MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — Octoer 13, 2007
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QUIZ 1 SOLUTIONS

Quiz Date: October 2, 2007

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following 5 questions are each worth 5 points:

(a) In the 1940's, three astrophysicists proposed a "steady state" theory of cosmology, in which the universe has always looked about the same as it does now. State the last name of at least one of these authors. (*Bonus points*: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)

Ans: (Weinberg, page 8, or Ryden, page 16): Hermann Bondi, Thomas Gold, and Fred Hoyle.

(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:

(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.

(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.

(iii) published a catalog, *Nebulae and Star Clusters*, listing 103 objects that astronomers should avoid when looking for comets.

$\boxed{\text{(iv)}}$ published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.

(v) discovered that the orbital periods of the planets are proportional to the 3/2 power of the semi-major axis of their elliptical orbits.

Discussion: (i) is false in part because de Sitter was not involved in the measurement of the size of the Milky Way, but the most obvious error is in the size of the Milky Way. Its actual diameter is reported by Weinberg (p. 16) to be about 100,000 light-years, although now it is believed to be about twice that large. (ii) is an accurate description of an observation by Edwin Hubble in 1923 (Weinberg, pp. 19-20). (iii) describes the work of Charles Messier in 1781 (Weinberg, p. 17). (v) is of course one of Kepler's laws of planetary motion.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted by a group of physicists at a neighboring institution as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were **not** part of the story behind this spectacular discovery:

(i) Bell Telephone Laboratory	(ii) MIT	(iii) Princeton University
(iv) pigeons	$\boxed{\text{(v)}}$ ground hogs	$\boxed{\text{(vi)}}$ Hubble's constant
(vii) liquid helium	(viii) 7.35 cm	

(Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer, but the minimum score is zero.)

Discussion: The discovery of the cosmic background radiation was described in some detail by Weinberg in Chapter 3. The observation was done at Bell Telephone Laboratories, in Holmdel, New Jersey. The detector was cooled with liquid helium to minimize electrical noise, and the measurements were made at a wavelength of 7.35 cm. During the course of the experiment the astronomers had to eject a pair of pigeons who were roosting in the antenna. Penzias and Wilson were not initially aware that the radiation they discovered might have come from the big bang, but Bernard Burke of MIT put them in touch with a group at Princeton University (Robert Dicke, James Peebles, P.G. Roll, and David Wilkinson) who were actively working on this hypothesis.

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for rotation about the Sun) and by the behavior of the Foucault pendulum (which showed that the Earth rotates). These discoveries were made

(i) during Copernicus' lifetime.

(ii) approximately two and three decades after Copernicus' death, respectively.

(iii) about one hundred years after Copernicus' death.

$\boxed{\text{(iv)}}$ approximately two and three centuries after Copernicus' death, respectively.

Ryden discusses this on p. 5. The aberration of starlight was discovered in 1728, while the Foucault pendulum was invented in 1851.

(e) If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this homogeneity and isotropy set in?

(i) 1 AU (1 AU $= 1.496 \times 10^{11}$ m).

(ii) 100 kpc (1 kpc $=$ 1000 pc, 1 pc $= 3.086 \times 10^{16}$ m $=$ 3.262 light-year).

(iii) 1 Mpc (1 Mpc $= 10^6$ pc).

(iv) 10 Mpc.

$\boxed{\text{(v)}}$ 100 Mpc.

(vi) 1000 Mpc.

This issue is discussed in Ryden's book on p. 11.

PROBLEM 2: AN EXPONENTIALLY EXPANDING UNIVERSE *(20 points)*

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by

$$H = \dot{R}/R \ .$$

So

$$H = \frac{\chi R_0 e^{\chi t}}{R_0 e^{\chi t}} = \boxed{\chi \ .}$$

(b) According to Eq. (3.8), the coordinate velocity of light is given by

$$\frac{dx}{dt} = \frac{c}{R(t)} = \frac{c}{R_0} e^{-\chi t} \ .$$

Integrating,

$$\begin{aligned} x(t) &= \frac{c}{R_0} \int_0^t e^{-\chi t'} dt' \\ &= \frac{c}{R_0} \left[-\frac{1}{\chi} e^{-\chi t'} \right]_0^t \\ &= \boxed{\frac{c}{\chi R_0} \left[1 - e^{-\chi t} \right] \ .} \end{aligned}$$

(c) From Eq. (3.11), or from the formula sheet given with the quiz, one has

$$1 + z = \frac{R(t_r)}{R(t_e)} \ .$$

Here $t_e = 0$, so

$$1 + z = \frac{R_0 e^{\chi t_r}}{R_0}$$

$$\Longrightarrow \quad e^{\chi t_r} = 1 + z$$

$$\Longrightarrow \quad \boxed{t_r = \frac{1}{\chi}\ln(1+z)\ .}$$

(d) The coordinate distance is $x(t_r)$, where $x(t)$ is the function found in part (b), and t_r is the time found in part (c). So

$$e^{\chi t_r} = 1 + z\ ,$$

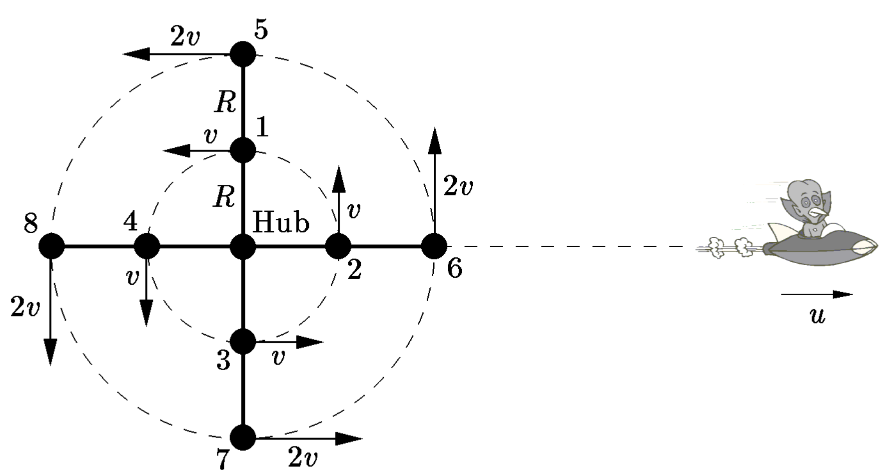
and

$$\begin{aligned} x(t_r) &= \frac{c}{\chi R_0}\left[1 - e^{-\chi t_r}\right] \\ &= \frac{c}{\chi R_0}\left[1 - \frac{1}{1+z}\right] \\ &= \frac{cz}{\chi R_0 (1+z)\ .} \end{aligned}$$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$\ell_p(t_r) = R(t_r)x(t_r) = \frac{cze^{\chi t_r}}{\chi(1+z)} = \boxed{\frac{cz}{\chi}\ .}$$

PROBLEM 3: A TWO-LEVEL HIGH-SPEED MERRY-GO-ROUND
(15 points)



(a) Since the relative positions of all the cars remain fixed as the merry-go-round rotates, each successive pulse from any given car to any other car takes the same amount of time to complete its trip. Thus there will be no Doppler shift caused by pulses taking different amounts of time; the only Doppler shift will come from time dilation.

We will describe the events from the point of view of an inertial reference frame at rest relative to the hub of the merry-go-round, which we will call the laboratory frame. This is the frame in which the problem is described, in which the inner cars are moving at speed v, and the outer cars are moving at speed $2v$. In the laboratory frame, the time interval between the wave crests emitted by the source $\Delta t_S^{\rm Lab}$ will be exactly equal to the time interval $\Delta t_O^{\rm Lab}$ between two crests reaching the observer:

$$\Delta t_O^{\rm Lab} = \Delta t_S^{\rm Lab} \ .$$

The clocks on the merry-go-round cars are moving relative to the laboratory frame, so they will appear to be running slowly by the factor

$$\gamma_1 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

for the inner cars, and by the factor

$$\gamma_2 = \frac{1}{\sqrt{1 - 4v^2/c^2}}$$

for the outer cars. Thus, if we let Δt_S denote the time between crests as measured by a clock on the source, and Δt_O as the time between crests as measured by a clock moving with the observer, then these quantities are related to the laboratory frame times by

$$\gamma_2 \Delta t_S = \Delta t_S^{\rm Lab} \quad \text{and} \quad \gamma_1 \Delta t_O = \Delta t_O^{\rm Lab} \ .$$

To make sure that the γ-factors are on the right side of the equation, you should keep in mind that any time interval should be measured as shorter on the moving clocks than on the lab clocks, since these clocks appear to run slowly. Putting together the equations above, one has immediately that

$$\Delta t_O = \frac{\gamma_2}{\gamma_1} \Delta t_S \ .$$

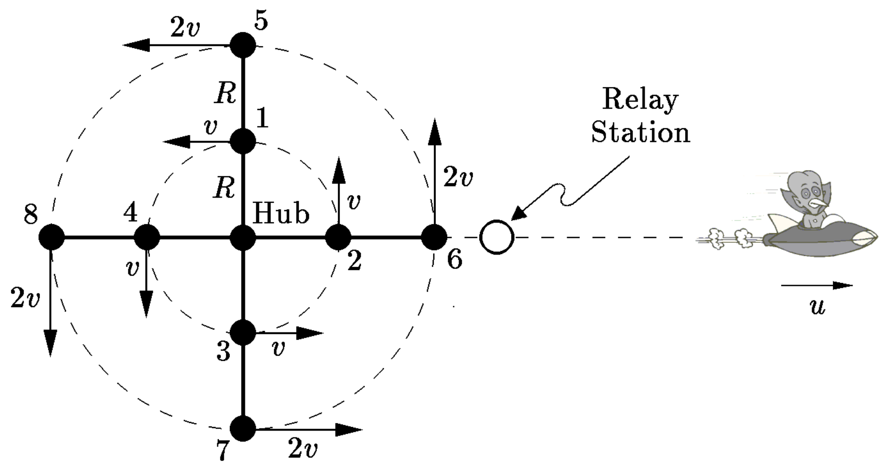
The redshift z is defined by

$$\Delta t_O \equiv (1 + z)\, \Delta t_S \ ,$$

so

$$\boxed{z = \frac{\gamma_2}{\gamma_1} - 1 = \sqrt{\frac{1 - \frac{v^2}{c^2}}{1 - \frac{4v^2}{c^2}}} - 1 \ .}$$

(b) For this part of the problem is useful to imagine a relay station located just to the right of car 6 in the diagram, at rest in the laboratory frame. The relay station rebroadcasts the waves as it receives them, and hence has no effect on the frequency received by the observer, but serves the purpose of allowing us to clearly separate the problem into two parts.



The first part of the discussion concerns the redshift of the signal as measured by the relay station. This calculation would involve both the time dilation and a change in path lengths between successive pulses, but we do not need to do it. It is the standard situation of a source and observer moving directly away from each other, as discussed at the end of Lecture Notes 1. The Doppler shift is given by Eq. (1.33), which was included in the formula sheet. Writing the formula for a recession speed u, it becomes

$$(1 + z)\big|_{\text{relay}} = \sqrt{\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}} \ .$$

If we again use the symbol Δt_S for the time between wave crests as measured by a clock on the source, then the time between the receipt of wave crests as measured by the relay station is

$$\Delta t_R = \sqrt{\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}} \ \Delta t_S \ .$$

The second part of the discussion concerns the transmission from the relay station to car 6. The velocity of car 6 is perpendicular to the direction from which the pulse is being received, so this is a transverse Doppler shift. Any change in path length between successive pulses is second order in Δt, so it can be ignored. The only effect is therefore the time dilation. As described in the laboratory frame, the time separation between crests reaching the observer is the same as the time separation measured by the relay station:

$$\Delta t_O^{\mathrm{Lab}} = \Delta t_R \ .$$

As in part (a), the time dilation implies that

$$\gamma_2 \Delta t_O = \Delta t_O^{\mathrm{Lab}} \ .$$

Combining the formulas above,

$$\Delta_O = \frac{1}{\gamma_2} \sqrt{\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}}\, \Delta t_S \ .$$

Again $\Delta t_O \equiv (1 + z)\, \Delta t_S$, so

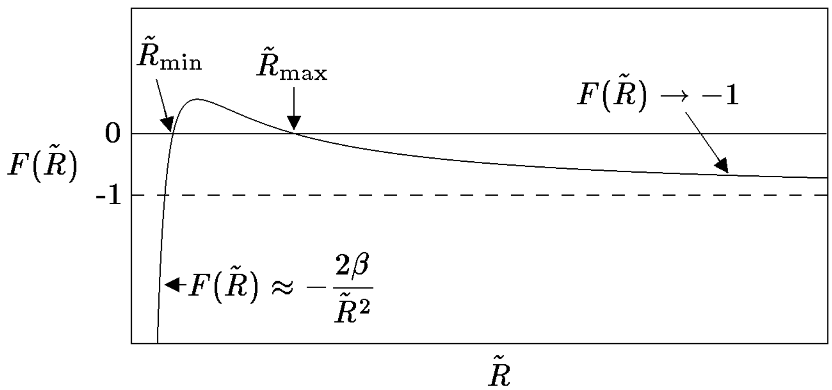
$$\boxed{z = \frac{1}{\gamma_2} \sqrt{\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}} - 1 = \sqrt{\frac{\left(1 - \frac{4v^2}{c^2}\right)\left(1 + \frac{u}{c}\right)}{1 - \frac{u}{c}}} - 1 \ .}$$

PROBLEM 4: A TOY UNIVERSE WITH MATTER AND PURPLE ENERGY *(40 points)*

(a) The function $F(\tilde{R})$ is defined by

$$F(\tilde{R}) \equiv \frac{2\alpha}{\tilde{R}} - \frac{2\beta}{\tilde{R}^2} - 1 ,$$

which looks like



For small $\tilde{R}$ the function is dominated by $-2\beta/\tilde{R}^2$, so it becomes negative and approaches $-\infty$ as $\tilde{R} \to 0$. For large $\tilde{R}$ the inverse powers of $\tilde{R}$ approach zero, so the function approaches -1. You were not required to say this, but it approaches -1 from above, since the next-to-leading term is $2\alpha/\tilde{R}$, which is positive. We are told that α and β have been chosen so that the function is positive somewhere, so the positive values must occur at intermediate values of $\tilde{R}$, as shown.

(b) Since

$$\left(\frac{d\tilde{R}}{d\tilde{t}}\right)^2 \geq 0 ,$$

the universe can satisfy Eq. (1) only if $F(\tilde{R}) \geq 0$. Thus $\tilde{R}_{\rm min}$ and $\tilde{R}_{\rm max}$ are located at the zeros of the function $F(\tilde{R})$, as shown in the diagram above. Analytically,

$$\begin{aligned} F(\tilde{R}) = \frac{2\alpha}{\tilde{R}} - \frac{2\beta}{\tilde{R}^2} - 1 = 0 \\ \Longrightarrow \quad \tilde{R}^2 - 2\alpha\tilde{R} + 2\beta = 0 \\ \Longrightarrow \quad \tilde{R} = \alpha \pm \sqrt{\alpha^2 - 2\beta} \ . \end{aligned}$$

Thus

$$\boxed{\begin{aligned}\tilde{R}_{\rm min} &= \alpha - \sqrt{\alpha^2 - 2\beta}\ ,\\ \tilde{R}_{\rm max} &= \alpha + \sqrt{\alpha^2 - 2\beta}\ .\end{aligned}}$$

(c) λ' can be found easily, since the relation between R and θ is determined by the relationship used to replace R by θ in the integration. This relation is stated in the problem as Eq. (3),

$$\tilde{R} = \alpha - \sqrt{\alpha^2 - 2\beta}\cos\theta\ .$$

By comparing with Eq. (4) of the problem statement,

$$\frac{R}{\sqrt{k}} = \tilde{R} = \alpha(1 - \lambda'\cos\theta)\ ,$$

one sees immediately that the two equations match only if

$$\boxed{\lambda' = \frac{\sqrt{\alpha^2 - 2\beta}}{\alpha}\ .}$$

To find λ one can carry out the integral, but much of the work has already been done in the statement of the problem. From the Friedmann equation in the form

$$\left(\frac{d\tilde{R}}{d\tilde{t}}\right)^2 = F(\tilde{R})\ ,$$

one finds

$$d\tilde{t} = \frac{d\tilde{R}}{\sqrt{F(\tilde{R})}}\ .$$

If one chooses to define $\tilde{t} = 0$ to be the time when $\tilde{R} = \tilde{R}_{\rm min}$, then the above equation can be integrated from $\tilde{t} = 0$ until some arbitrary final time $\tilde{t}_f$:

$$\begin{aligned}\int_0^{\tilde{t}_f} d\tilde{t} = \tilde{t}_f &= \int_{\tilde{R}_{\rm min}}^{\tilde{R}_f} \frac{d\tilde{R}}{\sqrt{F(\tilde{R})}}\\ &= \int_{\tilde{R}_{\rm min}}^{\tilde{R}_f} \frac{\tilde{R}\,d\tilde{R}}{\sqrt{2\alpha\tilde{R} - 2\beta - \tilde{R}^2}}\ .\end{aligned}$$

Using the substitution suggested in the problem,

$$\tilde{R} = \alpha - \sqrt{\alpha^2 - 2\beta}\cos\theta \ ,$$

with the identity

$$2\alpha\tilde{R} - 2\beta - \tilde{R}^2 = (\alpha^2 - 2\beta)\sin^2\theta$$

from the problem statement, this becomes

$$\begin{aligned}
\tilde{t}_f &= \int_0^{\theta_f} \frac{\left[\alpha - \sqrt{\alpha^2 - 2\beta}\cos\theta\right]\,\sqrt{\alpha^2 - 2\beta}\sin\theta\, d\theta}{\sqrt{\alpha^2 - 2\beta}\,\sin\theta} \\
&= \int_0^{\theta_f} \left[\alpha - \sqrt{\alpha^2 - 2\beta}\cos\theta\right]\, d\theta \\
&= \alpha\theta_f - \sqrt{\alpha^2 - 2\beta}\,\sin\theta_f \ .
\end{aligned}$$

We can now drop the subscript f, which was used only to distinguish the limits of integration from the variables of integration. Thus

$$\tilde{t} = \alpha\left(\theta - \frac{\sqrt{\alpha^2 - 2\beta}}{\alpha}\sin\theta\right) \ ,$$

so

$$\boxed{\lambda = \frac{\sqrt{\alpha^2 - 2\beta}}{\alpha} = \lambda' \ .}$$

(d) From

$$\frac{R}{\sqrt{k}} = \alpha(1 - \lambda'\cos\theta) \ ,$$

one can see that R goes through one cycle as θ goes through one cycle, say from $\theta = 0$ to $\theta = 2\pi$. Then from

$$ct = \alpha(\theta - \lambda\sin\theta) \ ,$$

one sees that as θ varies from 0 to 2π, ct increases by $2\pi\alpha$. (The term proportional to $\sin\theta$ returns to its original value, and so does not contribute to the increment in ct.) Thus, the period is

$$\boxed{P = \frac{2\pi\alpha}{c} \ .}$$

(e) We know that $H = \frac{1}{R}\frac{dR}{dt}$, where dR/dt can be evaluated by noting that the chain rule implies

$$\frac{dR}{d\theta} = \frac{dR}{dt}\frac{dt}{d\theta} .$$

Thus,

$$H = \frac{1}{R}\frac{dR/d\theta}{dt/d\theta} ,$$

where differentiation of Eqs. (4) gives

$$\frac{dR}{d\theta} = \alpha\sqrt{k}\lambda' \sin\theta$$

and

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \lambda\cos\theta) .$$

So

$$H = \frac{1}{\sqrt{k}\alpha(1 - \lambda'\cos\theta)} \frac{\alpha\sqrt{k}\lambda'\sin\theta}{\frac{\alpha}{c}(1 - \lambda\cos\theta)} ,$$

which simplifies to

$$\boxed{H = \frac{c\lambda'\sin\theta}{\alpha(1 - \lambda'\cos\theta)(1 - \lambda\cos\theta)} .}$$

There was no need to go further, but if you substituted the values for λ and λ', you would have found

$$H = \frac{c\sqrt{\alpha^2 - 2\beta}\,\sin\theta}{\left(\alpha - \sqrt{\alpha^2 - 2\beta}\,\cos\theta\right)^2} .$$