$$ds^2 = \rho^2(r_0)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \ .$$

To find the area element, consider first a path obtained by varying only θ. Then $ds = \rho(r_0)\, d\theta$. Similarly, a path obtained by varying only ϕ has length $ds = \rho(r_0)\sin\theta\, d\phi$. Furthermore, these two paths are perpendicular to each other, a fact that is incorporated into the metric by the absence of a $dr\, d\theta$ term. Thus, the area of a small rectangle constructed from these two paths is given by the product of their lengths, so

$$dA = \rho^2(r_0)\sin\theta\, d\theta\, d\phi \ .$$

The area is then obtained by integrating over the range of the coordinate variables:

$$\begin{aligned} A &= \rho^2(r_0)\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \\ &= \rho^2(r_0)(2\pi)\left(-\cos\theta\Big|_0^{\pi}\right) \\ \Longrightarrow &\quad \boxed{A = 4\pi\rho^2(r_0) \ .} \end{aligned}$$

As a check, notice that if $\rho(r) = r$, then the metric becomes the metric of Euclidean space, in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere, $4\pi r^2$.

(c) As in Problem 2 of Problem Set 3 (2000), we can imagine breaking up the volume into spherical shells of infinitesimal thickness, with a given shell extending from r to $r + dr$. By the previous calculation, the area of such a shell is $A(r) = 4\pi\rho^2(r)$. (In the previous part we considered only the case $r = r_0$, but the same argument applies for any value of r.) The thickness of the shell is just the path length ds of a radial path corresponding to the coordinate interval dr. For radial paths the metric reduces to $ds^2 = dr^2$, so the thickness of the shell is $ds = dr$. The volume of the shell is then

$$dV = 4\pi\rho^2(r)\, dr \ .$$

The total volume is then obtained by integration:

$$\boxed{V = 4\pi \int_0^{r_0} \rho^2(r)\, dr \ .}$$

Checking the answer for the Euclidean case, $\rho(r) = r$, one sees that it gives $V = (4\pi/3)r_0^3$, as expected.

(d) If r is replaced by a new coordinate $\sigma \equiv r^2$, then the infinitesimal variations of the two coordinates are related by

$$\frac{d\sigma}{dr} = 2r = 2\sqrt{\sigma}\ ,$$

so

$$dr^2 = \frac{d\sigma^2}{4\sigma}\ .$$

The function $\rho(r)$ can then be written as $\rho(\sqrt{\sigma}\,)$, so

$$\boxed{ds^2 = \frac{d\sigma^2}{4\sigma} + \rho^2(\sqrt{\sigma}\,)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\ .}$$

PROBLEM 5: VOLUMES IN A ROBERTSON-WALKER UNIVERSE

The product of differential length elements corresponding to infinitesimal changes in the coordinates r, θ and ϕ equals the differential volume element dV. Therefore

$$dV = R(t)\frac{dr}{\sqrt{1-kr^2}} \times R(t)rd\theta \times R(t)r\sin\theta d\phi$$

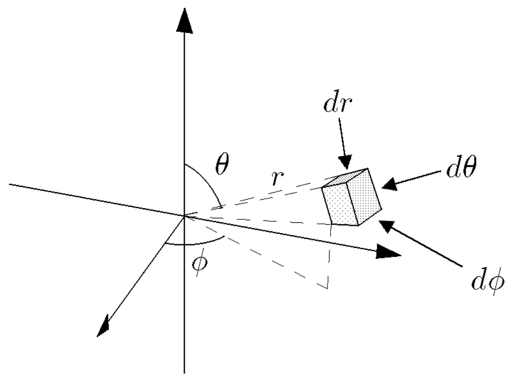
The total volume is then

$$V = \int dV = R^3(t)\int_0^{r_{\max}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2\sin\theta}{\sqrt{1-kr^2}}$$

We can do the angular integrations immediately:

$$\boxed{V = 4\pi R^3(t)\int_0^{r_{max}} \frac{r^2 dr}{\sqrt{1-kr^2}}\ .}$$

[Pedagogical Note: If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:



The cell includes the volume lying between r and $r+dr$, between θ and $\theta+d\theta$, and between ϕ and $\phi+d\phi$. In the limit as dr, $d\theta$, and $d\phi$ all approach zero, the cell approaches a rectangular solid with sides of length:

$$\begin{aligned} ds_1 &= R(t)\frac{dr}{\sqrt{1-kr^2}} \\ ds_2 &= R(t) r\, d\theta \\ ds_3 &= R(t) r \sin\theta\, d\theta \ . \end{aligned}$$

Here each ds is calculated by using the metric to find ds^2, in each case allowing only one of the quantities dr, $d\theta$, or $d\phi$ to be nonzero. The infinitesimal volume element is then $dV = ds_1 ds_2 ds_3$, resulting in the answer above. The derivation relies on the orthogonality of the dr, $d\theta$, and $d\phi$ directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as $dr\, d\theta$.]

[Extension: The integral can in fact be carried out, using the substitution

$$\begin{aligned} \sqrt{k}\, r &= \sin\psi \quad (\text{if } k>0) \\ \sqrt{-k}\, r &= \sinh\psi \quad (\text{if } k>0). \end{aligned}$$

The answer is

$$V = \begin{cases} 2\pi R^3(t)\left[\dfrac{\sin^{-1}\left(\sqrt{k}\, r_{\rm max}\right)}{k^{3/2}} - \dfrac{\sqrt{1-kr_{\rm max}^2}}{k}\right] & (\text{if } k>0) \\ 2\pi R^3(t)\left[\dfrac{\sqrt{1-kr_{\rm max}^2}}{(-k)} - \dfrac{\sinh^{-1}\left(\sqrt{-k}\, r_{\rm max}\right)}{(-k)^{3/2}}\right] & (\text{if } k<0) \quad .] \end{cases}$$

PROBLEM 6: THE SCHWARZSCHILD METRIC

a) The Schwarzschild horizon is the value of r for which the metric becomes singular. Since the metric contains the factor

$$\left(1 - \frac{2GM}{rc^2}\right) ,$$

it becomes singular at

$$\boxed{R_{\rm Sch} = \frac{2GM}{c^2} .}$$

b) The separation between A and B is purely in the radial direction, so the proper length of a segment along the path joining them is given by

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 ,$$

so

$$ds = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} .$$

The proper distance from A to B is obtained by adding the proper lengths of all the segments along the path, so

$$\boxed{s_{AB} = \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} .}$$

EXTENSION: The integration can be carried out explicitly. First use the expression for the Schwarzschild radius to rewrite the expression for s_{AB} as

$$s_{AB} = \int_{r_A}^{r_B} \frac{\sqrt{r}\, dr}{\sqrt{r - R_{\rm Sch}}} .$$

Then introduce the hyperbolic trigonometric substitution

$$r = R_{\rm Sch} \cosh^2 u .$$

One then has

$$\sqrt{r - R_{\rm Sch}} = \sqrt{R_{\rm Sch}}\, \sinh u$$

$$dr = 2R_{\rm Sch} \cosh u \sinh u \, du \ ,$$

and the indefinite integral becomes

$$\begin{aligned}
\int \frac{\sqrt{r}\, dr}{\sqrt{r - R_{\rm Sch}}} &= 2R_{\rm Sch} \int \cosh^2 u \, du \\
&= R_{\rm Sch} \int (1 + \cosh 2u) du \\
&= R_{\rm Sch} \left(u + \frac{1}{2} \sinh 2u \right) \\
&= R_{\rm Sch} (u + \sinh u \, \cosh u) \\
&= R_{\rm Sch} \sinh^{-1} \left(\sqrt{\frac{r}{R_{\rm Sch}} - 1} \right) + \sqrt{r(r - R_{\rm Sch})} \ .
\end{aligned}$$

Thus,

$$\begin{aligned}
s_{AB} = R_{\rm Sch} &\left[\sinh^{-1} \left(\sqrt{\frac{r_B}{R_{\rm Sch}} - 1} \right) - \sinh^{-1} \left(\sqrt{\frac{r_A}{R_{\rm Sch}} - 1} \right) \right] \\
&+ \sqrt{r_B(r_B - R_{\rm Sch})} - \sqrt{r_A(r_A - R_{\rm Sch})} \ .
\end{aligned}$$

c) A tick of the clock and the following tick are two events that differ only in their time coordinates. Thus, the metric reduces to

$$-c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2} \right) c^2 dt^2 \ ,$$

so

$$d\tau = \sqrt{1 - \frac{2GM}{rc^2}} \, dt \ .$$

The reading on the observer's clock corresponds to the proper time interval $d\tau$, so the corresponding interval of the coordinate t is given by

$$\boxed{\Delta t_A = \frac{\Delta \tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}} \ .}$$

d) Since the Schwarzschild metric does not change with time, each pulse leaving A will take the same length of time to reach B. Thus, the pulses emitted by A will arrive at B with a time coordinate spacing

$$\Delta t_B = \Delta t_A = \frac{\Delta \tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}} \ .$$

The clock at B, however, will read the proper time and not the coordinate time. Thus,

$$\Delta\tau_B = \sqrt{1 - \frac{2GM}{r_B c^2}}\,\Delta t_B$$

$$= \boxed{\sqrt{\frac{1 - \frac{2GM}{r_B c^2}}{1 - \frac{2GM}{r_A c^2}}}\,\Delta\tau_A\ .}$$

e) From parts (a) and (b), the proper distance between A and B can be rewritten as

$$s_{AB} = \int_{R_{\text{Sch}}}^{r_B} \frac{\sqrt{r}dr}{\sqrt{r - R_{\text{Sch}}}}\ .$$

The potentially divergent part of the integral comes from the range of integration in the immediate vicinity of $r = R_{\text{Sch}}$, say $R_{\text{Sch}} < r < R_{\text{Sch}} + \epsilon$. For this range the quantity $\sqrt{r}$ in the numerator can be approximated by $\sqrt{R_{\text{Sch}}}$, so the contribution has the form

$$\sqrt{R_{\text{Sch}}} \int_{R_{\text{Sch}}}^{R_{\text{Sch}}+\epsilon} \frac{dr}{\sqrt{r - R_{\text{Sch}}}}\ .$$

Changing the integration variable to $u \equiv r - R_{\text{Sch}}$, the contribution can be easily evaluated:

$$\sqrt{R_{\text{Sch}}} \int_{R_{\text{Sch}}}^{R_{\text{Sch}}+\epsilon} \frac{dr}{\sqrt{r - R_{\text{Sch}}}} = \sqrt{R_{\text{Sch}}} \int_0^{\epsilon} \frac{du}{\sqrt{u}} = 2\sqrt{R_{\text{Sch}}\epsilon} < \infty\ .$$

So, although the integrand is infinite at $r = R_{\text{Sch}}$, the integral is still finite.

$$\boxed{\text{The proper distance between } A \text{ and } B \text{ does not diverge.}}$$

Looking at the answer to part (d), however, one can see that when $r_A = R_{\text{Sch}}$,

$$\boxed{\text{The time interval } \Delta\tau_B \text{ diverges.}}$$

PROBLEM 7: GEODESICS

The geodesic equation for a curve $x^i(\lambda)$, where the parameter λ is the arc length along the curve, can be written as

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\,\frac{dx^\ell}{d\lambda} \ .$$

Here the indices j, k, and ℓ are summed from 1 to the dimension of the space, so there is one equation for each value of i.

(a) The metric is given by

$$ds^2 = g_{ij}dx^i dx^j = dr^2 + r^2\, d\theta^2 \ ,$$

so

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2 \ , \qquad g_{r\theta} = g_{\theta r} = 0 \ .$$

First taking $i = r$, the nonvanishing terms in the geodesic equation become

$$\frac{d}{d\lambda}\left\{g_{rr}\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r g_{\theta\theta}\right)\frac{d\theta}{d\lambda}\,\frac{d\theta}{d\lambda} \ ,$$

which can be written explicitly as

$$\frac{d}{d\lambda}\left\{\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r r^2\right)\left(\frac{d\theta}{d\lambda}\right)^2 \ ,$$

or

$$\boxed{\frac{d^2r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 \ .}$$

For $i = \theta$, one has the simplification that g_{ij} is independent of θ for all (i, j). So

$$\boxed{\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0 \ .}$$

(b) The first step is to parameterize the curve, which means to imagine moving along the curve, and expressing the coordinates as a function of the distance traveled. (I am calling the locus $y = 1$ a curve rather than a line, since the techniques that are used here are usually applied to curves. Since a line is a special case of a curve, there is nothing wrong with treating the line as a curve.) In Cartesian coordinates, the curve $y = 1$ can be parameterized as

$$x(\lambda) = \lambda \ , \qquad y(\lambda) = 1 \ .$$

(The parameterization is not unique, because one can choose $\lambda = 0$ to represent any point along the curve.) Converting to the desired polar coordinates,

$$r(\lambda) = \sqrt{x^2(\lambda) + y^2(\lambda)} = \sqrt{\lambda^2 + 1} \ ,$$

$$\theta(\lambda) = \tan^{-1} \frac{y(\lambda)}{x(\lambda)} = \tan^{-1}(1/\lambda) \ .$$

Calculating the needed derivatives,*

$$\begin{aligned} \frac{dr}{d\lambda} &= \frac{\lambda}{\sqrt{\lambda^2 + 1}} \\ \frac{d^2 r}{d\lambda^2} &= \frac{1}{\sqrt{\lambda^2 + 1}} - \frac{\lambda^2}{(\lambda^2 + 1)^{3/2}} = \frac{1}{(\lambda^2 + 1)^{3/2}} = \frac{1}{r^3} \\ \frac{d\theta}{d\lambda} &= -\frac{1}{1 + \left(\frac{1}{\lambda}\right)^2} \frac{1}{\lambda^2} = -\frac{1}{r^2} \ . \end{aligned}$$

Then, substituting into the geodesic equation for $i = r$,

$$\frac{d^2 r}{d\lambda^2} = r \left(\frac{d\theta}{d\lambda} \right)^2 \iff \frac{1}{r^3} = r \left(-\frac{1}{r^2} \right)^2 \ ,$$

which checks. Substituting into the geodesic equation for $i = \theta$,

$$\frac{d}{d\lambda} \left\{ r^2 \frac{d\theta}{d\lambda} \right\} = 0 \iff \frac{d}{d\lambda} \left\{ r^2 \left(-\frac{1}{r^2} \right) \right\} = 0 \ ,$$

which also checks.

---

* If you do not remember how to differentiate $\phi = \tan^{-1}(z)$, then you should know how to derive it. Write $z = \tan\phi = \sin\phi / \cos\phi$, so

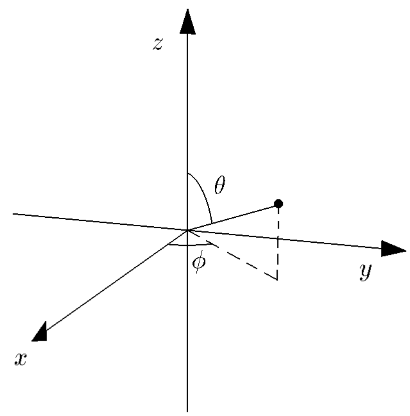
$$dz = \left(\frac{\cos\phi}{\cos\phi} + \frac{\sin^2\phi}{\cos^2\phi} \right) d\phi = (1 + \tan^2\phi) d\phi \ .$$

Then

$$\frac{d\phi}{dz} = \frac{1}{1 + \tan^2\phi} = \frac{1}{1 + z^2} \ .$$

PROBLEM 8: GEODESICS ON THE SURFACE OF A SPHERE

(a) Rotations are easy to understand in Cartesian coordinates. The relationship between the polar and Cartesian coordinates is given by

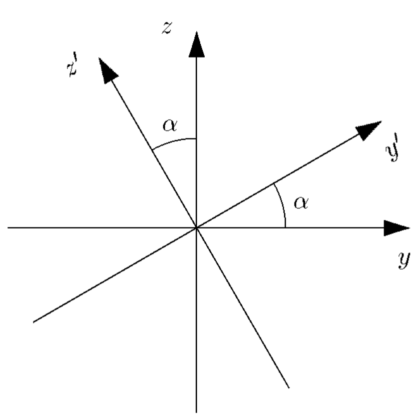


$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta \ .$$

The equator is then described by $\theta = \pi/2$, and $\phi = \psi$, where ψ is a parameter running from 0 to 2π. Thus, the equator is described by the curve $x^i(\psi)$, where

$$x^1 = x = r\cos\psi$$
$$x^2 = y = r\sin\psi$$
$$x^3 = z = 0 \ .$$

Now introduce a primed coordinate system that is related to the original system by a rotation in the y-z plane by an angle α:



$$x = x'$$
$$y = y'\cos\alpha - z'\sin\alpha$$
$$z = z'\cos\alpha + y'\sin\alpha \ .$$

The rotated equator, which we seek to describe, is just the standard equator in the primed coordinates:

$$x' = r\cos\psi\ , \qquad y' = r\sin\psi\ , \qquad z' = 0\ .$$

Using the relation between the two coordinate systems given above,

$$\begin{aligned} x &= r\cos\psi \\ y &= r\sin\psi\cos\alpha \\ z &= r\sin\psi\sin\alpha\ . \end{aligned}$$

Using again the relations between polar and Cartesian coordinates,

$$\boxed{\begin{aligned} \cos\theta &= \frac{z}{r} = \sin\psi\sin\alpha \\ \tan\phi &= \frac{y}{x} = \tan\psi\cos\alpha\ . \end{aligned}}$$

(b) A segment of the equator corresponding to an interval $d\psi$ has length $a\,d\psi$, so the parameter ψ is proportional to the arc length. Expressed in terms of the metric, this relationship becomes

$$ds^2 = g_{ij}\frac{dx^i}{d\psi}\frac{dx^j}{d\psi}d\psi^2 = a^2 d\psi^2\ .$$

Thus the quantity

$$A \equiv g_{ij}\frac{dx^i}{d\psi}\frac{dx^j}{d\psi}$$

is equal to a^2, so the geodesic equation (6.36) reduces to the simpler form of Eq. (6.38). (Note that we are following the notation of Lecture Notes 6, except that the variable used to parametrize the path is called ψ, rather than λ or s. Although A is not equal to 1 as we assumed in Lecture Notes 6, it is easily seen that Eq. (6.38) follows from (6.36) provided only that $A = \textit{constant}$.) Thus,

$$\frac{d}{d\psi}\left\{g_{ij}\frac{dx^j}{d\psi}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\psi}\,\frac{dx^\ell}{d\psi}\ .$$

For this problem the metric has only two nonzero components:

$$g_{\theta\theta} = a^2\ , \qquad g_{\phi\phi} = a^2\sin^2\theta\ .$$

Taking $i = \theta$ in the geodesic equation,

$$\frac{d}{d\psi}\left\{g_{\theta\theta}\frac{d\theta}{d\psi}\right\} = \frac{1}{2}\partial_\theta g_{\phi\phi}\frac{d\phi}{d\psi}\frac{d\phi}{d\psi} \quad \Longrightarrow$$

$$\boxed{\frac{d^2\theta}{d\psi^2} = \sin\theta\cos\theta\left(\frac{d\phi}{d\psi}\right)^2 \ .}$$

Taking $i = \phi$,

$$\frac{d}{d\psi}\left\{a^2\sin^2\theta\frac{d\phi}{d\psi}\right\} = 0 \quad \Longrightarrow$$

$$\boxed{\frac{d}{d\psi}\left\{\sin^2\theta\frac{d\phi}{d\psi}\right\} = 0 \ .}$$

(c) This part is mainly algebra. Taking the derivative of

$$\cos\theta = \sin\psi\sin\alpha$$

implies

$$-\sin\theta\, d\theta = \cos\psi\sin\alpha\, d\psi \ .$$

Then, using the trigonometric identity $\sin\theta = \sqrt{1-\cos^2\theta}$, one finds

$$\sin\theta = \sqrt{1-\sin^2\psi\sin^2\alpha} \ ,$$

so

$$\frac{d\theta}{d\psi} = -\frac{\cos\psi\sin\alpha}{\sqrt{1-\sin^2\psi\sin^2\alpha}} \ .$$

Similarly

$$\tan\phi = \tan\psi\cos\alpha \quad \Longrightarrow \quad \sec^2\phi\, d\phi = \sec^2\psi\, d\psi\cos\alpha \ .$$

Then

$$\begin{aligned}\sec^2\phi &= \tan^2\phi + 1 = \tan^2\psi\cos^2\alpha + 1\\ &= \frac{1}{\cos^2\psi}[\sin^2\psi\cos^2\alpha + \cos^2\psi]\\ &= \sec^2\psi[\sin^2\psi(1-\sin^2\alpha) + \cos^2\psi]\\ &= \sec^2\psi[1-\sin^2\psi\sin^2\alpha] \ ,\end{aligned}$$

So

$$\frac{d\phi}{d\psi} = \frac{\cos\alpha}{1 - \sin^2\psi\sin^2\alpha} \ .$$

To verify the geodesic equations of part (b), it is easiest to check the second one first:

$$\begin{aligned}\sin^2\theta\frac{d\phi}{d\psi} &= (1 - \sin^2\psi\sin^2\alpha)\frac{\cos\alpha}{1 - \sin^2\psi\sin^2\alpha} \\ &= \cos\alpha \ ,\end{aligned}$$

so clearly

$$\frac{d}{d\psi}\left\{\sin^2\theta\frac{d\phi}{d\psi}\right\} = \frac{d}{d\psi}(\cos\alpha) = 0 \ .$$

To verify the first geodesic equation from part (b), first calculate the left-hand side, $d^2\theta/d\psi^2$, using our result for $d\theta/d\psi$:

$$\frac{d^2\theta}{d\psi^2} = \frac{d}{d\psi}\left(\frac{d\theta}{d\psi}\right) = \frac{d}{d\psi}\left\{-\frac{\cos\psi\sin\alpha}{\sqrt{1 - \sin^2\psi\sin^2\alpha}}\right\} \ .$$

After some straightforward algebra, one finds

$$\frac{d^2\theta}{d\psi^2} = \frac{\sin\psi\sin\alpha\cos^2\alpha}{\left[1 - \sin^2\psi\sin^2\alpha\right]^{3/2}} \ .$$

The right-hand side of the first geodesic equation can be evaluated using the expression found above for $d\phi/d\psi$, giving

$$\begin{aligned}\sin\theta\cos\theta\left(\frac{d\phi}{d\psi}\right)^2 &= \sqrt{1 - \sin^2\psi\sin^2\alpha}\,\sin\psi\sin\alpha\,\frac{\cos^2\alpha}{\left[1 - \sin^2\psi\sin^2\alpha\right]^2} \\ &= \frac{\sin\psi\sin\alpha\cos^2\alpha}{\left[1 - \sin^2\psi\sin^2\alpha\right]^{3/2}} \ .\end{aligned}$$

So the left- and right-hand sides are equal.

PROBLEM 9: GEODESICS IN A CLOSED UNIVERSE

(a) *(7 points)* For purely radial motion, $d\theta = d\phi = 0$, so the line element reduces do

$$-c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-r^2}\right\} .$$

Dividing by dt^2,

$$-c^2\left(\frac{d\tau}{dt}\right)^2 = -c^2 + \frac{R^2(t)}{1-r^2}\left(\frac{dr}{dt}\right)^2 .$$

Rearranging,

$$\boxed{\frac{d\tau}{dt} = \sqrt{1 - \frac{R^2(t)}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2} .}$$

(b) *(3 points)*

$$\frac{dt}{d\tau} = \frac{1}{\dfrac{d\tau}{dt}} = \boxed{\frac{1}{\sqrt{1 - \dfrac{R^2(t)}{c^2(1-r^2)}\left(\dfrac{dr}{dt}\right)^2}}} .$$

(c) *(10 points)* During any interval of clock time dt, the proper time that would be measured by a clock moving with the object is given by $d\tau$, as given by the metric. Using the answer from part (a),

$$d\tau = \frac{d\tau}{dt}\,dt = \sqrt{1 - \frac{R^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\,dt .$$

Integrating to find the total proper time,

$$\boxed{\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{R^2(t)}{c^2(1-r_p^2)}\left(\frac{dr_p}{dt}\right)^2}\,dt .}$$

(d) *(10 points)* The physical distance $d\ell$ that the object moves during a given time interval is related to the coordinate distance dr by the spatial part of the metric:

$$d\ell^2 = ds^2 = R^2(t)\left\{\frac{dr^2}{1-r^2}\right\} \quad\Longrightarrow\quad d\ell = \frac{R(t)}{\sqrt{1-r^2}}\,dr .$$

Thus

$$\boxed{v_{\rm phys} = \frac{d\ell}{dt} = \frac{R(t)}{\sqrt{1-r^2}}\,\frac{dr}{dt}\ .}$$

Discussion: A common mistake was to include $-c^2\,dt^2$ in the expression for $d\ell^2$. To understand why this is not correct, we should think about how an observer would measure $d\ell$, the distance to be used in calculating the velocity of a passing object. The observer would place a meter stick along the path of the object, and she would mark off the position of the object at the beginning and end of a time interval $dt_{\rm meas}$. Then she would read the distance by subtracting the two readings on the meter stick. This subtraction is equal to the physical distance between the two marks, measured at the ***same*** time t. Thus, when we compute the distance between the two marks, we set $dt = 0$. To compute the speed she would then divide the distance by $dt_{\rm meas}$, which is nonzero.

(e) *(10 points)* We start with the standard formula for a geodesic, as written on the front of the exam:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\ .$$

This formula is true for each possible value of μ, while the Einstein summation convention implies that the indices ν, λ, and σ are summed. We are trying to derive the equation for r, so we set $\mu = r$. Since the metric is diagonal, the only contribution on the left-hand side will be $\nu = r$. On the right-hand side, the diagonal nature of the metric implies that nonzero contributions arise only when $\lambda = \sigma$. The term will vanish unless $dx^\lambda/d\tau$ is nonzero, so λ must be either r or t (i.e., there is no motion in the θ or ϕ directions). However, the right-hand side is proportional to

$$\frac{\partial g_{\lambda\sigma}}{\partial r}\ .$$

Since $g_{tt} = -c^2$, the derivative with respect to r will vanish. Thus, the only nonzero contribution on the right-hand side arises from $\lambda = \sigma = r$. Using

$$g_{rr} = \frac{R^2(t)}{1-r^2}\ ,$$

the geodesic equation becomes

$$\frac{d}{d\tau}\left\{g_{rr}\frac{dr}{d\tau}\right\} = \frac{1}{2}\left(\partial_r g_{rr}\right)\frac{dr}{d\tau}\frac{dr}{d\tau}\ ,$$

or

$$\frac{d}{d\tau}\left\{\frac{R^2}{1-r^2}\frac{dr}{d\tau}\right\} = \frac{1}{2}\left[\partial_r\left(\frac{R^2}{1-r^2}\right)\right]\frac{dr}{d\tau}\frac{dr}{d\tau}\ ,$$

or finally

$$\boxed{\frac{d}{d\tau}\left\{\frac{R^2}{1-r^2}\frac{dr}{d\tau}\right\} = R^2\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\tau}\right)^2\ .}$$

This matches the form shown in the question, with

$$A = \frac{R^2}{1-r^2}\ ,\ \text{and}\ C = R^2\frac{r}{(1-r^2)^2}\ ,$$

with $B = D = E = 0$.

(f) *(5 points EXTRA CREDIT)* The algebra here can get messy, but it is not too bad if one does the calculation in an efficient way. One good way to start is to simplify the expression for p. Using the answer from (d),

$$p = \frac{mv_{\text{phys}}}{\sqrt{1-\frac{v_{\text{phys}}^2}{c^2}}} = \frac{m\frac{R(t)}{\sqrt{1-r^2}}\frac{dr}{dt}}{\sqrt{1-\frac{R^2}{c^2(1-r^2)}\left(\frac{dr}{dt}\right)^2}}\ .$$

Using the answer from (b), this simplifies to

$$p = m\frac{R(t)}{\sqrt{1-r^2}}\frac{dr}{dt}\frac{dt}{d\tau} = m\frac{R(t)}{\sqrt{1-r^2}}\frac{dr}{d\tau}\ .$$

Multiply the geodesic equation by m, and then use the above result to rewrite it as

$$\frac{d}{d\tau}\left\{\frac{Rp}{\sqrt{1-r^2}}\right\} = mR^2\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\tau}\right)^2\ .$$

Expanding the left-hand side,

$$\begin{aligned} LHS = \frac{d}{d\tau}\left\{\frac{Rp}{\sqrt{1-r^2}}\right\} &= \frac{1}{\sqrt{1-r^2}}\frac{d}{d\tau}\{Rp\} + Rp\frac{r}{(1-r^2)^{3/2}}\frac{dr}{d\tau} \\ &= \frac{1}{\sqrt{1-r^2}}\frac{d}{d\tau}\{Rp\} + mR^2\frac{r}{(1-r^2)^2}\left(\frac{dr}{d\tau}\right)^2\ . \end{aligned}$$

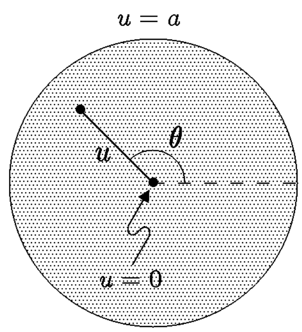
Inserting this expression back into left-hand side of the original equation, one sees that the second term cancels the expression on the right-hand side, leaving

$$\frac{1}{\sqrt{1-r^2}}\frac{d}{d\tau}\{Rp\} = 0\ .$$

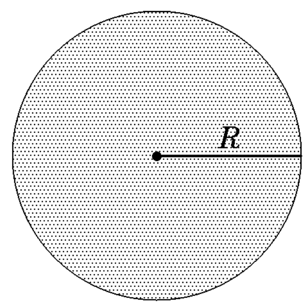
Multiplying by $\sqrt{1-r^2}$, one has the desired result:

$$\frac{d}{d\tau}\{Rp\} = 0 \quad \Longrightarrow \quad \boxed{p \propto \frac{1}{R(t)}} \ .$$

PROBLEM 10: A TWO-DIMENSIONAL CURVED SPACE *(40 points)*



(a) For $\theta = constant$, the expression for the metric reduces to



$$ds^2 = \frac{a\,du^2}{4u(a-u)} \quad \Longrightarrow$$

$$ds = \frac{1}{2}\sqrt{\frac{a}{u(a-u)}}\,du \ .$$

To find the length of the radial line shown, one must integrate this expression from the value of u at the center, which is 0, to the value of u at the outer edge, which is a. So

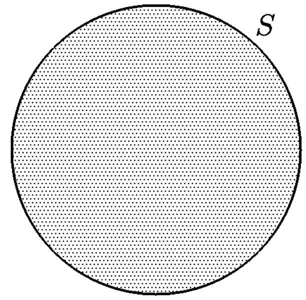
$$\boxed{R = \frac{1}{2}\int_0^a \sqrt{\frac{a}{u(a-u)}}\,du \ .}$$

You were not expected to do it, but the integral can be carried out, giving $R = (\pi/2)\sqrt{a}$.

(b) For $u = constant$, the expression for the metric reduces to

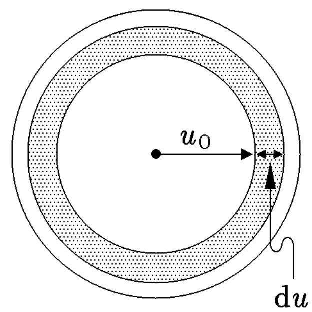
$$ds^2 = u\,\mathrm{d}\theta^2 \quad \Longrightarrow \quad ds = \sqrt{u}\,\mathrm{d}\theta \ .$$

Since θ runs from 0 to 2π, and $u = a$ for the circumference of the space,



$$\boxed{S = \int_0^{2\pi} \sqrt{a}\,\mathrm{d}\theta = 2\pi\sqrt{a} \ .}$$

(c) To evaluate the answer to first order in $\mathrm{d}u$ means to neglect any terms that would be proportional to $\mathrm{d}u^2$ or higher powers. This means that we can treat the annulus as if it were arbitrarily thin, in which case we can imagine bending it into a rectangle without changing its area. The area is then equal to the circumference times the width. Both the circumference and the width must be calculated by using the metric:



$$\begin{aligned}
\mathrm{d}A &= \text{circumference} \times \text{width} \\
&= [2\pi\sqrt{u_0}\,] \times \left[\frac{1}{2}\sqrt{\frac{a}{u_0(a-u_0)}}\,\mathrm{d}u\right] \\
&= \boxed{\pi\sqrt{\frac{a}{(a-u_0)}}\,\mathrm{d}u \ .}
\end{aligned}$$

(d) We can find the total area by imagining that it is broken up into annuluses, where a single annulus starts at radial coordinate u and extends to $u + \mathrm{d}u$. As in part (a), this expression must be integrated from the value of u at the center, which is 0, to the value of u at the outer edge, which is a.

$$\boxed{A = \pi \int_0^a \sqrt{\frac{a}{(a-u)}}\,\mathrm{d}u \ .}$$

You did not need to carry out this integration, but the answer would be $A = 2\pi a$.

(e) From the list at the front of the exam, the general formula for a geodesic is written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\,\frac{\mathrm{d}x^j}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{k\ell}}{\partial x^i}\frac{\mathrm{d}x^k}{\mathrm{d}s}\frac{\mathrm{d}x^\ell}{\mathrm{d}s} \; .$$

The metric components g_{ij} are related to $\mathrm{d}s^2$ by

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \; ,$$

where the Einstein summation convention (sum over repeated indices) is assumed. In this case

$$\begin{aligned} g_{11} \equiv g_{uu} &= \frac{a}{4u(a-u)} \\ g_{22} \equiv g_{\theta\theta} &= u \\ g_{12} = g_{21} &= 0 \; , \end{aligned}$$

where I have chosen $x^1 = u$ and $x^2 = \theta$. The equation with $\mathrm{d}u/\mathrm{d}s$ on the left-hand side is found by looking at the geodesic equations for $i = 1$. Of course j, k, and ℓ must all be summed, but the only nonzero contributions arise when $j = 1$, and k and ℓ are either both equal to 1 or both equal to 2:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{uu}\frac{\mathrm{d}u}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{uu}}{\partial u}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial u}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \; .$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{a}{4u(a-u)}\frac{\mathrm{d}u}{\mathrm{d}s}\right] &= \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{a}{4u(a-u)}\right)\right]\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}u}(u)\right]\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \\ &= \frac{1}{2}\left[\frac{a}{4u(a-u)^2} - \frac{a}{4u^2(a-u)}\right]\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \\ &= \boxed{\frac{1}{8}\frac{a(2u-a)}{u^2(a-u)^2}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \; .} \end{aligned}$$

(f) This part is solved by the same method, but it is simpler. Here we consider the geodesic equation with $i = 2$. The only term that contributes on the left-hand side is $j = 2$. On the right-hand side one finds nontrivial expressions when k and ℓ are either both equal to 1 or both equal to 2. However, the terms on the right-hand side both involve the derivative of the metric with respect to $x^2 = \theta$, and these derivatives all vanish. So

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = \frac{1}{2}\frac{\partial g_{uu}}{\partial\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 ,$$

which reduces to

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left[u\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = 0 .}$$

PROBLEM 11: EVOLUTION OF MODEL UNIVERSES *(30 points)*

(a) For an empty universe, the Friedmann equation is

$$\left(\frac{\dot{R}}{R}\right)^2 = -\frac{kc^2}{R^2} .$$

Since the left-hand side cannot be negative, an empty universe cannot have $k > 0$, i.e. it cannot be closed.

Now consider $k = 0$, i.e. a flat universe. In this case the above equation has the solution

$$R(t) = R_0 ,$$

where R_0 is independent of time. So an empty universe can be flat as long as it is static.

Finally consider $k < 0$, i.e. an open universe. From the Friedmann equation we get

$$\dot{R} = \sqrt{|k|}c \quad \Longrightarrow \quad R(t) = \sqrt{|k|}ct + \mathrm{const} ,$$

where the constant of integration *const* can be set to zero by using the convention that $t = 0$ when $R(t) = 0$. So, an empty universe can be open with a scale factor that increases linearly with time.

(b)

(i) Nonrelativistic matter: $w = 0$.

(ii) Relativistic matter: $w = 1/3$.

(iii) The cosmological constant: $w = -1$.

(c) The fluid equation is

$$\dot{\rho} = -3\frac{\dot{R}}{R}(1 + w)\rho ,$$

where we have used the equation of state to express p in terms of ρ. Now we're given that $\rho \propto R^{-b}$ and so $\dot{\rho} \propto -bR^{-b-1}\dot{R}$, with the same constant of proportionality. Plugging these expressions into the fluid equation,

$$-bR^{-b-1}\dot{R} = -3\frac{\dot{R}}{R}(1+w)R^{-b}$$

$$\Longrightarrow \quad \boxed{b = 3(1+w)\ .}$$

(d) We can use the Friedmann equation for a flat universe to determine $R(t)$:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho\ .$$

For $\rho \propto R^{-b}$, the above equation can be written as

$$\left(\frac{\dot{R}}{R}\right)^2 \propto R^{-b}\ .$$

First consider the case $b = 0$, for which we find

$$\frac{\dot{R}}{R} = \text{const} = H_0 \quad \Longrightarrow \quad \frac{\mathrm{d}R}{R} = H_0\,\mathrm{d}t\ .$$

Integrating,

$$\ln R = H_0 t + \text{const} \quad \Longrightarrow \quad \boxed{R(t) \propto e^{H_0 t}\ .}$$

Next consider the case $b \neq 0$, for which we find

$$\frac{\dot{R}}{R} \propto R^{-b/2} \quad \Longrightarrow \quad R^{b/2-1}\,\mathrm{d}R \propto \mathrm{d}t\ .$$

Integrating,

$$R^{b/2} \propto t + \text{const} \quad \Longrightarrow \quad R^{b/2} \propto t \quad \Longrightarrow \quad \boxed{R(t) \propto t^{2/b}\ ,}$$

where again the constant of integration was set to zero by our convention for choosing the origin of time t.

— *Problem and solution written by Vishesh Khemani.*

PROBLEM 12: ROTATING FRAMES OF REFERENCE *(35 points)*

(a) The metric was given as

$$-c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + \left[\mathrm{d}r^2 + r^2\,\left(\mathrm{d}\phi + \omega\,\mathrm{d}t\right)^2 + \mathrm{d}z^2\right]\ ,$$

and the metric coefficients are then just read off from this expression:

$$\begin{aligned}
&g_{11} \equiv g_{rr} = 1\\
&g_{00} \equiv g_{tt} = \text{coefficient of } \mathrm{d}t^2 = -c^2 + r^2\omega^2\\
&g_{20} \equiv g_{02} \equiv g_{\phi t} \equiv g_{t\phi} = \frac{1}{2} \times \text{coefficient of } \mathrm{d}\phi\,\mathrm{d}t = r^2\omega^2\\
&g_{22} \equiv g_{\phi\phi} = \text{coefficient of } \mathrm{d}\phi^2 = r^2\\
&g_{33} \equiv g_{zz} = \text{coefficient of } \mathrm{d}z^2 = 1\ .
\end{aligned}$$

Note that the off-diagonal term $g_{\phi t}$ must be multiplied by 1/2, because the expression

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\,dx^\mu\,dx^\nu$$

includes the two equal terms $g_{20}\,\mathrm{d}\phi\,\mathrm{d}t + g_{02}\,\mathrm{d}t\,\mathrm{d}\phi$, where $g_{20} \equiv g_{02}$.

(b) Starting with the general expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ ,$$

we set $\mu = r$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{r\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_r g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ .$$

When we sum over ν on the left-hand side, the only value for which $g_{r\nu} \neq 0$ is $\nu = 1 \equiv r$. Thus, the left-hand side is simply

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}\ .$$

The RHS includes every combination of λ and σ for which $g_{\lambda\sigma}$ depends on r, so that $\partial_r\, g_{\lambda\sigma} \neq 0$. This means g_{tt}, $g_{\phi\phi}$, and $g_{\phi t}$. So,

$$\begin{aligned}
RHS &= \frac{1}{2}\partial_r(-c^2 + r^2\omega^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\partial_r(r^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + \partial_r(r^2\omega)\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\\
&= r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\\
&= r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\ .
\end{aligned}$$

Note that the final term in the first line is really the sum of the contributions from $g_{\phi t}$ and $g_{t\phi}$, where the two terms were combined to cancel the factor of 1/2 in the general expression. Finally,

$$\boxed{\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 .}$$

If one expands the RHS as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + r\omega^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2r\omega\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} ,$$

then one can identify the term proportional to ω^2 as the centrifugal force, and the term proportional to ω as the Coriolis force.

(c) Substituting $\mu = \phi$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\phi\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\phi g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} .$$

But none of the metric coefficients depend on ϕ, so the right-hand side is zero. The left-hand side receives contributions from $\nu = \phi$ and $\nu = t$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\phi\phi}\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + g_{\phi t}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0 ,$$

so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\omega\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = 0 .}$$

Note that one cannot "factor out" r^2, since r can depend on τ. If this equation is expanded to give an equation for $\mathrm{d}^2\phi/\mathrm{d}\tau^2$, the term proportional to ω would be identified as the Coriolis force. There is no term proportional to ω^2, since the centrifugal force has no component in the ϕ direction.

(d) If Eq. (1) of the problem is divided by $c^2\mathrm{d}t^2$, one obtains

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right] .$$

Then using

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)} ,$$

one has

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \omega\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]}} \ .}$$

Note that this equation is really just

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} \ ,$$

adapted to the rotating cylindrical coordinate system.

PROBLEM 13: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} \ .$$

(The above expression is proportional to $1/\sqrt{V + \Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1 + \frac{\Delta V}{V}}} = \frac{u_0}{1 + \frac{1}{2}\frac{\Delta V}{V}} = u_0\left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) \ .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0\left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0\left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) \ ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V}U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2}u_0 \ .}$$

PROBLEM 14: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

In general, the number density of a particle in the black-body radiation is given by

$$n = g^*\frac{\xi(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$$

For photons, one has $g^* = 2$. Then

$$\left.\begin{aligned} k &= 1.381 \times 10^{-16}\text{erg}/^\circ\text{K} \\ T &= 2.7\,^\circ\text{K} \\ \hbar &= 1.055 \times 10^{-27}\text{erg-sec} \\ c &= 2.998 \times 10^{10}\text{cm/sec} \end{aligned}\right\} \Longrightarrow \left(\frac{kT}{\hbar c}\right)^3 = 1.638 \times 10^3\text{cm}^{-3} \ .$$

Then using $\xi(3) \simeq 1.202$, one finds

$$\boxed{n_\gamma = 399/\text{cm}^3 \ .}$$

For the neutrinos,

$$g_\nu^* = 2 \times \frac{3}{4} = \frac{3}{2} \quad \text{per species.}$$

The factor of 2 is to account for ν and $\bar{\nu}$, and the factor of $3/4$ arises from the Pauli exclusion principle. So for three species of neutrinos one has

$$g_\nu^* = \frac{9}{2} \ .$$

Using the result

$$T_\nu^3 = \frac{4}{11}T_\gamma^3$$

from Problem 8 of Problem Set 3 (2000), one finds

$$\begin{aligned} n_\nu &= \left(\frac{g_\nu^*}{g_\gamma^*}\right)\left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma \\ &= \left(\frac{9}{4}\right)\left(\frac{4}{11}\right) 399\text{cm}^{-3} \\ \Longrightarrow &\quad \boxed{n_\nu = 326/\text{cm}^3 \text{ (for all three species combined).}} \end{aligned}$$

PROBLEM 15: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$\begin{aligned} E &= \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \boxed{\frac{\pi^4}{30\zeta(3)}kT\ .} \end{aligned}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\, kT\ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$\begin{aligned} S &= \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} \\ &= \boxed{\frac{2\pi^4}{45\zeta(3)}k\ .} \end{aligned}$$

Numerically, this gives $3.602\,k$, where k is the Boltzman constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be $\boxed{\text{no difference from the answers given in parts (a) and (b).}}$

(d) For a fermion, g is $7/8$ times the number of spin states, and g^* is $3/4$ times the number of spin states. So the average energy per particle is

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8}\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{180\zeta(3)}\,kT\ .}$$

Numerically, $E = 3.1514\,kT$.

Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \frac{\dfrac{7}{8}\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}}$$

$$= \boxed{\frac{7\pi^4}{135\zeta(3)}\,k\ .}$$

Numerically, this gives $S = 4.202\,k$.

PROBLEM 16: A NEW SPECIES OF LEPTON

a) The number density is given by the formula at the start of the exam,

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\ .$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$n = 3\frac{\zeta(3)}{\pi^2}\times\left(\frac{3\ \cancel{\text{MeV}}}{6.582\times10^{-16}\ \cancel{\text{eV-sec}}\times 2.998\times10^{10}\ \cancel{\text{cm-sec}}^{-1}}\right)^3$$

$$\times\left(\frac{10^6\ \cancel{\text{eV}}}{1\ \cancel{\text{MeV}}}\right)^3\times\left(\frac{10^2\ \cancel{\text{cm}}}{1\ \text{m}}\right)^3$$

$$= 3\frac{\zeta(3)}{\pi^2}\times\left(\frac{3\times10^6\times10^2}{6.582\times10^{-16}\times2.998\times10^{10}}\right)^3\ \text{m}^{-3}\ .$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2}\times\left(\frac{3\times10^6\times10^2}{6.582\times10^{-16}\times2.998\times10^{10}}\right)^3\ .}$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho \quad .$$

During the radiation-dominated era $R(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot{R}}{R} = \frac{1}{2t} \quad .$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3} G\rho \quad .$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \quad .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}$ cm^3 sec^{-2} g^{-1}, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732 \times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9$ g/cm^3.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \quad .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion – anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\rm tot} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2}\frac{1}{g_{\rm tot}}\hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/R$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\rm rest} T^3(t) R^3(t)$$

where $g_{\rm rest} = g_{\rm tot} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 R^3(t) = g_{\rm rest} T^3(t) R^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\rm rest}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 17: DID YOU DO THE READING?

(a) This is a total trick question. Lepton number is, of course, conserved, so the factor is just 1. See Weinberg chapter 4, pages 91-4.

(b) The correct answer is (i). The others are all real reasons why it's hard to measure, although Weinberg's book emphasizes reason (v) a bit more than modern astrophysicists do: astrophysicists have been looking for other ways that deuterium might be produced, but no significant mechanism has been found. See Weinberg chapter 5, pages 114-7.

(c) The most obvious answers would be proton, neutron, and pi meson. However, there are many other possibilities, including many that were not mentioned by Weinberg. See Weinberg chapter 7, pages 136-8.

(d) The correct answers were the $\boxed{\text{neutrino}}$ and the $\boxed{\text{antiproton.}}$ The neutrino was first hypothesized by Wolfgang Pauli in 1932 (in order to explain the kinematics of beta decay), and first detected in the 1950s. After the positron was discovered in 1932, the antiproton was thought likely to exist, and the Bevatron in Berkeley was built to look for antiprotons. It made the first detection in the 1950s.

(e) The correct answers were $\boxed{\text{(ii), (v) and (vi).}}$ The others were incorrect for the following reasons:

(i) the earliest prediction of the CMB temperature, by Alpher and Herman in 1948, was 5 degrees, not 0.1 degrees.

(iii) Weinberg quotes his experimental colleagues as saying that the $3°$ K radiation could have been observed "long before 1965, probably in the mid-1950s and perhaps even in the mid-1940s." To Weinberg, however, the historically interesting question is not when the radiation could have been observed, but why radio astronomers did not know that they ought to try.

(iv) Weinberg argues that physicists at the time did not pay attention to either the steady state model or the big bang model, as indicated by the sentence in item (v) which is a direct quote from the book: "It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe".