MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 2, 2007
Prof. Alan Guth

# QUIZ 1

## Reformatted to Remove Blank Pages

**A SUMMARY OF USEFUL INFORMATION IS AT THE END OF THE EXAM**

______________________________
Your Name

| Problem | Maximum | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 20 | |
| 3 | 15 | |
| 4 | 40 | |
| **TOTAL** | 100 | |

**PROBLEM 1: DID YOU DO THE READING?** *(25 points)*

*The following 5 questions are each worth 5 points:*

(a) In the 1940's, three astrophysicists proposed a "steady state" theory of cosmology, in which the universe has always looked about the same as it does now. State the last name of at least one of these authors. (*Bonus points*: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)

(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:

(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.

(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.

(iii) published a catalog, *Nebulae and Star Clusters*, listing 103 objects that astronomers should avoid when looking for comets.

(iv) published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.

(v) discovered that the orbital periods of the planets are proportional to the 3/2 power of the semi-major axis of their elliptical orbits.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted by a group of physicists at a neighboring institution as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were **not** part of the story behind this spectacular discovery:

| | | |
|---|---|---|
| (i) Bell Telephone Laboratory | (ii) MIT | (iii) Princeton University |
| (iv) pigeons | (v) ground hogs | (vi) Hubble's constant |
| (vii) liquid helium | (viii) 7.35 cm | |

(Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer, but the minimum score is zero.)

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for rotation about the Sun) and by the behavior of the Foucault pendulum (which showed that the Earth rotates). These discoveries were made

(i) during Copernicus' lifetime.

(ii) approximately two and three decades after Copernicus' death, respectively.

(iii) about one hundred years after Copernicus' death.

(iv) approximately two and three centuries after Copernicus' death, respectively.

(e) If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this homogeneity and isotropy set in?

(i) 1 AU (1 AU $= 1.496 \times 10^{11}$ m).

(ii) 100 kpc (1 kpc = 1000 pc, 1 pc $= 3.086 \times 10^{16}$ m = 3.262 light-year).

(iii) 1 Mpc (1 Mpc $= 10^6$ pc).

(iv) 10 Mpc.

(v) 100 Mpc.

(vi) 1000 Mpc.

**PROBLEM 2: AN EXPONENTIALLY EXPANDING UNIVERSE** *(20 points)*

*The following problem was Problem 4 on the Review Problems for Quiz 1.*

Consider a flat (i.e., a $k = 0$, or a Euclidean) universe with scale factor given by

$$R(t) = R_0 e^{\chi t} ,$$

where $R_0$ and $\chi$ are constants.

(a) *(5 points)* Find the Hubble constant $H$ at an arbitrary time $t$.

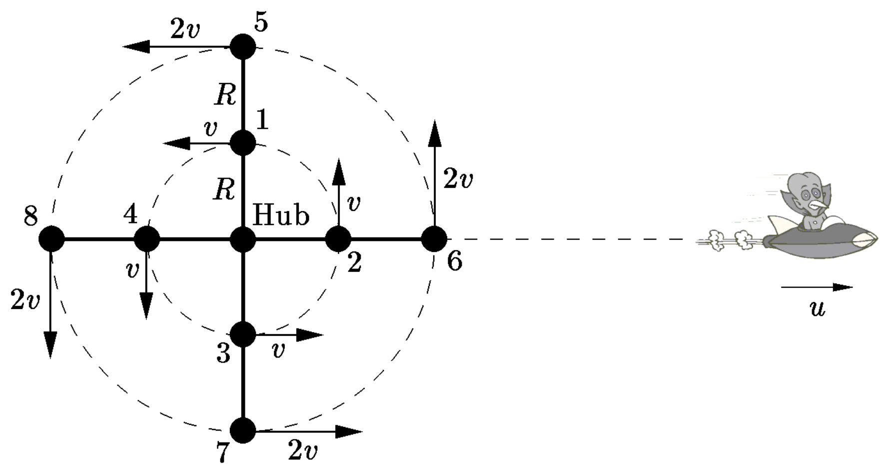
(b) *(5 points)* Let $(x, y, z, t)$ be the coordinates of a comoving coordinate system. Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive $x$-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) *(5 points)* Suppose that we are living on a galaxy along the positive $x$-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift $z$. Express the time $t_r$ at which we receive the pulse in terms of $z$, $\chi$, and any relevant physical constants.

(d) *(5 points)* At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of $z$, $\chi$, and any relevant physical constants.

**PROBLEM 3: A TWO-LEVEL HIGH-SPEED MERRY-GO-ROUND** *(15 points)*

Consider a high-speed merry-go-round which is similar to the one discussed in Problem 3 of Problem Set 1, but which has two levels. That is, there are four evenly spaced cars which travel around a central hub at speed $v$ at a distance $R$ from a central hub, and also another four cars that are attached to extensions of the four radial arms, each moving at a speed $2v$ at a distance $2R$ from the center. In this problem we will consider only light waves, not sound waves, and we will assume that $v$ is not negligible compared to $c$, but that $2v < c$.




We learned in Problem Set 1 that there is no redshift when light from one car at radius $R$ is received by an observer on another car at radius $R$.

(a) *(5 points)* Suppose that cars 5–8 are all emitting light waves in all directions. If an observer in car 1 receives light waves from each of these cars, what redshift $z$ does she observe for each of the four signals?

(b) *(10 points)* Suppose that a spaceship is receding to the right at a relativistic speed $u$ along a line through the hub, as shown in the diagram. Suppose that an observer in car 6 receives a radio signal from the spaceship, at the time when the car is in the position shown in the diagram. What redshift $z$ is observed?

**PROBLEM 4: A TOY UNIVERSE WITH MATTER AND PURPLE ENERGY** *(40 points)*

In this problem we examine the behavior of a toy-model closed universe that includes both ordinary nonrelativistic matter, plus a new form of matter which we will call purple energy. Nothing like purple energy is known or even suspected to exist, but one never knows what might someday be discovered. Dark energy, after all, was not thought to exist until 1998.

The universe will obey the usual Friedmann equation,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} ,$$

but the mass density $\rho$ will include two terms:

$$\rho(t) = \rho_m(t) + \rho_p(t) ,$$

where $\rho_m(t)$ is the mass density of normal matter,

$$\rho_m(t) = \frac{\bar{\rho}_m}{R^3(t)} ,$$

and $\rho_p(t)$ is the mass density of purple energy, given by

$$\rho_p(t) = -\frac{\bar{\rho}_p}{R^4(t)} .$$

Here $\bar{\rho}_m$ and $\bar{\rho}_p$ are positive constants, so the purple matter contributes negatively to the total mass density.

We consider a closed universe, so $k > 0$, and we define

$$\tilde{R}(t) \equiv \frac{R(t)}{\sqrt{k}} , \quad \text{and} \quad \tilde{t} \equiv ct ,$$

as in Lecture Notes 5. After some algebra which you are not asked to repeat, the Friedmann equation can be rewritten as

$$\left(\frac{d\tilde{R}}{d\tilde{t}}\right)^2 = F(\tilde{R}) , \quad \text{where} \quad F(\tilde{R}) \equiv \frac{2\alpha}{\tilde{R}} - \frac{2\beta}{\tilde{R}^2} - 1 , \tag{1}$$

and

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho_m(t)\tilde{R}^3(t)}{c^2} = \frac{4\pi}{3}\frac{G\bar{\rho}_m}{k^{3/2}c^2} > 0 ,$$

$$\beta \equiv -\frac{4\pi}{3}\frac{G\rho_p(t)\tilde{R}^4(t)}{c^2} = \frac{4\pi}{3}\frac{G\bar{\rho}_p}{k^2c^2} > 0 .$$

Note that $\alpha$ and $\beta$ are both positive and independent of time. Moreover, $\alpha$ and $\beta$ are chosen so that $F(\tilde{R}) > 0$ for some values of $\tilde{R}$.

(a) *(10 points)* Sketch a graph of the function $F(\tilde{R})$ for $\tilde{R} > 0$. How does $F(\tilde{R})$ behave for very large $\tilde{R}$? How does it behave for very small (positive) $\tilde{R}$?

(b) *(10 points)* What are the minimum and maximum values $\tilde{R}_{\rm min}$ and $\tilde{R}_{\rm max}$ that are attained by $\tilde{R}(t)$ during the evolution of the universe? Show these values on the graph that you drew in part (a), and write analytic expressions in terms of $\alpha$ and/or $\beta$.

As in Lecture Notes 5, one can write a solution to the differential equation in the form

$$\tilde{t}_f = \int_{\tilde{R}_{\rm min}}^{\tilde{R}_f} \ldots \, d\tilde{R} \ . \tag{2}$$

The integral can be carried out by a method very similar to that used in Lecture Notes 5, introducing the variable $\theta$ defined in this case by

$$\tilde{R} = \alpha - \sqrt{\alpha^2 - 2\beta}\cos\theta \ . \tag{3}$$

After some algebra (which you are not asked to reproduce), Eq. (3) leads to

$$2\alpha\tilde{R} - 2\beta - \tilde{R}^2 = (\alpha^2 - 2\beta)\sin^2\theta \ ,$$

which is useful in simplifying Eq. (2). By carrying out the integration in Eq. (2), one derives a parametric form of the solution to the Friedmann equation, which can be written as

$$\begin{aligned} ct &= \alpha(\theta - \lambda\sin\theta) \ , \\ \frac{R}{\sqrt{k}} &= \alpha(1 - \lambda'\cos\theta) \ , \end{aligned} \tag{4}$$

where $\lambda$ and $\lambda'$ are constants in the range $0 < \lambda, \lambda' < 1$.

(c) *(10 points)* Express $\lambda$ and $\lambda'$ in terms of $\alpha$ and/or $\beta$.

(d) *(5 points)* This model universe has neither a big bang nor a big crunch, but instead oscillates forever. What is the period $P$ of these oscillations? Your answer may depend upon $\lambda$ and/or $\lambda'$, as well as $\alpha$ and/or $\beta$.

(e) *(5 points)* Find the value of the Hubble parameter $H(\theta)$ as a function of $\theta$. Your answer may depend upon any of the variables listed in part (d).

## USEFUL INFORMATION:

### DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

### COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

### SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: $\gamma$

Relativity of Simultaneity:

Trailing clock reads later by an amount $\beta \ell_0 / c$ .

— *Continued on next page* —

**EVOLUTION OF A MATTER-DOMINATED UNIVERSE:**

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} \ , \quad \ddot{R} = -\frac{4\pi}{3}G\rho R \ ,$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i) \ ,$$

$$\Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat ($k = 0$):

$$R(t) \propto t^{2/3} \ ,$$

$$\Omega = 1 \ .$$

Closed ($k > 0$):

$$ct = \alpha(\theta - \sin\theta) \ , \quad \frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

$$\Omega = \frac{2}{1 + \cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$):

$$ct = \alpha\left(\sinh\theta - \theta\right) \ , \quad \frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$