MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 26, 2009
Prof. Alan Guth

QUIZ 3 SOLUTIONS

Quiz Date: December 6, 2007

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following parts are each worth 5 points.

(a) (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

$\boxed{\text{(ii)}}$ After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

Explanation: After subtracting the dipole contribution, the temperature fluctuation is about 1.1×10^{-5}.

(b) (CMB experiments) The current mean energy per CMB photon, about 6×10^{-4} eV, is comparable to the energy of vibration or rotation for a small molecule such as H_2O. Thus microwaves with wavelengths shorter than $\lambda \sim 3$ cm are strongly absorbed by water molecules in the atmosphere. To measure the CMB at $\lambda < 3$ cm, which one of the following methods is *not* a feasible solution to this problem?

(i) Measure CMB from high-altitude balloons, e.g. MAXIMA.

(ii) Measure CMB from the South Pole, e.g. DASI.

$\boxed{\text{(iii)}}$ Measure CMB from the North Pole, e.g. BOOMERANG.

(iv) Measure CMB from a satellite above the atmosphere of the Earth, e.g. COBE, WMAP and PLANCK.

Explanation: The North Pole is at sea level. In contrast, the South Pole is nearly 3 kilometers above sea level. BOOMERANG is a balloon-borne experiment launched from Antarctica.

(c) (Temperature fluctuations) The creation of temperature fluctuations in CMB by variations in the gravitational potential is known as the Sachs-Wolfe effect. Which one of the following statements is *not* correct concerning this effect?

(i) A CMB photon is redshifted when climbing out of a gravitational potential well, and is blueshifted when falling down a potential hill.

(ii) At the time of last scattering, the nonbaryonic dark matter dominated the energy density, and hence the gravitational potential, of the universe.

(iii) The large-scale fluctuations in CMB temperatures arise from the gravitational effect of primordial density fluctuations in the distribution of nonbaryonic dark matter.

$\boxed{\text{(iv)}}$ The peaks in the plot of temperature fluctuation Δ_T vs. multipole l are due to variations in the density of nonbaryonic dark matter, while the contributions from baryons alone would not show such peaks.

Explanation: These peaks are due to the acoustic oscillations in the photon-baryon fluid.

(d) (Dark matter candidates) Which one of the following is *not* a candidate of nonbaryonic dark matter?

(i) massive neutrinos

(ii) axions

$\boxed{\text{(iii)}}$ matter made of top quarks (a type of quarks with heavy mass of about 171 GeV).

(iv) WIMPs (Weakly Interacting Massive Particles)

(v) primordial black holes

Explanation: Matter made of top quarks is so unstable that it is seen only fleetingly as a product in high energy particle collisions.

(e) (Signatures of dark matter) By what methods can signatures of dark matter be detected? List two methods. (Grading: 3 points for one correct answer, 5 points for two correct answers. If you give more than two answers, your score will be based on the number of right answers minus the number of wrong answers, with a lower bound of zero.)

Answers:

(i) Galaxy rotation curves. (I.e., measurements of the orbital speed of stars in spiral galaxies as a function of radius R show that these curves remain flat at radii far beyond the visible stellar disk. If most of the matter were contained in the disk, then these velocities should fall off as $1/\sqrt{R}$*.)*

(ii) *Use the virial theorem to estimate the mass of a galaxy cluster. (For example, the virial analysis shows that only 2% of the mass of the Coma cluster consistes of stars, and only 10% consists of hot intracluster gas.*

(iii) *Gravitational lensing. (For example, the mass of a cluster can be estimated from the distortion of the shapes of the galaxies behind the cluster.)*

(iv) *CMB temperature fluctuations. (I.e., the analysis of the intensity of the fluctuations as a function of multipole number shows that* $\Omega_{\rm tot} \approx 1$, *and that dark energy contributes* $\Omega_\Lambda \approx 0.7$, *baryonic matter contributes* $\Omega_{\rm bary} \approx 0.04$, *and dark matter contributes* $\Omega_{\rm dark\ matter} \approx 0.26$.*)*

There are other possible answers as well, but these are the ones discussed by Ryden in Chapters 8 and 9.

PROBLEM 2: NEUTRINO NUMBER AND THE NEUTRON/PROTON EQUILIBRIUM

(a) From the chemical equilibrium equation on the front of the exam, the number densities of neutrons and protons can be written as

$$n_n = g_n \frac{(2\pi m_n kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_n - m_n c^2)/kT}$$

$$n_p = g_p \frac{(2\pi m_p kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_p - m_p c^2)/kT} ,$$

where $g_n = g_p = 2$. Dividing,

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(\Delta E + \mu_p - \mu_n)/kT} ,$$

where $\Delta E = (m_n - m_p)c^2$ is the proton-neutron mass-energy difference. Approximating $m_n/m_p \approx 1$, one has

$$\boxed{\frac{n_n}{n_p} = e^{-(\Delta E + \mu_p - \mu_n)/kT} .}$$

The approximation $m_n/m_p \approx 1$ is very accurate (0.14%), but is clearly not necessary. Full credit was given whether or not this approximation was used.

(b) For any allowed chemical reaction, the sum of the chemical potentials on the two sides must be equal. So, from

$$e^+ + n \longleftrightarrow p + \bar{\nu}_e ,$$

we can infer that

$$-\mu_e + \mu_n = \mu_p - \mu_\nu ,$$

which implies that

$$\boxed{\mu_n - \mu_p = \mu_e - \mu_\nu .}$$

(c) Applying the formula given in the problem to the number densities of electron neutrinos and the corresponding antineutrinos,

$$n_\nu = g_\nu^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{\mu_\nu/kT}$$

$$\bar{n}_\nu = g_\nu^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{-\mu_\nu/kT} ,$$

since the chemical potential for the antineutrinos ($\bar{\nu}$) is the negative of the chemical potential for neutrinos. A neutrino has only one spin state, so $g_\nu = 3/4$, where the factor of 3/4 arises because neutrinos are fermions. Setting

$$x \equiv e^{-\mu_\nu/kT}$$

and

$$A \equiv \frac{3}{4} \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} ,$$

the number density equations can be written compactly as

$$n_\nu = \frac{A}{x} , \qquad \bar{n}_\nu = xA .$$

To express x in terms of the ratio $\bar{n}_\nu/n_\nu$, divide the second equation by the first to obtain

$$\frac{\bar{n}_\nu}{n_\nu} = x^2 \quad \Longrightarrow \quad x = \sqrt{\frac{\bar{n}_\nu}{n_\nu}} .$$

Alternatively, x can be expressed in terms of the difference in number densities $\bar{n}_\nu - n_\nu$ by starting with

$$\Delta n = \bar{n}_\nu - n_\nu = xA - \frac{A}{x} .$$

Rewriting the above formula as an explicit quadratic,

$$Ax^2 - \Delta n\, x - A = 0 ,$$

one finds

$$x = \frac{\Delta n \pm \sqrt{\Delta n^2 + 4A^2}}{2A} \ .$$

Since the definition of x implies $x > 0$, only the positive root is relevant. Since the number of electrons is still assumed to be equal to the number of positrons, $\mu_e = 0$, so the answer to (b) reduces to $\mu_n - \mu_p = -\mu_\nu$. From (a),

$$\begin{aligned} \frac{n_n}{n_p} &= e^{-(\Delta E + \mu_p - \mu_n)/kT} \\ &= e^{-(\Delta E + \mu_\nu)/kT} \\ &= x e^{-\Delta E/kT} \\ &= \boxed{\sqrt{\frac{\bar{n}_\nu}{n_\nu}}\, e^{-\Delta E/kT} \ .} \end{aligned}$$

Alternatively, one can write the answer as

$$\boxed{\frac{n_n}{n_p} = \frac{\sqrt{\Delta n^2 + 4A^2} + \Delta n}{2A}\, e^{-\Delta E/kT} \ ,}$$

where

$$\boxed{A \equiv \frac{3}{4}\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} \ .}$$

(d) For $\Delta n > 0$, the answer to (c) implies that the ratio n_n/n_p would be larger than in the usual case ($\Delta n = 0$). This is consistent with the expectation that an excess of antineutrinos will tend to cause p's to turn into n's according to the reaction

$$p + \bar{\nu}_e \longrightarrow e^+ + n \ .$$

Since the amount of helium produced is proportional to the number of neutrons that survive until the breaking of the deuterium bottleneck, starting with a higher equilibrium abundance of neutrons will $\boxed{\text{increase}}$ the production of helium.

PROBLEM 3: SECOND HUBBLE CROSSING *(40 points)*

(a) From the formula sheets, we know that for a flat radiation-dominated universe,

$$a(t) \propto t^{1/2} \ .$$

Since

$$H = \frac{\dot{a}}{a} \ ,$$

(which is also on the formula sheets),

$$H = \frac{1}{2t} \ .$$

Then

$$\ell_H(t) \equiv cH^{-1}(t) = \boxed{2ct \ .}$$

(b) We are told that the energy density is dominated by photons and neutrinos, so we need to add together these two contributions to the energy density. For photons, the formula sheet reminds us that $g_\gamma = 2$, so

$$u_\gamma = 2\,\frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3} \ .$$

For neutrinos the formula sheet reminds us that

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} \ ,$$

so

$$u_\nu = \frac{21}{4}\frac{\pi^2}{30}\frac{(kT_\nu)^4}{(\hbar c)^3} \ .$$

Combining these two expressions and using $T_\nu = (4/11)^{1/3}\, T_\gamma$, one has

$$u = u_\gamma + u_\nu = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3} \ ,$$

so finally

$$\boxed{g_1 = 2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3} \ .}$$

(c) The Friedmann equation tells us that, for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho ,$$

where in this case $H = 1/(2t)$ and

$$\rho = \frac{u}{c^2} = g_1 \frac{\pi^2}{30} \frac{(kT_\gamma)^4}{\hbar^3 c^5} .$$

Thus

$$\left(\frac{1}{2t}\right)^2 = \frac{8\pi G}{3} g_1 \frac{\pi^2}{30} \frac{(kT_\gamma)^4}{\hbar^3 c^5} .$$

Solving for T_γ,

$$\boxed{T_\gamma = \frac{1}{k}\left(\frac{45\hbar^3 c^5}{16\pi^3 g_1 G}\right)^{1/4} \frac{1}{\sqrt{t}} .}$$

(d) The condition for Hubble crossing is

$$\lambda(t) = cH^{-1}(t) ,$$

and the first Hubble crossing always occurs during the inflationary era. Thus any Hubble crossing during the radiation-dominated era must be the second Hubble crossing.

If λ is the present physical wavelength of the density perturbations under discussion, the wavelength at time t is scaled by the scale factor $a(t)$:

$$\lambda(t) = \frac{a(t)}{a(t_0)} \lambda .$$

Between the second Hubble crossing and now, there have been no freeze-outs of particle species. Today the entropy of the universe is still dominated by photons and neutrinos, so the conservation of entropy implies that aT_γ has remained essentially constant between then and now. Thus,

$$\lambda(t) = \frac{T_{\gamma,0}}{T_\gamma(t)} \lambda .$$

Using the previous results for $cH^{-1}(t)$ and for $T_\gamma(t)$, the condition $\lambda(t) = cH^{-1}(t)$ can be rewritten as

$$kT_{\gamma,0}\left(\frac{16\pi^3 g_1 G}{45\hbar^3 c^5}\right)^{1/4} \sqrt{t}\,\lambda = 2ct .$$

Solving for t, the time of second Hubble crossing is found to be

$$\boxed{t_{H2}(\lambda) = (kT_{\gamma,0}\lambda)^2 \left(\frac{\pi^3 g_1 G}{45\hbar^3 c^9}\right)^{1/2} .}$$

Extension: You were not asked to insert numbers, but it is of course interesting to know where the above formula leads. If we take $\lambda = 10^6$ lt-yr, it gives

$$t_{H2}(10^6 \text{ lt-yr}) = 1.04 \times 10^7 \text{ s} = 0.330 \text{ year} .$$

For $\lambda = 1$ Mpc,

$$t_{H2}(1 \text{ Mpc}) = 1.11 \times 10^8 \text{ s} = 3.51 \text{ year} .$$

Taking $\lambda = 2.5 \times 10^6$ lt-yr, the distance to Andromeda, the nearest spiral galaxy,

$$t_{H2}(2.5 \times 10^6 \text{ lt-yr}) = 6.50 \times 10^7 \text{ sec} = 2.06 \text{ year} .$$