PROBLEM 1: DID YOU DO THE READING?

(24 points)

(a) (6 points) In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow. In this model, the universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled, the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about $10^9$. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but the neutron is thought to be absolutely stable.

(ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)

(iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about $10^3$, not $10^9$ as Alpher and Herman concluded.

(b) (6 points) In Weinberg's "Recipe for a Hot Universe," he described the primordial composition of the universe in terms of three conserved quantities: electric charge, baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons (and vice versa)?

Electric Charge: (i) $\sim 10^9$ (ii) $\sim 1000$ (iii) $\sim 1$ (iv) $\sim 10^{-6}$ (v) either zero or negligible

Baryon Number: (i) $\sim 10^{-20}$ (ii) $\sim 10^{-9}$ (iii) $\sim 10^{-6}$ (iv) $\sim 1$ (v) anywhere from $10^{-5}$ to 1

Lepton Number: (i) $\sim 10^9$ (ii) $\sim 1000$ (iii) $\sim 1$ (iv) $\sim 10^{-6}$ (v) could be as high as $\sim 1$, but is assumed to be very small

(c) (12 points) The figure below comes from Weinberg's Chapter 5, and is labeled The Shifting Neutron-Proton Balance.

(i) (3 points) The period labeled "thermal equilibrium," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(ii) (6 points) In Weinberg's "Recipe for a Hot Universe," the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(iii) (3 points) At some point the universe cooled to a temperature of 5 K. The figure was an attempt to illustrate this background with a fragmentation of 5 K. The figure was read as a consequence of the initial ratio of the number density of the nuclei to the number density of photons.
(D) Neutrons and protons can be converted from one into the other through reactions such as

\begin{align*}
\text{antineutrino} + \text{proton} &\rightarrow \text{electron} + \text{neutron} \\
\text{neutrino} + \text{neutron} &\rightarrow \text{positron} + \text{proton}
\end{align*}

(E) Neutrons and protons can be converted from one into the other through reactions such as

\begin{align*}
\text{antineutrino} + \text{proton} &\rightarrow \text{positron} + \text{neutron} \\
\text{neutrino} + \text{neutron} &\rightarrow \text{electron} + \text{proton}
\end{align*}

(F) Neutrons and protons can be created and destroyed by reactions such as

\begin{align*}
\text{proton} + \text{neutrino} &\rightarrow \text{positron} + \text{antineutrino} \\
\text{neutron} + \text{antineutrino} &\rightarrow \text{electron} + \text{positron}
\end{align*}

(ii) (3 points)

During the period labeled "neutron decay," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as

\begin{align*}
\text{antineutrino} + \text{proton} &\rightarrow \text{electron} + \text{neutron} \\
\text{neutrino} + \text{neutron} &\rightarrow \text{positron} + \text{proton}
\end{align*}

(E) Neutrons and protons can be converted from one into the other through reactions such as

\begin{align*}
\text{antineutrino} + \text{proton} &\rightarrow \text{positron} + \text{neutron} \\
\text{neutrino} + \text{neutron} &\rightarrow \text{electron} + \text{proton}
\end{align*}

(F) Neutrons and protons can be converted from one into the other through reactions such as

\begin{align*}
\text{proton} + \text{neutrino} &\rightarrow \text{positron} + \text{antineutrino} \\
\text{neutron} + \text{antineutrino} &\rightarrow \text{electron} + \text{positron}
\end{align*}

(iii) (3 points)

The masses of the neutron and proton are not exactly equal, but instead

(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV = 10^9 eV).

(B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV = 10^6 eV).

(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV = 10^3 eV).

(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.

(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

(iv) (3 points)

During the period labeled "era of nucleosynthesis," (choose one):

(A) Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.

(B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.

(C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.

(F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.
PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION

(a) This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by
\[ \frac{dU}{dt} = -pdV. \]
Substituting the fact that the energy density \( u \) is equal to \( \rho c^2 \), the energy conservation relation can be written
\[ \frac{dU}{dt} = -pdV dt = \frac{d}{dt} \left( \rho c^2 R^3 \right) = -pd R^3 dt. \]
Setting \( \rho = \alpha R^6 \) for some constant \( \alpha \), the conservation of energy formula becomes
\[ \frac{d}{dt} \left( \alpha c^2 R^3 \right) = -pd R^3 dt, \]
which implies
\[ -3\alpha c^2 R^4 \frac{dR}{dt} = -3pR^2 \frac{dR}{dt}. \]
Thus \( p = \rho c^2 \).

Alternatively, one may start from the equation for the time derivative of \( \rho \),
\[ \dot{\rho} = -3 \dot{R} R (\rho + p c^2). \]
Since \( \rho = \alpha R^6 \), we take the time derivative to find
\[ \dot{\rho} = -3(\dot{R}/R) \rho, \]
and therefore
\[ -6(\dot{R}/R) \rho = -3\dot{R} R (\rho c^2). \]
Thus \( p = \rho c^2 \).

(b) For a flat universe, the Friedmann equation reduces to
\[ \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi}{3} \frac{G}{\rho}. \]
Using \( \rho \propto \frac{1}{R^6} \), this implies that
\[ \frac{\dot{R}}{R} = \beta R^2, \]
for some constant \( \beta \). Rewriting this as
\[ \frac{R^2}{\dot{R}} \frac{dR}{dt} = \beta, \]
we can integrate the equation to give
\[ \frac{1}{3} R^3 = \beta t + \text{const}. \]
The arbitrary constant of proportionality in this answer is consistent with the

}\( x \propto H \)

\[ \frac{x}{t} \propto H \]

We start from the conservation of energy equation in the form
and therefore
\[ \frac{d}{dt} \frac{3}{t} = \frac{\dot{H}}{H} = \frac{\ddot{H}}{H} = \frac{1}{t} \]
we can rearrange the equation to give
\[ \frac{dp}{dt} = \frac{\dot{p} c^2}{c^2} \]
for some constant \( \dot{p} \). Rewriting this as
\[ \frac{\dot{p} c^2}{c^2} = \dot{p}, \]
which implies
\[ \dot{p} = \frac{\dot{c} c^2}{c^2} \dot{c}. \]
Setting
\[ \left( \frac{\dot{c} c^2}{c^2} \right) \frac{dp}{dt} = \left( \frac{\dot{c} c^2}{c^2} \right) \frac{p}{t} \]
implies that
\[ \frac{\dot{c} c^2}{c^2} \frac{dp}{dt} = \left( \frac{\dot{c} c^2}{c^2} \right) \frac{p}{t} \]
for energy conservation can be written
\[ \frac{dp}{dt} = \dot{p} \]
for energy conservation, which is immediate most easily in the comoving formalism

This problem was presented at a panel meeting of the Problem Set G.

THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION (20 points)

PROBLEM 2
PROBLEM 3: A NEW SPECIES OF MESON

(a) Since \( kT \gg m_X c^2 \), we can treat the \( X \) particles as if they were massless, so the number density of massless particles is given by

\[
\rho = \frac{g^* \zeta(3) \pi^2}{(kT)^3 (\hbar c)^3}.
\]

Since \( X \) is a boson, \( g^* = 1 \) per spin state. There are three species, no additional antiparticles, and only one spin state per species, so \( g^* = 3 \). Thus, the mass density is given by

\[
\rho = \frac{3 g^* \zeta(3) \pi^2}{(kT)^3 (\hbar c)^3} = \frac{3 \cdot 3 \cdot 3 \cdot 3}{3} \left( \frac{\pi}{6} \right)^2 \left( \frac{c}{\hbar} \right)^3 = \frac{3 \pi^2}{4 \hbar^3 c^3} kT^3.
\]

Thus, the mass density of a flat radiation-dominated universe does not depend on the number of particle species present, but is always given by

\[
\rho = \frac{3 \pi^2}{4 \hbar^3 c^3} kT^3.
\]

(b) The contributions from the other particles are

\[
\begin{align*}
\text{photon:} & \quad \rho_{\gamma} = \frac{9}{8 \pi^2} \left( \frac{\hbar c}{g^*} \right)^3 kT^5, \\
\text{neutrino:} & \quad \rho_{\nu} = \frac{9}{8 \pi^2} \left( \frac{\hbar c}{g^*} \right)^3 kT^5, \\
\text{e-pair:} & \quad \rho_{e^\pm} = \frac{9}{8 \pi^2} \left( \frac{\hbar c}{g^*} \right)^3 kT^5.
\end{align*}
\]

Since the number of particle species present, \( g^* \), is always given by the number density of massless particles, the mass density of a flat radiation-dominated universe does not depend on the number of particle species present, but is always given by

\[
\rho = \frac{3 \pi^2}{4 \hbar^3 c^3} kT^3.
\]

(c) The number density of a flat radiation-dominated universe does not depend on the number of particle species present, but is always given by

\[
\rho = \frac{3 \pi^2}{4 \hbar^3 c^3} kT^3.
\]

Thus, the number density of a flat radiation-dominated universe does not depend on the number of particle species present, but is always given by

\[
\rho = \frac{3 \pi^2}{4 \hbar^3 c^3} kT^3.
\]
\[
\frac{\partial^2}{\partial x^2} \phi = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi - \frac{1}{c^2} \frac{\partial}{\partial x} \phi
\]

where \( \phi \) denotes the entropy density. We know that for each particle species

\[ \rho \left[ \phi \right] = \rho \left[ \phi \right] \]

where \( i \) denotes a type of matter. It follows that, at any time \( t \), (a) the temperature of the \( X \)-s, and (b) the entropy of the \( X \)-s, are zero.

Thus, if follows that at time \( t \) after the disappearance of the \( X \)-s

\[ \eta \left[ \phi \right] = \rho \left[ \phi \right] \]

while from (b) we learn that

\[ \eta \left[ \phi \right] = \frac{\eta \left[ \phi \right]}{\rho \left[ \phi \right]} \]

Note that, before the freeze-out of the \( X \)-s all particles were in equilibrium.

We are told that the orbit has a phase angle \( \theta \), so on the orbit \( \phi \) and the velocity

\[ \eta \left[ \phi \right] = \frac{\eta \left[ \phi \right]}{\rho \left[ \phi \right]} \]

and the convolution of the matter components are:

\[ \left\{ \eta \left[ \phi \right] \right\} = \left\{ \eta \left[ \phi \right] \right\} \]

From the matter:

Problem 4: THE STABILITY OF SCHWARZSCHILD ORBITS.
dτ
dh

\begin{align}
(16) & \quad 0 = \varepsilon T \frac{\partial}{\partial \omega} f + (\omega \partial) f = (\omega \partial) H
\end{align}

where we have introduced the function \( f \) with follows. The differential equation then takes the form

\begin{align}
(17) & \quad (\omega \partial) H = \varepsilon T \frac{\partial}{\partial \omega} f + (\omega \partial) f = \varepsilon T \frac{\partial}{\partial \omega} f
\end{align}

This is the most useful form of the answer. Of course, we also have

\begin{align}
(18) & \quad (\omega \partial) e^{\phi} = (\omega \partial) f + (\omega \partial) e^{\phi} = \varepsilon T \frac{\partial}{\partial \omega} f + (\omega \partial) f
\end{align}

where we have introduced the function \( f \) where \( T \equiv L \), recall that (17) \( \omega \partial \) takes the form

\begin{align}
(19) & \quad (\omega \partial) H \equiv T \frac{\partial}{\partial \omega} f + (\omega \partial) f = \varepsilon T \frac{\partial}{\partial \omega} f
\end{align}

The quantity \( \omega \partial \) is a constant of the motion, namely, it is a number independent of

\begin{align}
(20) & \quad \frac{\partial}{\partial \phi} e^{\phi} \equiv \omega \partial, 0 = T \frac{\partial}{\partial \omega} f - 0 = \left[ \frac{\partial}{\partial \phi} e^{\phi} \right] \frac{\partial}{\partial \phi} f
\end{align}

Since no metric component depends on \( \phi \), the right-hand side vanishes and we get:

\begin{align}
(21) & \quad \frac{\partial}{\partial \phi} e^{\phi} \equiv \omega \partial, 0 = T \frac{\partial}{\partial \omega} f - 0 = (\omega \partial) f
\end{align}

In the notation of the problem statement, we have

\begin{align}
(22) & \quad \frac{\partial}{\partial \phi} e^{\phi} \equiv \omega \partial, (\omega \partial) f = \frac{\partial}{\partial \phi} e^{\phi} \frac{\partial}{\partial \phi} f
\end{align}

This is an acceptable answer. One can simplify further by noting that (10) \( \omega \partial e^{\phi} \equiv \omega \partial \) takes the form

\begin{align}
(23) & \quad \frac{\partial}{\partial \phi} e^{\phi} \equiv \omega \partial, 0 = T \frac{\partial}{\partial \omega} f - 0 = (\omega \partial) f
\end{align}

Expanding out, the terms with \( \frac{\partial}{\partial \phi} e^{\phi} \) cancel and we find:

\begin{align}
\frac{\partial}{\partial \phi} e^{\phi} \equiv \omega \partial, (\omega \partial) f = \frac{\partial}{\partial \phi} e^{\phi} \frac{\partial}{\partial \phi} f
\end{align}

The quantity \( \omega \partial \) is a constant of the motion, namely, it is a number independent of

\begin{align}
\omega \partial \equiv \omega \partial, 0 = T \frac{\partial}{\partial \omega} f - 0 = \left[ \frac{\partial}{\partial \phi} e^{\phi} \right] \frac{\partial}{\partial \phi} f
\end{align}

Since no metric component depends on \( \phi \), the right-hand side vanishes and we get:

\begin{align}
\omega \partial \equiv \omega \partial, 0 = T \frac{\partial}{\partial \omega} f - 0 = (\omega \partial) f
\end{align}

In the notation of the problem statement, we have

\begin{align}
\omega \partial \equiv \omega \partial, (\omega \partial) f = \frac{\partial}{\partial \phi} e^{\phi} \frac{\partial}{\partial \phi} f
\end{align}

This is an acceptable answer. One can simplify further by noting that (10) \( \omega \partial e^{\phi} \equiv \omega \partial \) takes the form

\begin{align}
\omega \partial \equiv \omega \partial, 0 = T \frac{\partial}{\partial \omega} f - 0 = (\omega \partial) f
\end{align}

Expanding out, the terms with \( \frac{\partial}{\partial \phi} e^{\phi} \) cancel and we find:
For students interested in getting the famous result that orbits are stable for $r > 3R_S$, we complete this part of the analysis below. First we evaluate $H(r)$ in (18) using the values of $f_0$ and $f_1$ in (2):

$$H(r) = \frac{d}{dr} \left[ f_0(r) + f_1(r) \right] = R_S^2 - \frac{3L^2}{2(r_0 - 2R_S)}. \tag{19}$$

This is the desired condition for stable orbits in the Schwarzschild geometry.

The inequality in (18) then gives us

$$R_S^2 - \frac{3L^2}{2(r_0 - 2R_S)} > 0,$$

which is equivalent to

$$\frac{3}{2} < \frac{r_0}{r_0 - 2R_S} > 1.$$

This is the answer to part (d).

For $r_0 > \frac{3}{2}R_S$, we get

$$3(r_0 - 2R_S) > 2(r_0 - \frac{3}{2}R_S) \Rightarrow r_0 > 3R_S.$$

Stability Condition: $H(r_0) = \frac{d}{dr} [f_0(r) + f_1(r)] = R_S^2 - \frac{3L^2}{2(r_0 - 2R_S)} < 0.$ \tag{18}

Note, incidentally, that the equality to the right demands that for a circular orbit $r_0 > \frac{3}{2}R_S$: Substituting this into (15) we get, to first nontrivial approximation

$$\frac{d^2r}{dr^2} = H(r_0) + \delta H(r_0) = \delta H(r_0),$$

where $H(r_0) = \frac{d}{dr} f_0(r_0) + f_1(r_0)$ and we used $H(r_0) = 0$ from (16). The resulting equation $d^2r = \delta H(r_0)$ is just a number. The condition of stability is that this number is negative: $H(r_0) < 0$. Indeed, in this case (17) is the harmonic oscillator equation

$$\frac{d^2r}{dr^2} = -\omega^2 r,$$

which is equivalent to

$$\frac{d^2r}{dr^2} = -\omega^2 (r - r_0).$$

For $\omega > 0$, we have

$$\omega^2 (r - r_0)^2 > 0.$$

Cancelling the common factors of $R_S^2$, we find

$$1 - \frac{3}{2} \left( r_0 - 2R_S \right) > 0,$$

or

$$\frac{3}{2} > \frac{r_0}{r_0 - 2R_S} > 1.$$