MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 2, 2007
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 3

QUIZ DATE: Thursday, December 6, 2007, during the normal class time.

COVERAGE: Problem Sets 7, 8, and 9; Lecture Notes 8, 10, and 13 (where Lecture Notes 10 and 13 can be found on the 2005 Lecture Notes page); Ryden, *Introduction to Cosmology*, Chapters 8 and 9, excluding Section 9.3. The quiz will include big bang nucleosynthesis, at the level discussed in lecture. You may want to review this topic in Weinberg's *The First Three Minutes*, and/or Ryden's Chapter 10, but the quiz will not include any questions specifically aimed at testing this reading. Similarly, the quiz will include inflation at the level discussed in lecture, through Tuesday, December 4. You may want to review this topic in my article, "Inflation and the New Era of High-Precision Cosmology," at

http://web.mit.edu/physics/alumniandfriends/physicsjournal_fall_02_cosmology.pdf.

and/or in Ryden's Chapter 11, but the quiz will also not include any questions specifically aimed at testing this reading.

One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

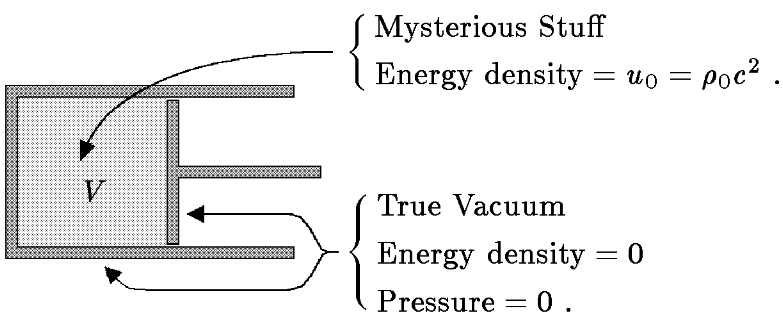
REVIEW SESSION AND OFFICE HOURS: To help you study for the quiz, Barton Zwiebach will hold a review session on Wednesday, December 5, at 7:00 pm. Yi Mao will have office hours on Tuesday, December 4, from 1:00 – 2:00 pm (not quite the usual time), and also on Wednesday, December 5, from 5:00 - 6:00 pm. Barton Zwiebach will have office hours on Tuesday evening, December 4, at 7:00 pm. The rooms for all of these office hours and the review session will be announced by email and on the website.

INFORMATION TO BE GIVEN ON QUIZ: There will be a formula sheet, which will include all the formulas from the Quiz 2 formula sheet, plus a few more. It will be posted in advance of the quiz, but possibly not until late Wednesday afternoon.

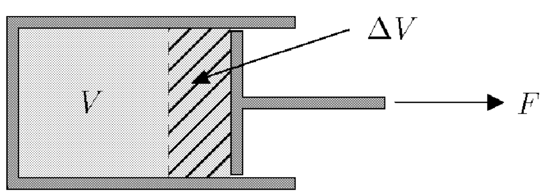
∗ PROBLEM 1: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

In Lecture Notes 13, a thought experiment involving a piston was used to show that $p = -\rho c^2$ for any substance for which the energy density remains constant under expansion. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

∗ PROBLEM 2: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

Consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and the same ***mysterious stuff*** that was introduced in the previous problem. Since the mass density of mysterious stuff falls off as $1/\sqrt{V}$, where V is the volume, it follows that in an expanding universe the mass density of mysterious stuff falls off as $1/R^{3/2}(t)$.

Suppose that you are given the present value of the Hubble parameter H_0, and also the present values of the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\mathrm{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe t_0 in terms of these quantities.

(a) *(8 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{R(t)}{R(t_0)}$$

for an arbitrary time t. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(b) *(7 points)* Write an integral expression for the age of the universe t_0. The expression should depend only on H_0 and the various contributions to Ω_0 listed above ($\Omega_{m,0}$, $\Omega_{r,0}$, etc.), but it might include x as a variable of integration.

Extra Credit for Super-Sharpies (no partial credit): For 5 points extra credit, you can calculate the angular diameter $\Delta\theta$ of the image of a spherical object at redshift z which had a physical diameter w at the time of emission. You should assume that $\Omega_{\mathrm{tot}} < 1$, and also that $\Delta\theta \ll 1$. The calculation is to be carried out for the same model universe described above.

PROBLEM 3: TIME SCALES IN COSMOLOGY

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-37} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the beginning of the processes involved in big bang nucleosynthesis;

(b) the end of the processes involved in big bang nucleosynthesis;

(c) the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{16}$ GeV;

(d) "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

∗ PROBLEM 4: EVOLUTION OF FLATNESS *(15 points)*

The following problem was Problem 3, Quiz 3, 2004.

The "flatness problem" is related to the fact that during the evolution of the standard cosmological model, Ω is always driven away from 1.

(a) *(9 points)* During a period in which the universe is matter-dominated (meaning that the only relevant component is nonrelativistic matter), the quantity

$$\frac{\Omega - 1}{\Omega}$$

grows as a power of t. Show that this is true, and derive the power. (Stating the right power without a derivation will be worth 3 points.)

(b) *(6 points)* During a period in which the universe is radiation-dominated, the same quantity will grow like a different power of t. Show that this is true, and derive the power. (Stating the right power without a derivation will again be worth 3 points.)

In each part, you may assume that the universe was *always* dominated by the specified form of matter.

∗ PROBLEM 5: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

The following problem was Problem 4, Quiz 3, 2004.

On April 13, 2000, the Sloan Digital Sky Survey announced the discovery of what was then the most distant object known in the universe: a quasar at $z = 5.82$. To explain to the public how this object fits into the universe, the SDSS posted on their website an article by Michael Turner and Craig Wiegert titled "How Can An Object We See Today be 27 Billion Light Years Away If the Universe is only 14 Billion Years Old?" Using a model with $H_0 = 65$ km-s^{-1}-Mpc^{-1}, $\Omega_m = 0.35$, and $\Omega_\Lambda = 0.65$, they claimed

(a) that the age of the universe is 13.9 billion years.

(b) that the light that we now see was emitted when the universe was 0.95 billion years old.

(c) that the distance to the quasar, as it would be measured by a ruler today, is 27 billion light-years.

(d) that the distance to the quasar, at the time the light was emitted, was 4.0 billion light-years.

(e) that the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing, is 1.8 times the velocity of light.

The goal of this problem is to check all of these conclusions, although you are of course not expected to actually work out the numbers. Your answers can be expressed in terms of H_0, Ω_m, Ω_Λ, and z. Definite integrals need not be evaluated.

Note that Ω_m represents the present density of nonrelativistic matter, expressed as a fraction of the critical density; and Ω_Λ represents the present density of vacuum energy, expressed as a fraction of the critical density. In answering each of the following questions, you may consider the answer to any previous part — whether you answered it or not — as a given piece of information, which can be used in your answer.

(a) *(15 points)* Write an expression for the age t_0 of this model universe?

(b) *(5 points)* Write an expression for the time t_e at which the light which we now receive from the distant quasar was emitted.

(c) *(10 points)* Write an expression for the present physical distance $\ell_{\rm phys,0}$ to the quasar.

(d) *(5 points)* Write an expression for the physical distance $\ell_{\rm phys,e}$ between us and the quasar at the time that the light was emitted.

(e) *(5 points)* Write an expression for the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing.

∗ PROBLEM 6: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS *(20 points)*

The following problem was on Quiz 4, 2000, except that part (c) has been modified. For 2004 a problem about nucleosynthesis, like this one, would be considered a difficult problem, since the topic was covered only in your readings of Weinberg. If I were to use a question like this on the coming quiz, I would probably try to make it easier by adding some hints.

(a) *(5 points)* When the temperature of the early universe was 5×10^{10} K, what was the ratio of neutrons to protons? You may assume thermal equilibrium, and that the mass difference is given by $(m_n - m_p)c^2 = 1.293$ MeV.

Questions (b), (c), and (d) all refer to calculations that describe a hypothetical world, which differs from the real world in a specified way. In each case you are asked about the calculation of the predicted helium abundance for this hypothetical world. Each of these three parts are to be answered independently; that is, in each part you are to consider a hypothetical world that differs from the real world only by the characteristics stated in that part.

(b) *(5 points)* Suppose the proton-neutron mass difference were larger than the actual value of 1.293 MeV/c^2. Would the predicted helium abundance be larger or smaller than in the standard calculation? Explain your answer in a sentence, or in a few sentences.

(c) *(5 points)* Suppose that the nucleosynthesis calculations were carried out with an electron mass given by $m_e c^2 = 1$ KeV, instead of the physical value of 0.511 MeV. This change would affect the production of helium in several ways. Describe one way in which the helium production process would be affected, and explain in a few sentences whether this change would increase or decrease the predicted helium abundance.

(d) *(5 points)* Suppose, due to some significant difference in the nuclear reaction rates, that nucleosynthesis occurred suddenly at a temperature of 5×10^{10} K. In that case, what would be the predicted value of Y, the fraction of the baryonic mass density of the universe which is helium?

∗ PROBLEM 7: NEUTRINO NUMBER AND THE NEUTRON/PROTON EQUILIBRIUM *(35 points)*

The following problem was 1998 Quiz 4, Problem 4.

In the standard treatment of big bang nucleosynthesis it is assumed that at early times the ratio of neutrons to protons is given by the Boltzmann formula,

$$\frac{n_n}{n_p} = e^{-\Delta E/kT} \; , \tag{1}$$

where k is Boltzmann's constant, T is the temperature, and $\Delta E = 1.29$ MeV is the proton-neutron mass-energy difference. This formula is believed to be very accurate, but it assumes that the chemical potential for neutrons μ_n is the same as the chemical potential for protons μ_p.

(a) *(10 points)* Give the correct version of Eq. (1), allowing for the possibility that $\mu_n \neq \mu_p$.

The equilibrium between protons and neutrons in the early universe is sustained mainly by the following reactions:

$$\begin{aligned} e^+ + n &\longleftrightarrow p + \bar{\nu}_e \\ \nu_e + n &\longleftrightarrow p + e^- \; . \end{aligned}$$

Let μ_e and μ_ν denote the chemical potentials for the electrons (e^-) and the electron neutrinos (ν_e) respectively. The chemical potentials for the positrons (e^+) and the anti-electron neutrinos ($\bar{\nu}_e$) are then $-\mu_e$ and $-\mu_\nu$, respectively, since the chemical potential of a particle is always the negative of the chemical potential for the antiparticle.*

(b) *(10 points)* Express the neutron/proton chemical potential difference $\mu_n - \mu_p$ in terms of μ_e and μ_ν.

The black-body radiation formulas at the beginning of the quiz did not allow for the possibility of a chemical potential, but they can easily be generalized. For example, the formula for the number density n_i (of particles of type i) becomes

$$n_i = g_i^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{\mu_i/kT} \; .$$

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* This fact is a consequence of the principle that the chemical potential of a particle is the sum of the chemical potentials associated with its conserved quantities, while particle and antiparticle always have the opposite values of all conserved quantities.

(c) *(10 points)* Suppose that the density of anti-electron neutrinos $\bar{n}_\nu$ in the early universe was higher than the density of electron neutrinos n_ν. Express the thermal equilibrium value of the ratio n_n/n_p in terms of ΔE, T, and the antineutrino excess $\Delta n = \bar{n}_\nu - n_\nu$. (Your answer may also contain fundamental constants, such as k, $\hbar$, and c.)

(d) *(5 points)* Would an excess of anti-electron neutrinos, as considered in part (c), increase or decrease the amount of helium that would be produced in the early universe? Explain your answer.

SOLUTIONS

PROBLEM 1: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} \ .$$

(The above expression is proportional to $1/\sqrt{V + \Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1 + \frac{\Delta V}{V}}} = \frac{u_0}{1 + \frac{1}{2}\frac{\Delta V}{V}} = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) \ .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) \ ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2}u_0 \ .}$$

PROBLEM 2: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF

(a) The critical density ρ_c is defined as that density for which $k = 0$, where the Friedmann equation from the front of the exam implies that

$$H^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} .$$

Thus the critical density today is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} .$$

The mass density today of any species X is then related to $\Omega_{X,0}$ by

$$\rho_{X,0} = \rho_c \Omega_{X,0} = \frac{3H_0^2 \Omega_{X,0}}{8\pi G} .$$

The total mass density today is then expressed in terms of its four components as

$$\rho_0 = \frac{3H_0^2}{8\pi G} \left[\Omega_{m,0} + \Omega_{r,0} + \Omega_{v,0} + \Omega_{\text{ms},0} \right] .$$

But we also know how each of these contributions to the mass density scales with $x(t)$: $\rho_m \propto 1/x^3$, $\rho_r \propto 1/x^4$, $\rho_v \propto 1$, and $\rho_{\text{ms}} \propto 1/\sqrt{V} \propto 1/x^{3/2}$. Inserting these factors,

$$\rho(t) = \frac{3H_0^2}{8\pi G} \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} \right] .$$

(b) The Friedmann equation then becomes

$$\left(\frac{\dot{x}}{x} \right)^2 = \frac{8\pi G}{3} \frac{3H_0^2}{8\pi G} \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} \right] - \frac{kc^2}{R^2} .$$

Defining

$$H_0^2 \Omega_{k,0} = -\frac{kc^2}{R^2(t_0)} ,$$

so

$$-\frac{kc^2}{R^2(t)} = -\frac{kc^2}{R^2(t_0)} \frac{1}{x^2} = \frac{H_0^2 \Omega_{k,0}}{x^2} ,$$

and then the Friedmann equation becomes

$$\left(\frac{\dot{x}}{x} \right)^2 = H_0^2 \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2} \right] .$$

Applying this equation today, when $\dot{x}/x = H_0$, one finds that

$$\boxed{\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{v,0} - \Omega_{\mathrm{ms},0}\ .}$$

Rearranging the equation for $(\dot{x}/x)^2$ above,

$$H_0\,\mathrm{d}t = \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}}\ .$$

The age of the universe is found by integrating over the full range of x, which starts from 0 when the universe is born, and is equal to 1 today. So

$$\boxed{t_0 = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}}\ .}$$

Extra Credit for Super-Sharpies (no partial credit):

Since $\Omega_{\mathrm{tot}} < 1$, we use the Robertson-Walker open universe form

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + R^2(t)\left\{\frac{\mathrm{d}r^2}{1+r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\}\ ,$$

where I have started with the general form from the front of the exam, and replaced k by -1. To discuss the radial transmission of light rays it is useful to define a new radial coordinate

$$r = \sinh\psi\ ,$$

so

$$\mathrm{d}r = \cosh\psi\,\mathrm{d}\psi = \sqrt{1+r^2}\,\mathrm{d}\psi\ ,$$

where I used the identity that $\cosh^2\psi = 1 + \sinh^2\psi$. The metric can then be rewritten as

$$\mathrm{d}s^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\}\ .$$

Light rays then travel with $\mathrm{d}\tau^2 = 0$, so

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{c}{R(t)}\ .$$

If a light ray leaves the object at time t_e and arrives at Earth today, then it will travel an interval of ψ given by

$$\psi = \int_{t_e}^{t_0} \frac{c}{R(t')} \, \mathrm{d}t' \; .$$

We will need to know ψ, but we don't know either t_e or $R(t)$. So we need to manipulate the right-hand side of the above equation to express it in terms of things that we do know. Changing integration variables from t' to x, where $x = R(t')/R(t_0)$, one finds $\mathrm{d}x = \dot{x}\,\mathrm{d}t'$, and then

$$\psi = \int_{x_e}^{1} \frac{c}{R(t_0)} \frac{1}{x} \frac{\mathrm{d}x}{\dot{x}} \; .$$

Using $H = \dot{x}/x$,

$$\psi = \frac{c}{R(t_0)} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2 H} \; .$$

Now use the formula for $H = \dot{x}/x$ from part (b), so

$$\psi = \frac{c}{R(t_0) H_0} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \; .$$

Here

$$\boxed{x_e = \frac{R(t_e)}{R(t_0)} = \frac{1}{1+z} \; ,}$$

and the coefficient in front of the integral can be evaluated using the Friedman equation for $k = -1$:

$$H_0^2 = \frac{8\pi}{3} G\rho_0 + \frac{c^2}{R^2(t_0)} = H_0^2 \Omega_0 + \frac{c^2}{R^2(t_0)} \; ,$$

so

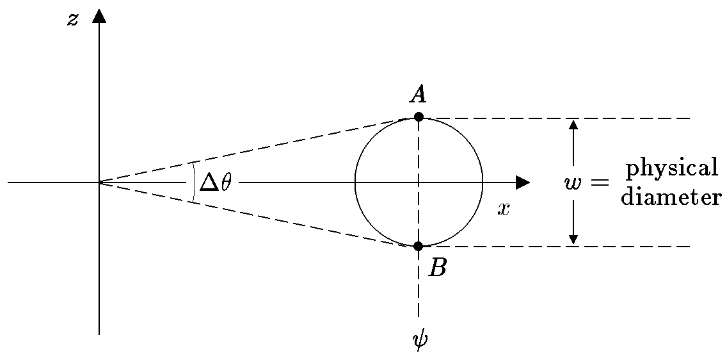
$$\frac{c^2}{R^2(t_0)\, H_0^2} = 1 - \Omega_0 = \Omega_{k,0} \; .$$

Finally, then, the expression for ψ can be written

$$\boxed{\psi = \sqrt{\Omega_{k,0}} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \; ,}$$

where x_e is given by the boxed equation above.

Once we know ψ, the rest is straightforward. We draw a picture in comoving coordinates of the light rays leaving the object and arriving at Earth:



In this picture $\Delta\theta$ is the angular size that would be measured. Using the $\mathrm{d}\theta^2$ part of the metric,

$$\mathrm{d}s^2 = R^2(t)\sinh^2\psi\,\mathrm{d}\theta^2\ ,$$

we can relate w, the physical size of the object at the time of emission, to $\Delta\theta$:

$$w = R(t_e)\sinh\psi\,\Delta\theta\ .$$

To evaluate $R(t_e)$ we can use

$$R(t_e) = x_e R(t_0) = \frac{x_e c}{H_0\sqrt{\Omega_{k,0}}}\ .$$

Finally, then,

$$\boxed{\Delta\theta = \frac{wH_0\sqrt{\Omega_{k,0}}}{x_e c\,\sinh\psi}\ ,}$$

where ψ is given by the boxed equation above.

PROBLEM 3: TIME SCALES IN COSMOLOGY

(a) 1 sec. *[This is the time at which the weak interactions begin to "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) 10^{-37} sec. *[We learned in Lecture Notes 7 that kT was about 1 MeV at $t = 1$ sec. Since 1 GeV = 1000 MeV, the value of kT that we want is 10^{19} times higher. In the radiation-dominated era $T \propto R^{-1} \propto t^{-1/2}$, so we get 10^{-38} sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) 10^{-5} sec. *[As in (c), we can use $t \propto T^{-2}$, with $kT \approx 1$ MeV at $t = 1$ sec.]*

PROBLEM 4: EVOLUTION OF FLATNESS *(15 points)*

(a) We start with the Friedmann equation from the formula sheet on the quiz:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \ .$$

The critical density is the value of ρ corresponding to $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c \ .$$

Using this expression to replace H^2 on the left-hand side of the Friedmann equation, and then dividing by $8\pi G/3$, one finds

$$\rho_c = \rho - \frac{3kc^2}{8\pi G R^2} \ .$$

Rearranging,

$$\frac{\rho - \rho_c}{\rho} = \frac{3kc^2}{8\pi G R^2 \rho} \ .$$

On the left-hand side we can divide the numerator and denominator by ρ_c, and then use the definition $\Omega \equiv \rho/\rho_c$ to obtain

$$\frac{\Omega - 1}{\Omega} = \frac{3kc^2}{8\pi G R^2 \rho} \ . \qquad (1)$$

For a matter-dominated universe we know that $\rho \propto 1/R^3(t)$, and so

$$\frac{\Omega - 1}{\Omega} \propto R(t) \ .$$

If the universe is nearly flat we know that $R(t) \propto t^{2/3}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t^{2/3} \ .}$$

(b) Eq. (1) above is still true, so our only task is to re-evaluate the right-hand side. For a radiation-dominated universe we know that $\rho \propto 1/R^4(t)$, so

$$\frac{\Omega - 1}{\Omega} \propto R^2(t) \; .$$

If the universe is nearly flat then $R(t) \propto t^{1/2}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t \; .}$$

PROBLEM 5: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

(a) Since $\Omega_m + \Omega_\Lambda = 0.35 + 0.65 = 1$, the universe is flat. It therefore obeys a simple form of the Friedmann equation,

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G(\rho_m + \rho_\Lambda) \; ,$$

where the overdot indicates a derivative with respect to t, and the term proportional to k has been dropped. Using the fact that $\rho_m \propto 1/R^3(t)$ and $\rho_\Lambda =$ const, the energy densities on the right-hand side can be expressed in terms of their present values $\rho_{m,0}$ and $\rho_\Lambda \equiv \rho_{\Lambda,0}$. Defining

$$x(t) \equiv \frac{R(t)}{R(t_0)} \; ,$$

one has

$$\begin{aligned} \left(\frac{\dot{x}}{x}\right)^2 &= \frac{8\pi}{3} G \left(\frac{\rho_{m,0}}{x^3} + \rho_\Lambda\right) \\ &= \frac{8\pi}{3} G \rho_{c,0} \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \\ &= H_0^2 \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \; . \end{aligned}$$

Here we used the facts that

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}}; \qquad \Omega_{\Lambda,0} \equiv \frac{\rho_\Lambda}{\rho_{c,0}} \; ,$$

and

$$H_0^2 = \frac{8\pi}{3} G\rho_{c,0} \ .$$

The equation above for $(\dot{x}/x)^2$ implies that

$$\dot{x} = H_0 \, x \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}} \ ,$$

which in turn implies that

$$\mathrm{d}t = \frac{1}{H_0} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .$$

Using the fact that x changes from 0 to 1 over the life of the universe, this relation can be integrated to give

$$t_0 = \int_0^{t_0} \mathrm{d}t = \boxed{\frac{1}{H_0} \int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .}$$

The answer can also be written as

$$\boxed{t_0 = \frac{1}{H_0} \int_0^1 \frac{x \, \mathrm{d}x}{\sqrt{\Omega_{m,0} x + \Omega_{\Lambda,0} x^4}}}$$

or

$$\boxed{t_0 = \frac{1}{H_0} \int_0^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}} \ ,}$$

where in the last answer I changed the variable of integration using

$$x = \frac{1}{1+z} \ ; \qquad \mathrm{d}x = -\frac{\mathrm{d}z}{(1+z)^2} \ .$$

Note that the minus sign in the expression for $\mathrm{d}x$ is canceled by the interchange of the limits of integration: $x = 0$ corresponds to $z = \infty$, and $x = 1$ corresponds to $z = 0$.

Your answer should look like one of the above boxed answers. You were not expected to complete the numerical calculation, but for pedagogical purposes I will continue. The integral can actually be carried out analytically, giving

$$\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{\Omega_{m,0}x + \Omega_{\Lambda,0}x^4}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}} + \sqrt{\Omega_{\Lambda,0}}}{\sqrt{\Omega_m}}\right) .$$

Using

$$\frac{1}{H_0} = \frac{9.778 \times 10^9}{h_0}\ \text{yr} ,$$

where $H_0 = 100\,h_0\,\text{km-sec}^{-1}\text{-Mpc}^{-1}$, one finds for $h_0 = 0.65$ that

$$\frac{1}{H_0} = 15.043 \times 10^9\ \text{yr} .$$

Then using $\Omega_m = 0.35$ and $\Omega_{\Lambda,0} = 0.65$, one finds

$$\boxed{t_0 = 13.88 \times 10^9\ \text{yr} .}$$

So the SDSS people were right on target.

(b) Having done part (a), this part is very easy. The dynamics of the universe is of course the same, and the question is only slightly different. In part (a) we found the amount of time that it took for x to change from 0 to 1. The light from the quasar that we now receive was emitted when

$$x = \frac{1}{1+z} ,$$

since the cosmological redshift is given by

$$1 + z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} .$$

Using the expression for $\mathrm{d}t$ from part (a), the amount of time that it took the universe to expand from $x = 0$ to $x = 1/(1+z)$ is given by

$$t_e = \int_0^{t_e} \mathrm{d}t = \boxed{\frac{1}{H_0}\int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Again one could write the answer other ways, including

$$\boxed{t_0 = \frac{1}{H_0}\int_z^\infty \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3+\Omega_{\Lambda,0}}}\ .}$$

Again you were expected to stop with an expression like the one above. Continuing, however, the integral can again be done analytically:

$$\int_0^{x_{\max}} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3}+\Omega_{\Lambda,0}}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}}\ln\left(\frac{\sqrt{\Omega_m+\Omega_{\Lambda,0}x_{\max}^3}+\sqrt{\Omega_{\Lambda,0}}\,x_{\max}^{3/2}}{\sqrt{\Omega_m}}\right)\ .$$

Using $x_{\max} = 1/(1+5.82) = .1466$ and the other values as before, one finds

$$\boxed{t_e = \frac{0.06321}{H_0} = 0.9509\times 10^9\ \mathrm{yr}\ .}$$

So again the SDSS people were right.

(c) To find the physical distance to the quasar, we need to figure out how far light can travel from $z = 5.82$ to the present. Since we want the present distance, we multiply the coordinate distance by $R(t_0)$. For the flat metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + R^2(t)\left\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\right\}\ ,$$

the coordinate velocity of light (in the radial direction) is found by setting $\mathrm{d}s^2 = 0$, giving

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{R(t)}\ .$$

So the total coordinate distance that light can travel from t_e to t_0 is

$$\ell_c = \int_{t_e}^{t_0}\frac{c}{R(t)}\,\mathrm{d}t\ .$$

This is not the final answer, however, because we don't explicitly know $R(t)$. We can, however, change variables of integration from t to x, using

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \frac{\mathrm{d}x}{\dot{x}}\ .$$

So

$$\ell_c = \frac{c}{R(t_0)} \int_{x_e}^{1} \frac{\mathrm{d}x}{x\,\dot{x}} \ ,$$

where x_e is the value of x at the time of emission, so $x_e = 1/(1+z)$. Using the equation for $\dot{x}$ from part (a), this integral can be rewritten as

$$\ell_c = \frac{c}{H_0 R(t_0)} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .$$

Finally, then

$$\boxed{\ell_{\rm phys,0} = R(t_0)\,\ell_c = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .}$$

Alternatively, this result can be written as

$$\boxed{\ell_{\rm phys,0} = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}\,x + \Omega_{\Lambda,0}\,x^4}} \ ,}$$

or by changing variables of integration to obtain

$$\boxed{\ell_{\rm phys,0} = \frac{c}{H_0} \int_{0}^{z} \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}} \ .}$$

Continuing for pedagogical purposes, this time the integral has no analytic form, so far as I know. Integrating numerically,

$$\int_0^{5.82} \frac{\mathrm{d}z'}{\sqrt{0.35\,(1+z')^3 + 0.65}} = 1.8099 \ ,$$

and then using the value of $1/H_0$ from part (a),

$$\boxed{\ell_{\rm phys,0} = 27.23 \text{ light-yr} \ .}$$

Right again.

(d) $\ell_{\text{phys},e} = R(t_e)\ell_c$, so

$$\ell_{\text{phys},e} = \frac{R(t_e)}{R(t_0)}\,\ell_{\text{phys},0} = \boxed{\frac{\ell_{\text{phys},0}}{1+z}\ .}$$

Numerically this gives

$$\boxed{\ell_{\text{phys},e} = 3.992 \times 10^9 \text{ light-yr}\ .}$$

The SDSS announcement is still okay.

(e) The speed defined in this way obeys the Hubble law exactly, so

$$v = H_0\,\ell_{\text{phys},0} = c\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .$$

Then

$$\boxed{\frac{v}{c} = \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}}\ .}$$

Numerically, we have already found that this integral has the value

$$\boxed{\frac{v}{c} = 1.8099\ .}$$

The SDSS people get an A.

PROBLEM 6: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS

(a) In thermal equilibrium, the ratio of neutrons to protons is given by a Boltzmann factor,

$$\boxed{\frac{n_n}{n_p} = e^{-\Delta m\, c^2/kT}\ ,}$$

where $\Delta m = (m_n - m_p)$. For $\Delta m\, c^2 = 1.293 \times 10^6$ eV, $k = 8.617 \times 10^{-5}$ eV/K, and $T = 5 \times 10^{10}$ K, this gives

$$\frac{n_n}{n_p} = \exp\left\{-1.293 \times 10^6/(8.617 \times 10^{-5} \times 5 \times 10^{10})\right\} = \boxed{\;0.741\;.\;}$$

Caveat (for stat mech experts): The above formula would be a precise consequence of statistical mechanics if the neutron and proton were two possible energy levels of the same system. In this case one would describe the system using the *canonical ensemble*, which implies that the probability of the system existing in any specific state i is proportional to $\exp(-E_i/kT)$, where E_i is the energy of the state. However, the neutron and proton are not really different energy levels of the same system, because the conversion between neutrons and protons involves other particles as well; a sample conversion reaction would be

$$n + \nu_e \longleftrightarrow p + e^- \;,$$

where ν_e is the electron neutrino, and e^- is the electron. This means that if the universe contained a very large density of electron neutrinos, then n-ν_e collisions would occur more frequently, and the reaction would be driven in the forward direction. Thus, a large density of electron neutrinos would lead to a lower ratio of neutrons to protons than the Boltzmann factor given above. Similarly, if the universe contained a large density of electrons, then the reaction would be driven in the reverse direction, and the ratio of neutrons to protons would be higher than the Boltzmann factor. A complete statistical mechanical treatment of this situation would use the *grand canonical ensemble*, which describes systems in which the number of particles of a given type can change by chemical reactions. In this formalism the density of each type of particle is related to a quantity called the *chemical potential* μ, where in general the relationship is given by

$$n = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp\left[(E-\mu)/(kT)\right] \pm 1} E\,\mathrm{d}E$$

where the $+$ sign holds for Fermi particles, the $-$ sign holds for Bose particles, and the factor g has the same meaning as in Lecture Notes 7. The ratio of neutrons to protons is then given by

$$\frac{n_n}{n_p} = e^{-(\Delta m\, c^2 + \mu_\nu - \mu_e)/kT} \;,$$

where μ_ν and μ_e represent the chemical potentials for electron neutrinos and electrons, respectively. In the early universe, however, the standard theories imply that the chemical potentials for electrons and neutrinos were both negligible.

(b) A larger Δm would mean that the Boltzmann factor described in the previous answer would be smaller, so that there would be fewer neutrons at any given temperature. Fewer neutrons implies less helium, since essentially all the neutrons that exist when the temperature falls enough for deuterium to become stable become bound into helium.

(c) There are at least four effects that occur when the electron mass/energy is taken as 1 KeV instead of 0.511 MeV:

(i) For the real mass/energy of 0.511 MeV the electron-positron pairs freeze out before nucleosynthesis, but a mass/energy of 1 KeV would mean that electron-positron pairs would behave as massless particles throughout the nucleosynthesis process. Just like adding an extra species of neutrino, this additional massless particle would mean that the expansion rate would be larger, since for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho \ ,$$

and

$$\rho = \frac{u}{c^2} = g\frac{\pi^2}{30}\frac{(kT)^4}{\hbar^3 c^5} \ .$$

Faster expansion means that the weak interactions "freeze out" earlier, since the freeze-out point is the time at which the interactions can no longer maintain equilibrium as the universe expands. An earlier freeze-out means a higher temperature of freeze-out and hence more neutrons at the time of freeze-out. In addition, the faster expansion rate means faster cooling, which means less time before the temperature of nucleosynthesis is reached, and therefore less time for neutrons to decay. Thus, faster expansion means more neutrons. Since essentially all the neutrons present when the deuterium bottleneck breaks are collected into helium, this implies more helium.

(ii) The most important reactions that keep protons and neutrons in thermal equilibrium all involve electrons and positrons:

$$n + e^+ \longleftrightarrow p + \bar{\nu}_e$$

$$n + \nu_e \longleftrightarrow p + e^- \ .$$

If the electron-positron mass/energy were smaller, then the rates of all of these reactions would be enhanced. The reactions in which an e^+ or e^- appears in the initial state will be enhanced by the presence of more e^+'s and e^-'s, and the reactions in which they appear in the final state will be enhanced because a lighter final state is easier to produce. The enhanced rate for these reactions will keep neutrons and protons in thermal equilibrium longer, and hence to lower temperatures, and this would decrease the final abundance of neutrons. Thus this effect will go in the opposite direction as effect (i), leading to the production of less helium.

(iii) If the electron mass is decreased, then the neutron decay

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

becomes more exothermic, so it will happen more quickly. Thus more neutrons can decay, leading to less helium.

(iv) As mentioned in (i), lowering the mass/energy of electron-positron pairs to 1 KeV would mean that their freeze-out would not occur until after nucleosynthesis is over. In the real case, however, with $m_e c^2 = 0.511$ MeV, the electron-positron pairs start to freeze out at $t \approx 10$ sec. The energy released by this freeze-out heats the photons, protons, and neutrons, and this extra heat delays the time when the universe cools enough to break the deuterium bottleneck so that helium production can proceed. The delay allows more time for the neutrons to decay, resulting in less helium. Since the freeze-out that occurs for $m_e c^2 = 0.511$ MeV results in less helium, the absence of this freeze-out if $m_e c^2 = 1$ KeV would result in more helium.

Since the effects point in different directions, there is no easy way to know what the net effect will be. I (AHG) tried carrying out a full numerical integration, using the equations from P.J.E. Peebles, "Primordial helium abundance and the primordial fireball II," *Astrophysical Journal* **146**, 542-552 (1966). I found that the net effect of changing $m_e c^2$ to 1 KeV was to produce less helium. Apparently effects (ii) and (iii) above are the most significant. Of course I did not expect students to figure this out in doing their problem sets.

(d) Part (a) asked for the ratio of neutrons to protons, so its answer is

$$A = \frac{n_{\text{neutron}}}{n_{\text{proton}}} .$$

The fraction of the baryonic mass in neutrons is then

$$\frac{n_{\text{neutron}}}{n_B} = \frac{n_{\text{neutron}}}{n_{\text{neutron}} + n_{\text{proton}}} = \frac{\frac{n_{\text{neutron}}}{n_{\text{proton}}}}{\frac{n_{\text{neutron}}}{n_{\text{proton}}} + 1} = \frac{A}{1+A} .$$

The fraction of the baryonic mass in helium is twice this number, since after nucleosynthesis essentially all neutrons are in helium, and the mass of each helium nucleus is twice the mass of the neutrons within it. Thus

$$Y = \frac{2A}{1+A} .$$

This gives $\boxed{Y = 0.851.}$

PROBLEM 7: NEUTRINO NUMBER AND THE NEUTRON/PROTON EQUILIBRIUM

(a) From the chemical equilibrium equation on the front of the exam, the number densities of neutrons and protons can be written as

$$n_n = g_n \frac{(2\pi m_n kT)^{3/2}}{(2\pi\hbar)^3}\, e^{(\mu_n - m_n c^2)/kT}$$

$$n_p = g_p \frac{(2\pi m_p kT)^{3/2}}{(2\pi\hbar)^3}\, e^{(\mu_p - m_p c^2)/kT} \ ,$$

where $g_n = g_p = 2$. Dividing,

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(\Delta E + \mu_p - \mu_n)/kT} \ ,$$

where $\Delta E = (m_n - m_p)c^2$ is the proton-neutron mass-energy difference. Approximating $m_n/m_p \approx 1$, one has

$$\boxed{\frac{n_n}{n_p} = e^{-(\Delta E + \mu_p - \mu_n)/kT} \ .}$$

The approximation $m_n/m_p \approx 1$ is very accurate (0.14%), but is clearly not necessary. Full credit was given whether or not this approximation was used.

(b) For any allowed chemical reaction, the sum of the chemical potentials on the two sides must be equal. So, from

$$e^+ + n \longleftrightarrow p + \bar{\nu}_e \ ,$$

we can infer that

$$-\mu_e + \mu_n = \mu_p - \mu_\nu \ ,$$

which implies that

$$\boxed{\mu_n - \mu_p = \mu_e - \mu_\nu \ .}$$

(c) Applying the formula given in the problem to the number densities of electron neutrinos and the corresponding antineutrinos,

$$n_\nu = g_\nu^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{\mu_\nu/kT}$$

$$\bar{n}_\nu = g_\nu^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{-\mu_\nu/kT} \ ,$$

since the chemical potential for the antineutrinos ($\bar{\nu}$) is the negative of the chemical potential for neutrinos. A neutrino has only one spin state, so $g_\nu = 3/4$, where the factor of 3/4 arises because neutrinos are fermions. Setting

$$x \equiv e^{-\mu_\nu/kT}$$

and

$$A \equiv \frac{3}{4}\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} ,$$

the number density equations can be written compactly as

$$n_\nu = \frac{A}{x} , \qquad \bar{n}_\nu = xA .$$

The quantity x can then be determined from

$$\Delta n = \bar{n}_\nu - n_\nu = xA - \frac{A}{x} .$$

Rewriting the above formula as an explicit quadratic,

$$Ax^2 - \Delta n\, x - A = 0 ,$$

one finds

$$x = \frac{\Delta n \pm \sqrt{\Delta n^2 + 4A^2}}{2A} .$$

Since the definition of x implies $x > 0$, only the positive root is relevant. Since the number of electrons is still assumed to be equal to the number of positrons, $\mu_e = 0$, so the answer to (b) reduces to $\mu_n - \mu_p = -\mu_\nu$. From (a),

$$\begin{aligned}\frac{n_n}{n_p} &= e^{-(\Delta E + \mu_p - \mu_n)/kT} \\ &= e^{-(\Delta E + \mu_\nu)/kT} \\ &= xe^{-\Delta E/kT} \\ &= \boxed{\frac{\sqrt{\Delta n^2 + 4A^2} + \Delta n}{2A}\, e^{-\Delta E/kT} ,}\end{aligned}$$

where

$$\boxed{A \equiv \frac{3}{4}\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} .}$$

(d) For $\Delta n > 0$, the answer to (c) implies that the ratio n_n/n_p would be larger than in the usual case ($\Delta n = 0$). This is consistent with the expectation that an excess of antineutrinos will tend to cause p's to turn into n's according to the reaction

$$p + \bar{\nu}_e \longrightarrow e^+ + n \ .$$

Since the amount of helium produced is proportional to the number of neutrons that survive until the breaking of the deuterium bottleneck, starting with a higher equilibrium abundance of neutrons will $\boxed{\text{increase}}$ the production of helium.