MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 1, 2007
Prof. Alan Guth

QUIZ 2 SOLUTIONS

Quiz Date: November 6, 2007

PROBLEM 1: DID YOU DO THE READING? *(24 points)*

(a) *(6 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

(ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)

(iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

(b) *(6 points)* In Weinberg's "Recipe for a Hot Universe," he described the primordial composition of the universe in terms of three conserved quantities: electric charge, baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For

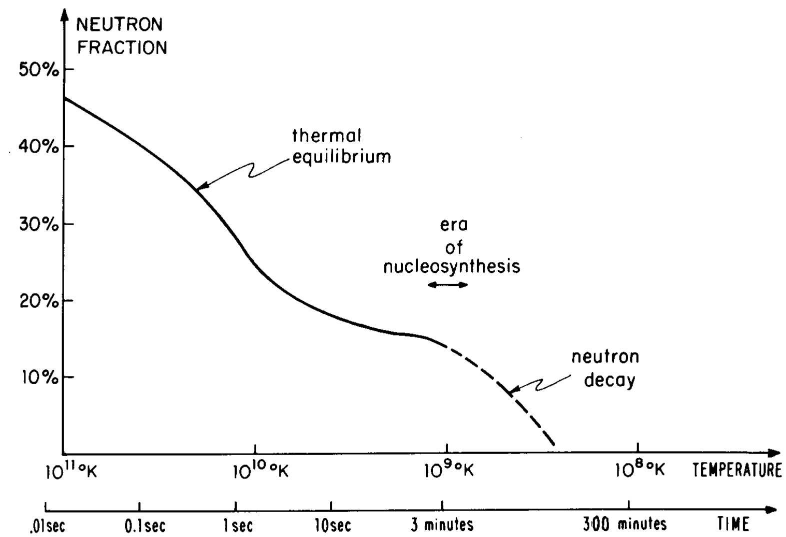
each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons:

Electric Charge: (i) $\sim 10^9$ (ii) ~ 1000 (iii) ~ 1 (iv) $\sim 10^{-6}$ **(v) either zero or negligible**

Baryon Number: (i) $\sim 10^{-20}$ **(ii) $\sim 10^{-9}$** (iii) $\sim 10^{-6}$ (iv) ~ 1 (v) anywhere from 10^{-5} to 1

Lepton Number: (i) $\sim 10^9$ (ii) ~ 1000 (iii) ~ 1 (iv) $\sim 10^{-6}$ **(v) could be as high as ~ 1, but is assumed to be very small**

(c) *(12 points)* The figure below comes from Weinberg's Chapter 5, and is labeled *The Shifting Neutron-Proton Balance.*



(i) *(3 points)* During the period labeled "thermal equilibrium," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{positron} + \text{proton}.$$

[(E)] Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{electron} + \text{proton}.$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \longleftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \longleftrightarrow \text{electron} + \text{positron}.$$

(ii) *(3 points)* During the period labeled "neutron decay," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

[(C)] The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{positron} + \text{proton}.$$

(E) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{electron} + \text{proton}.$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \longleftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \longleftrightarrow \text{electron} + \text{positron}.$$

(iii) *(3 points)* The masses of the neutron and proton are not exactly equal, but instead

(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV = 10^9 eV).

[(B)] The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV = 10^6 eV).

(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV = 10^3 eV).

(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.

(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

(iv) *(3 points)* During the period labeled "era of nucleosynthesis," (choose one:)

[(A)] Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.

(B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.

(C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.

(F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.

PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(20 points)*

This problem was Problem 2 of Problem Set 6.

(a) This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by $dU = -p\,dV$. Using the fact that the energy density u is equal to ρc^2, the energy conservation relation can be written

$$\frac{dU}{dt} = -p\frac{dV}{dt} \quad \Longrightarrow \quad \frac{d}{dt}\left(\rho c^2 R^3\right) = -p\frac{d}{dt}\left(R^3\right) \ .$$

Setting

$$\rho = \frac{\alpha}{R^6}$$

for some constant α, the conservation of energy formula becomes

$$\frac{d}{dt}\left(\frac{\alpha c^2}{R^3}\right) = -p\frac{d}{dt}\left(R^3\right) \ ,$$

which implies

$$-3\frac{\alpha c^2}{R^4}\frac{dR}{dt} = -3pR^2\frac{dR}{dt} \ .$$

Thus

$$p = \frac{\alpha c^2}{R^6} = \boxed{\rho c^2 \ .}$$

Alternatively, one may start from the equation for the time derivative of ρ,

$$\dot\rho = -3\frac{\dot R}{R}\left(\rho + \frac{p}{c^2}\right) .$$

Since $\rho = \frac{\alpha}{R^6}$, we take the time derivative to find $\dot\rho = -6(\dot R/R)\rho$, and therefore

$$-6\frac{\dot R}{R}\rho = -3\frac{\dot R}{R}\left(\rho + \frac{p}{c^2}\right) ,$$

and therefore

$$p = \rho c^2 .$$

(b) For a flat universe, the Friedmann equation reduces to

$$\left(\frac{\dot R}{R}\right)^2 = \frac{8\pi}{3}G\rho \ .$$

Using $\rho \propto 1/R^6$, this implies that

$$\dot R = \frac{\beta}{R^2} \ ,$$

for some constant β. Rewriting this as

$$R^2\,dR = \beta\,dt \ ,$$

we can integrate the equation to give

$$\frac{1}{3}R^3 = \beta t + \text{const} \ ,$$

where the constant of integration has no effect other than to shift the origin of the time variable t. Using the standard big bang convention that $R = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$\boxed{R \propto t^{1/3} \ .}$$

The arbitrary constant of proportionality in this answer is consistent with the wording of the problem, which states that "You should be able to determine the function $R(t)$ up to a constant factor." Note that we could have expressed the constant of proportionality in terms of the constant α that we used in part (a), but there would not really be any point in doing that. The constant α was not a given variable. If the comoving coordinates are measured in "notches," then R is measured in meters per notch, and the constant of proportionality in our answer can be changed by changing the arbitrary definition of the notch.

(c) We start from the conservation of energy equation in the form

$$\dot\rho = -3\frac{\dot R}{R}\left(\rho + \frac{p}{c^2}\right) .$$

Substituting $\dot\rho = -n(\dot R/R)\rho$ and $p = (1/2)\rho c^2$, we have

$$-nH\rho = -3H\left(\frac{3}{2}\rho\right)$$

and therefore

$$\boxed{n = \frac{9}{2}.}$$

PROBLEM 3: A NEW SPECIES OF MESON *(26 points)*

(a) Since $kT \gg m_X c^2$, we can treat the X particles as if they were massless, so we can use the thermal equilibrium formula for the number density of massless particles. From the formula sheet, the number density is given by

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} .$$

The X is a boson, so g^* is 1 per spin state. There are three species, no additional antiparticles, and only one spin state per species, so $g^* = 3$. Thus,

$$n_X = 3\,\frac{1.202}{\pi^2} \left(\frac{30 \cancel{\text{MeV}}}{6.582 \times 10^{-16} \cancel{\text{eV-s}} \times 2.998 \times 10^{10} \cancel{\text{cm-s}^{-1}}} \right)^3 \times \left(\frac{10^6 \cancel{\text{eV}}}{1 \cancel{\text{MeV}}} \frac{100 \cancel{\text{cm}}}{1 \text{ m}} \right)^3 = \boxed{1.28 \times 10^{42} \text{ m}^{-3}}$$

(b) The mass density of a flat radiation-dominated universe does not depend on the number of particle species present, but is always given by

$$\rho = \frac{3}{32\pi G t^2} ,$$

as written in the formula sheet. Numerically,

$$\rho = \frac{3}{32\pi \times 6.673 \times 10^{-8} \text{ cm}^3\text{-g}^{-1}\text{-s}^{-2} \times (10^{-3} \text{ s})^2} = \boxed{4.472 \times 10^{11} \frac{\text{g}}{\text{cm}^3} .}$$

(c) There are three sensible ways to attack this problem, so I will show all three. All the methods require that we know the value of g for this range of temperatures. The contribution from the X particles is 3, the same as for g^*, and the contributions from the other particles are

$$\underbrace{2}_{\text{photons}} + \underbrace{\frac{21}{4}}_{\text{neutrinos}} + \underbrace{\frac{7}{2}}_{e^+e^- \text{ pairs}} = 10\frac{3}{4} .$$

Thus, for the X's and the other particles together, $g = 13\frac{3}{4}$.

The first method equates the mass density found in part (b) to the thermal mass density given by

$$\rho = \frac{u}{c^2} = g \frac{\pi^2}{30} \frac{(kT)^4}{\hbar^3 c^5} .$$

Thus

$$kT = \left[\frac{30 \hbar^3 c^5 \rho}{\pi^2 g} \right]^{1/4}$$

$$= \left[\frac{30 (6.582 \times 10^{-16} \cancel{\text{eV}}\text{-}\cancel{\text{s}})^3 (2.998 \times 10^{10} \cancel{\text{cm-s}^{-1}})^5 \times 4.472 \times 10^{11} \cancel{\frac{\text{g}}{\text{cm}^3}}}{\pi^2 \times (13.75)} \right]^{1/4} \times \left[\frac{1 \cancel{\text{eV}}}{1.602 \times 10^{-12} \cancel{\text{erg}}} \frac{1 \cancel{\text{erg}}}{1 \cancel{\text{g-cm}^2\text{-s}^{-2}}} \left(\frac{1 \text{ MeV}}{10^6 \cancel{\text{eV}}} \right)^4 \right]^{1/4} = \boxed{25.55 \text{ MeV} .}$$

The second method uses the general formula for the energy density of a flat, radiation-dominated universe. From the formula sheet,

$$kT = \left[\frac{45 \hbar^3 c^5}{16 \pi^3 g G} \right]^{1/4} \frac{1}{\sqrt{t}}$$

$$= \left[\frac{45 (6.582 \times 10^{-16} \cancel{\text{eV}}\text{-}\cancel{\text{s}})^3 (2.998 \times 10^{10} \cancel{\text{cm-s}^{-1}})^5}{16 \pi^3 (13.75)(6.673 \times 10^{-8} \cancel{\text{cm}^3\text{-g}^{-1}}\text{-}\cancel{\text{s}}^{-2})} \right]^{1/4} \frac{1}{\sqrt{10^{-3} \cancel{\text{s}}}} \times \left[\frac{1 \cancel{\text{eV}}}{1.602 \times 10^{-12} \cancel{\text{erg}}} \frac{1 \cancel{\text{erg}}}{1 \cancel{\text{g-cm}^2\text{-s}^{-2}}} \left(\frac{1 \text{ MeV}}{10^6 \cancel{\text{eV}}} \right)^4 \right]^{1/4} = \boxed{25.55 \text{ MeV} .}$$

The final method uses the formula from the formula sheet which is an evaluation of the formula above for the special case of $g = 10.75$, which applies to the real universe for 106 MeV $\gg kT \gg$ 0.511 MeV:

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} .$$

Knowing that the answer is proportional to $1/g^{1/4}$, the formula above can be corrected for $g = 13.75$:

$$kT = \frac{0.860\ \text{MeV}}{\sqrt{10^{-3}}}\left(\frac{10.75}{13.75}\right)^{1/4} = \boxed{25.57\ \text{MeV}\ .}$$

Note that unit conversions were crucial to first two methods, but that the right answer can be found by multiplying by one and keeping track of the unit cancellations.

(d) As with the freeze-out of electron-positron pairs, we use entropy to calculate the temperature shifts. When the X's disappear they give essentially all their entropy to the electrons and photons, and none to the neutrinos. Thus, if we let t_i denote a time before the disappearance of the X's, and t_f a time after the disappearance, then

$$[R^3(s_X + s_\gamma + s_{e^+e^-})]\big|_{t_i} = [R^3(s_\gamma + s_{e^+e^-})]\big|_{t_f}\ ,$$

while

$$[R^3 s_\nu]\big|_{t_i} = [R^3 s_\nu]\big|_{t_f}\ ,$$

where s denotes the entropy density. We know that for each particle species

$$s \propto gT^3\ ,$$

so the relations above imply that

$$[R^3(g_X + g_\gamma + g_{e^+e^-})T^3]\big|_{t_i} = [R^3(g_\gamma + g_{e^+e^-})T_\gamma^3]\big|_{t_f}\ , \tag{1}$$

and

$$[R^3 g_\nu T^3]\big|_{t_i} = [R^3 g_\nu T_\nu^3]\big|_{t_f}\ . \tag{2}$$

Note that before the freeze-out of the X's all particles were in equilibrium, so they were described by one common temperature T. Afterward, T_γ describes the temperature of the photons and e^+e^- pairs, while T_ν describes the temperature of the neutrinos. From Eq. (1) we learn that

$$\begin{aligned}[R^3T_\gamma^3]\big|_{t_f} &= \frac{g_X + g_\gamma + g_{e^+e^-}}{g_\gamma + g_{e^+e^-}}\,[R^3T^3]\big|_{t_i} \\ &= \frac{3 + 2 + \frac{7}{2}}{2 + \frac{7}{2}}\,[R^3T^3]\big|_{t_i} \\ &= \frac{17}{11}\,[R^3T^3]\big|_{t_i}\ ,\end{aligned}$$

while from Eq. (2) we learn that

$$[R^3T_\nu^3]\big|_{t_f} = [R^3T^3]\big|_{t_i}\ .$$

Thus, it follows that at time t_f (after the disappearance of the X's),

$$T_\gamma = \left(\frac{17}{11}\right)^{1/3} T_\nu\ ,$$

or

$$\boxed{\frac{T_\gamma}{T_\nu} = \left(\frac{17}{11}\right)^{1/3}\ .}$$

PROBLEM 4: THE STABILITY OF SCHWARZSCHILD ORBITS* *(30 points)*

From the metric:

$$ds^2 = -c^2d\tau^2 = -h(r)\,c^2dt^2 + h(r)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\,, \tag{1}$$

and the convention $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ we read the nonvanishing metric components:

$$g_{tt} = -h(r)c^2\,,\quad g_{rr} = \frac{1}{h(r)}\,,\quad g_{\theta\theta} = r^2\,,\quad g_{\phi\phi} = r^2\sin^2\theta\,. \tag{2}$$

We are told that the orbit has $\theta = \pi/2$, so on the orbit $d\theta = 0$ and the relevant metric and metric components are:

$$ds^2 = -c^2d\tau^2 = -h(r)\,c^2dt^2 + h(r)^{-1}dr^2 + r^2d\phi^2\,, \tag{3}$$

$$g_{tt} = -h(r)c^2\,,\quad g_{rr} = \frac{1}{h(r)}\,,\quad g_{\phi\phi} = r^2\,. \tag{4}$$

We also know that

$$h(r) = 1 - \frac{R_S}{r}\,. \tag{5}$$

(a) The geodesic equation

$$\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\,, \tag{6}$$

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* Solution by Barton Zwiebach.

for the index value $\mu = r$ takes the form

$$\frac{d}{d\tau}\left[g_{rr}\,\frac{dr}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial r}\,\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\;.$$

Expanding out

$$\frac{d}{d\tau}\left[\frac{1}{h}\,\frac{dr}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\,\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{rr}}{\partial r}\,\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\,\left(\frac{d\phi}{d\tau}\right)^2 .$$

Using the values in (4) to evaluate the right-hand side and taking the derivatives on the left-hand side:

$$\underline{-\frac{h'}{h^2}\left(\frac{dr}{d\tau}\right)^2} + \frac{1}{h}\,\frac{d^2r}{d\tau^2} = -\frac{1}{2}c^2h'\,\left(\frac{dt}{d\tau}\right)^2 \underline{-\frac{1}{2}\frac{h'}{h^2}\,\left(\frac{dr}{d\tau}\right)^2} + r\,\left(\frac{d\phi}{d\tau}\right)^2 .$$

Here $h' \equiv \frac{dh}{dr}$ and we have supressed the arguments of h and h' to avoid clutter. Collecting the underlined terms to the right and multiplying by h, we find

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\,h'\,hc^2\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}\frac{h'}{h}\,\left(\frac{dr}{d\tau}\right)^2 + rh\,\left(\frac{d\phi}{d\tau}\right)^2 . \qquad (7)$$

(b) Dividing the expression (3) for the metric by $d\tau^2$ we readily find

$$-c^2 = -hc^2\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{h}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2,$$

and rearranging,

$$hc^2\left(\frac{dt}{d\tau}\right)^2 = c^2 + \frac{1}{h}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 . \qquad (8)$$

This is the most useful form of the answer. Of course, we also have

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{h} + \frac{1}{h^2c^2}\left(\frac{dr}{d\tau}\right)^2 + \frac{r^2}{hc^2}\left(\frac{d\phi}{d\tau}\right)^2 . \qquad (9)$$

We use now (8) to simplify (7):

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\,h'\left(c^2 + \frac{1}{h}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right) + \frac{1}{2}\frac{h'}{h}\,\left(\frac{dr}{d\tau}\right)^2 + rh\,\left(\frac{d\phi}{d\tau}\right)^2 .$$

Expanding out, the terms with $(\frac{dr}{d\tau})^2$ cancel and we find

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\,h'\,c^2 + \left(rh - \frac{1}{2}\,h'r^2\right)\,\left(\frac{d\phi}{d\tau}\right)^2 . \qquad (10)$$

This is an acceptable answer. One can simplify (10) further by noting that $h' = R_S/r^2$ and $rh = r - R_S$:

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{R_Sc^2}{r^2} + \left(r - \frac{3}{2}R_S\right)\,\left(\frac{d\phi}{d\tau}\right)^2 . \qquad (11)$$

In the notation of the problem statement, we have

$$f_0(r) = -\frac{1}{2}\frac{R_Sc^2}{r^2}\,, \quad f_1(r) = r - \frac{3}{2}R_S\,. \qquad (12)$$

(c) The geodesic equation (6) for $\mu = \phi$ gives

$$\frac{d}{d\tau}\left[g_{\phi\phi}\,\frac{d\phi}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial\phi}\,\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\,.$$

Since no metric component depends on ϕ, the right-hand side vanishes and we get:

$$\frac{d}{d\tau}\left[r^2\,\frac{d\phi}{d\tau}\right] = 0 \quad\to\quad \frac{d}{d\tau}L = 0\,, \ \text{ where } \ L \equiv r^2\,\frac{d\phi}{d\tau}\,. \qquad (13)$$

The quantity L is a constant of the motion, namely, it is a number independent of τ.

(d) Using (13) the second-order differential equation (11) for $r(\tau)$ takes the form stated in the problem:

$$\frac{d^2r}{d\tau^2} = f_0(r) + \frac{f_1(r)}{r^4}L^2 \equiv H(r)\,, \qquad (14)$$

where we have introduced the function $H(r)$ (recall that L is a constant!). The differential equation then takes the form

$$\frac{d^2r}{d\tau^2} = H(r)\,. \qquad (15)$$

Since we are told that a circular orbit with radius r_0 exists, the function $r(\tau) = r_0$ must solve this equation. Being the constant function, the left-hand side vanishes and, consequently, the right-hand side must also vanish:

$$H(r_0) = f_0(r_0) + \frac{f_1(r_0)}{r_0^4}L^2 = 0\,. \qquad (16)$$

To investigate stability we consider a small perturbation $\delta r(\tau)$ of the orbit:

$$r(\tau) = r_0 + \delta r(\tau)\,, \quad \text{with} \quad \delta r(\tau) \ll r_0 \ \text{at some initial } \tau\,.$$

Substituting this into (15) we get, to first nontrivial approximation

$$\frac{d^2\delta r}{d\tau^2} = H(r_0 + \delta r) \simeq H(r_0) + \delta r H'(r_0) = \delta r\, H'(r_0)\,,$$

where $H'(r) = \frac{dH(r)}{dr}$ and we used $H(r_0) = 0$ from (16). The resulting equation

$$\frac{d^2\delta r(\tau)}{d\tau^2} = H'(r_0)\,\delta r(\tau)\,, \qquad (17)$$

is familiar because $H'(r_0)$ is just a number. The condition of stability is that this number is negative: $H'(r_0) < 0$. Indeed, in this case (17) is the harmonic oscillator equation

$$\frac{d^2x}{dt^2} = -\omega^2 x\,, \quad \text{with replacements} \quad x \leftrightarrow \delta r, \quad t \leftrightarrow \tau\,, \quad -\omega^2 \leftrightarrow H'(r_0)\,,$$

and the solution describes bounded oscillations. So stability requires:

$$\text{Stability Condition:} \quad H'(r_0) = \frac{d}{dr}\left[f_0(r) + \frac{f_1(r)}{r^4}L^2\right]_{r=r_0} < 0\,. \qquad (18)$$

This is the answer to part (d).

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For students interested in getting the famous result that orbits are stable for $r > 3R_S$ we complete this part of the analysis below. First we evaluate $H'(r_0)$ in (18) using the values of f_0 and f_1 in (12):

$$H'(r_0) = \frac{d}{dr}\left[-\frac{1}{2}\frac{R_S c^2}{r^2} + \left(\frac{1}{r^3} - \frac{3R_S}{2r^4}\right)L^2\right]_{r=r_0} = \frac{R_S c^2}{r_0^3} - \frac{3L^2}{r_0^5}(r_0 - 2R_S)\,.$$

The inequality in (18) then gives us

$$R_S c^2 - \frac{3L^2}{r_0^2}(r_0 - 2R_S) < 0\,, \qquad (19)$$

where we multiplied by $r_0^3 > 0$. To complete the calculation we need the value of L^2 for the orbit with radius r_0. This value is determined by the vanishing of $H(r_0)$:

$$-\frac{1}{2}\frac{R_S c^2}{r_0^2} + (r_0 - \frac{3}{2}R_S)\frac{L^2}{r_0^4} = 0 \quad \rightarrow \quad \frac{L^2}{r_0^2} = \frac{1}{2}\frac{R_S c^2}{(r_0 - \frac{3}{2}R_S)}\,.$$

Note, incidentally, that the equality to the right demands that for a circular orbit $r_0 > \frac{3}{2}R_S$. Substituting the above value of L^2/r_0^2 in (19) we get:

$$R_S c^2 - \frac{3}{2}\frac{R_S c^2}{(r_0 - \frac{3}{2}R_S)}(r_0 - 2R_S) < 0\,.$$

Cancelling the common factors of $R_S c^2$ we find

$$1 - \frac{3}{2}\frac{(r_0 - 2R_S)}{(r_0 - \frac{3}{2}R_S)} < 0\,,$$

which is equivalent to

$$\frac{3}{2}\frac{(r_0 - 2R_S)}{(r_0 - \frac{3}{2}R_S)} > 1\,.$$

For $r_0 > \frac{3}{2}R_S$, we get

$$3(r_0 - 2R_S) > 2(r_0 - \frac{3}{2}R_S) \quad \rightarrow \quad r_0 > 3R_S\,. \qquad (20)$$

This is the desired condition for stable orbits in the Schwarzschild geometry.