QUIZ 1 SOLUTIONS

Problem 1: Did you do the reading?

(a) In the 1940's, three astrophysicists proposed a "steady state" theory of cosmology, in which the universe has always looked about the same as it does now. State the last name of at least one of these authors. (Bonus points: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)


(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:

(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.
(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.
(iii) published a catalog, Nebulae and Star Clusters, listing 103 objects that astronomers should avoid when looking for comets.
(iv) published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.
(v) discovered that the orbital periods of the planets are proportional to the square of their average distance from the Sun.

Discussion: (i) is false in part because de Sitter was not involved in the measurement of the size of the Milky Way, but the most obvious error is in the size of the Milky Way. Its actual diameter is reported by Weinberg (p. 16) to be about 100,000 light-years, although now it is believed to be about twice that size. (ii) is an accurate description of an observation by Edwin Hubble in the late 1920's, who showed that the galaxies are moving away from us. (iii) describes the work of Charles Messier in 1781 (Weinberg, p. 17). (iv) is one of Kepler's laws of planetary motion.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were not part of the story behind this spectacular discovery:

(i) Bell Telephone Laboratory
(ii) MIT
(iii) Princeton University
(iv) pigeons
(v) ground hogs
(vi) Hubble's constant
(vii) liquid helium
(viii) 7.35 cm

Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer, but the minimum score is zero.

Discussion: The discovery of the cosmic background radiation was described in some detail by Weinberg in Chapter 3. The observation was done at Bell Telephone Laboratories, in Holmdel, New Jersey. The detector was cooled with liquid helium to minimize electrical noise, and Wilson was a member of the team that built the radio telescope used to make the observation. The data were analyzed by the team of astronomers working at the Bell Laboratories, and there is no evidence that pigeons or ground hogs were involved. Wilson later became the director of Bell Labs.

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for planetary motion), by the discovery of the precession of the perihelion of Mercury (which showed that gravity is not instantaneous), by the discovery of the existence of Neptune (which showed that the orbit of Uranus was not as elliptical as had been thought), and by the discovery of the existence of the planet Pluto (which showed that there were more planets in the solar system than had been thought).

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(e) If one average over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this property becomes evident?

Problem 2: Quiz Date: October 2, 2007

Quiz 1 Solutions

Physics 8.286: The Early Universe

October 13, 2007

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Problem 2: An Exponentially Expanding Universe

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by
\[ \frac{d}{dt} = \frac{\chi}{R} \]
So
\[ H = \frac{\chi}{R_0} e^{\chi t} \]
(b) According to Eq. (3.8), the coordinate velocity of light is given by
\[ \frac{dx}{dt} = c R(t) = c R_0 e^{-\chi t} \]
Integrating,
\[ x(t) = c R_0 \int_0^t e^{-\chi t} dt \]
\[ = c R_0 \left[ -\frac{1}{\chi} e^{-\chi t} \right]_0^t \]
\[ = c R_0 \left[ 1 - e^{-\chi t} \right] \]
(c) From Eq. (3.11), or from the formula sheet given with the quiz, one has
\[ 1 + z = \frac{R(t_r)}{R(t_e)} \]
Here \( t_e = 0 \), so
\[ 1 + z = \frac{R_0 e^{\chi t_r}}{R_0} = \Rightarrow e^{\chi t_r} = 1 + z \]
(d) The coordinate distance is
\[ x(t_r), \text{ where } x(t) \text{ is the function found in part (b),} \]
and \( t_r \) is the time found in part (c). So
\[ e^{\chi t_r} = 1 + z \]
and
\[ x(t_r) = c R_0 \left[ 1 - e^{-\chi t_r} \right] = c R_0 \left[ 1 - \frac{1}{1 + z} \right] \]
\[ = c z R_0 \frac{1 + z}{1 + z} = cz \]
(e) The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so
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(a) Since the relative positions of all the cars remain fixed as the merry-go-round rotates, each successive pulse from any given car to any other car takes the same amount of time to complete its trip. Thus there will be no Doppler shift caused by pulses taking different amounts of time; the only Doppler shift will come from time dilation.

We will describe the events from the point of view of an inertial reference frame at rest relative to the hub of the merry-go-round, which we will call the laboratory frame. This is the frame in which the problem is described, in which the inner cars are moving at speed $v$, and the outer cars are moving at speed $2v$. In the laboratory frame, the time interval between the wave crests emitted by the source $\Delta t_{\text{Lab}}$ will be exactly equal to the time interval $\Delta t_{\text{Lab}}$ between two crests reaching the observer:

$$\Delta t_{\text{Lab}} = \Delta t_{\text{Lab}}.$$

The clocks on the merry-go-round cars are moving relative to the laboratory frame, so they will appear to be running slowly by the factor $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ for the inner cars, and by the factor $\gamma' = \frac{1}{\sqrt{1 - \frac{4v^2}{c^2}}}$ for the outer cars. Thus, if we let $\Delta t_S$ denote the time between crests as measured by a clock on the source, and $\Delta t_O$ as the time between crests as measured by a clock moving with the observer, then these quantities are related to the laboratory frame times by

$$\gamma \Delta t_S = \Delta t_{\text{Lab}}$$

and

$$\gamma' \Delta t_O = \Delta t_{\text{Lab}}.$$

To make sure that the $\gamma$-factors are on the right side of the equation, you should keep in mind that any time interval should be measured shorter on the moving clocks than on the lab clocks, since these clocks appear to run slow as observed by the laboratory frame.

(b) For this part of the problem it is useful to imagine a relay station located just to the right of car 6 in the diagram, at rest in the laboratory frame. The relay station rebroadcasts the waves as it receives them, and hence has no effect on the frequency received by the observer, but serves the purpose of clearly separating the problem into two parts.

The first part of the discussion concerns the redshift of the signal as measured by the relay station. This calculation would involve both the time dilation and a change in path lengths between successive pulses, but we do not need to do that. The second part of the discussion concerns the redshift of the signal as measured by the relay station.

The redshift $z$ is defined by

$$\frac{\Delta t_O}{\Delta t_S} = (1 + z),$$

so

$$z = \frac{\gamma \Delta t_S}{\gamma' \Delta t_O} - 1 = \frac{\sqrt{1 + \frac{u^2}{c^2}} - 1}{\sqrt{1 - \frac{u^2}{c^2}} - 1}.$$
The second part of the discussion concerns the transmission from the relay station to car 6. The velocity of car 6 is perpendicular to the direction from which the pulse is being received, so this is a transverse Doppler shift. Any change in path length between successive pulses is second order in $\Delta t$, so we can ignore it. The only effect is therefore the time dilation. As described in the laboratory frame, the time separation between crests reaching the observer is the same as that measured by the relay station:

$$\Delta t_{\text{Lab}} = \Delta t_{\text{R}}.$$

As in part (a), the time dilation implies that

$$\gamma^2 \Delta t_{\text{O}} = \Delta t_{\text{Lab}}.$$ 

Combining the formulas above,

$$\Delta \approx \frac{1}{\gamma^2} \sqrt{1 + \frac{u}{c}} \frac{1}{1 - \frac{u}{c}}.$$

Again $\Delta \approx (1 + z) \Delta_S$, so

$$\gamma^2 \frac{\Delta}{\Delta_S} = \frac{1}{\sqrt{1 - \frac{u}{c}}} \frac{1}{1 - \frac{u}{c}}.$$
Thus \( \overline{\alpha - \sqrt{\alpha^2 - 2\beta}} = \lambda \).

(c) \( \lambda' \) can be found easily, since the relation between \( R \) and \( \theta \) is determined by the relationship used to replace \( R \) by \( \theta \) in the integration. This relation is stated in the problem as Eq. (3),

\[ \overline{\alpha - \sqrt{\alpha^2 - 2\beta}} = \cos \theta. \]

By comparing with Eq. (4) of the problem statement,

\[ R = \overline{\alpha (1 - \lambda' \cos \theta)} = \frac{\gamma^2}{Y}, \]

one sees immediately that the two equations match only if

\[ \lambda' = \sqrt{\alpha^2 - 2\beta}. \]

To find \( \lambda \) one can carry out the integration, but much of the work has already been done in the statement of the problem. From the Friedmann equation in the form

\[ \left( \frac{d\overline{\alpha}}{d\overline{\alpha}} \right)^2 = \frac{F}{\overline{\alpha}}, \]

one finds

\[ \overline{\alpha} = \frac{\gamma^2}{Y} \overline{\alpha}. \]

If one chooses to define \( \overline{\alpha} = 0 \) to be the time when \( \overline{\alpha} = \overline{\alpha}_{\text{min}} \), then the above equation can be integrated from \( \overline{\alpha} = 0 \) to \( \overline{\alpha} = \overline{\alpha}_{\text{final}} \) to find

\[ \int_{\overline{\alpha}}^{\overline{\alpha}_{\text{final}}} \frac{d\overline{\alpha}}{\overline{\alpha}} = \int_{\overline{\alpha}}^{\overline{\alpha}_{\text{final}}} \frac{d\overline{\alpha}}{\overline{\alpha}} = \overline{\alpha}_{\text{final}} - \overline{\alpha}. \]

Thus

\[ \overline{\alpha} = \overline{\alpha}_{\text{final}} - \overline{\alpha} = \overline{\alpha}_{\text{final}}. \]
\[ \frac{\ell (\theta \cos \chi - \theta_0 ^2 \, \chi - \theta_0 ^2)}{\theta \sin \chi} = H \]

We know that \( H = \frac{1}{R} \frac{dR}{dt} \), where \( \frac{dR}{dt} \) can be evaluated by noting that the chain rule implies
\[ \frac{dR}{d\theta} = \frac{dR}{dt} \frac{dt}{d\theta}. \]

Thus, \( H = \frac{1}{R} \frac{dR}{d\theta} \frac{dt}{d\theta} \), where differentiation of Eq. (4) gives
\[ \frac{dR}{d\theta} = \alpha \sqrt{k\lambda'} \sin \theta \]
and
\[ \frac{dt}{d\theta} = \alpha c (1 - \lambda \cos \theta). \]

So
\[ \frac{(\theta \cos \chi - \theta_0 ^2)(\theta \cos \chi - \theta_0 ^2) \chi}{\theta \sin \chi} = H \]
which simplifies to
\[ \frac{(\theta \cos \chi - \theta_0 ^2)(\theta \cos \chi - \theta_0 ^2) \chi}{\theta \sin \chi} = H \]

where differentiation of Eq. (4) gives
\[ \frac{\theta p}{\partial p} \]
and
\[ \theta \sin \chi \chi = \frac{\theta p}{\partial p} \]

Thus
\[ \frac{\theta p}{\partial p} \frac{H}{\partial p} \frac{H}{\partial p} = H \]
which reduces to
\[ \frac{\theta p}{\partial p} \frac{H}{\partial p} \frac{H}{\partial p} = H \]

(6)