MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 6, 2007
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QUIZ 2

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BLANK PAGES AND A SUMMARY OF USEFUL INFORMATION ARE AT THE END OF THE EXAM.

Problem	Maximum	Score
1	24	
2	20	
3	26	
4	30	
TOTAL	100	

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Your Name

PROBLEM 1: DID YOU DO THE READING? *(24 points)*

(a) *(6 points)* In 1948 Ralph A. Alpher and Robert Herman wrote a paper predicting a cosmic microwave background with a temperature of 5 K. The paper was based on a cosmological model that they had developed with George Gamow, in which the early universe was assumed to have been filled with hot neutrons. As the universe expanded and cooled the neutrons underwent beta decay into protons, electrons, and antineutrinos, until at some point the universe cooled enough for light elements to be synthesized. Alpher and Herman found that to account for the observed present abundances of light elements, the ratio of photons to nuclear particles must have been about 10^9. Although the predicted temperature was very close to the actual value of 2.7 K, the theory differed from our present theory in two ways. Circle the two correct statements in the following list. (3 points for each right answer; circle at most 2.)

(i) Gamow, Alpher, and Herman assumed that the neutron could decay, but now the neutron is thought to be absolutely stable.

(ii) In the current theory, the universe started with nearly equal densities of protons and neutrons, not all neutrons as Gamow, Alpher, and Herman assumed.

(iii) In the current theory, the universe started with mainly alpha particles, not all neutrons as Gamow, Alpher, and Herman assumed. (Note: an alpha particle is the nucleus of a helium atom, composed of two protons and two neutrons.)

(iv) In the current theory, the conversion of neutrons into protons (and vice versa) took place mainly through collisions with electrons, positrons, neutrinos, and antineutrinos, not through the decay of the neutrons.

(v) The ratio of photons to nuclear particles in the early universe is now believed to have been about 10^3, not 10^9 as Alpher and Herman concluded.

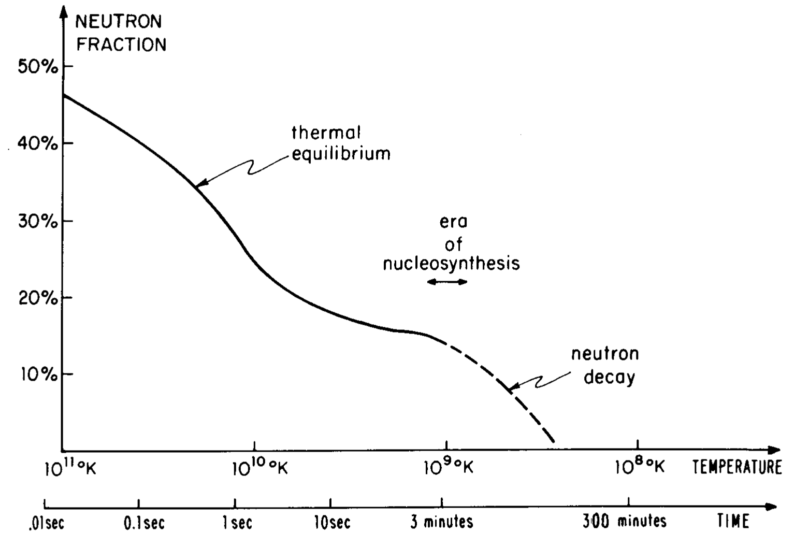
(b) *(6 points)* In Weinberg's "Recipe for a Hot Universe," he described the primordial composition of the universe in terms of three conserved quantities: electric charge, baryon number, and lepton number. If electric charge is measured in units of the electron charge, then all three quantities are integers for which the number density can be compared with the number density of photons. For each quantity, which choice most accurately describes the initial ratio of the number density of this quantity to the number density of photons:

Electric Charge: (i) $\sim 10^9$ (ii) ~ 1000 (iii) ~ 1
(iv) $\sim 10^{-6}$ (v) either zero or negligible

Baryon Number: (i) $\sim 10^{-20}$ (ii) $\sim 10^{-9}$ (iii) $\sim 10^{-6}$
(iv) ~ 1 (v) anywhere from 10^{-5} to 1

Lepton Number: (i) $\sim 10^9$ (ii) ~ 1000 (iii) ~ 1
(iv) $\sim 10^{-6}$ (v) could be as high as ~ 1, but is assumed to be very small

(c) *(12 points)* The figure below comes from Weinberg's Chapter 5, and is labeled *The Shifting Neutron-Proton Balance.*



(i) *(3 points)* During the period labeled "thermal equilibrium," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{positron} + \text{proton}.$$

(E) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{electron} + \text{proton}.$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \longleftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \longleftrightarrow \text{electron} + \text{positron}.$$

(ii) *(3 points)* During the period labeled "neutron decay," the neutron fraction is changing because (choose one):

(A) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 1 second.

(B) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 seconds.

(C) The neutron is unstable, and decays into a proton, electron, and antineutrino with a lifetime of about 15 minutes.

(D) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{electron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{positron} + \text{proton}.$$

(E) Neutrons and protons can be converted from one into the other through reactions such as

$$\text{antineutrino} + \text{proton} \longleftrightarrow \text{positron} + \text{neutron}$$
$$\text{neutrino} + \text{neutron} \longleftrightarrow \text{electron} + \text{proton}.$$

(F) Neutrons and protons can be created and destroyed by reactions such as

$$\text{proton} + \text{neutrino} \longleftrightarrow \text{positron} + \text{antineutrino}$$
$$\text{neutron} + \text{antineutrino} \longleftrightarrow \text{electron} + \text{positron}.$$

(iii) *(3 points)* The masses of the neutron and proton are not exactly equal, but instead

(A) The neutron is more massive than a proton with a rest energy difference of 1.293 GeV (1 GeV $= 10^9$ eV).

(B) The neutron is more massive than a proton with a rest energy difference of 1.293 MeV (1 MeV $= 10^6$ eV).

(C) The neutron is more massive than a proton with a rest energy difference of 1.293 KeV (1 KeV $= 10^3$ eV).

(D) The proton is more massive than a neutron with a rest energy difference of 1.293 GeV.

(E) The proton is more massive than a neutron with a rest energy difference of 1.293 MeV.

(F) The proton is more massive than a neutron with a rest energy difference of 1.293 KeV.

(iv) *(3 points)* During the period labeled "era of nucleosynthesis," (choose one:)

(A) Essentially all the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time.

(B) Essentially all the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time.

(C) About half the neutrons present combine with protons to form helium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(D) About half the neutrons present combine with protons to form deuterium nuclei, which mostly survive until the present time, and the other half of the neutrons remain free.

(E) Essentially all the protons present combine with neutrons to form helium nuclei, which mostly survive until the present time.

(F) Essentially all the protons present combine with neutrons to form deuterium nuclei, which mostly survive until the present time.

PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(20 points)*

This problem was Problem 2 of Problem Set 6.

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(10 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $R(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{R^6(t)} .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(5 points)* Find the behavior of the scale factor $R(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $R(t)$ up to a constant factor.

(c) *(5 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{2}\rho c^2 .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{R^n(t)} .$$

Find the power n.

PROBLEM 3: A NEW SPECIES OF MESON *(26 points)*

Suppose the calculations describing the early universe were modified by including an additional, hypothetical meson, called an X. The X has roughly the same properties as the pion, except that its mass is given by $mc^2 = 5$ MeV. That is, the particle is a boson, it has no spin, and it has three charge states: the X^+ has charge $Q = +e$, the X^- has charge $Q = -e$, and the X^0 has charge $Q = 0$, where e is the magnitude of the charge of an electron. There are no additional antiparticles: the X^+ is the antiparticle of the X^-, and the X^0 is its own antiparticle. The X is much heavier than the electron, which has a rest energy of 0.511 MeV, and it is much lighter than the muon, which has a rest energy of 106 MeV.

(a) *(6 points)* What would be the number density of X's, in particles per cubic meter, when the temperature T was given by $kT = 30$ MeV? Include all the X's: the X^+, the X^-, and the X^0.

(b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the value of the mass density at $t = 10^{-3}$ sec? Express your answer in the units of g-cm^{-3}.

(c) *(5 points)* What would be the value of kT, in MeV, at $t = 10^{-3}$ sec? You may assume that 5 MeV $\ll kT \ll$ 100 MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, electron-positron pairs, and the three charge states of the X.

(d) *(10 points)* When kT falls below 5 MeV, the X's will disappear from the thermal equilibrium mix. For realistic parameters the neutrinos would still be interacting significantly with the other particles at this temperature, but for the purpose of this problem we will discuss an imaginary world which has X's, and in which the neutrinos interact somewhat more weakly than in the real world. In the imaginary world the neutrinos have decoupled from the rest of matter by the time kT falls near 5 MeV, so the disappearance of the X's transfers no energy to the neutrinos. After the disappearance of the X's, the photons and the electron positron pairs will be in thermal equilibrium with each other, at a temperature that that we will call T_γ. After the disappearance of the X's, but before the disappearance of the electron-positron pairs, what is the ratio T_γ/T_ν, where T_ν is the temperature of the neutrinos?

PROBLEM 4: THE STABILITY OF SCHWARZSCHILD ORBITS *(30 points)*

This problem is an elaboration of Problem 1 of Problem Set 5, for which both the statement and the solution are reproduced at the end of this quiz. This material is reproduced for your reference, but you should be aware that the solution to the present problem has important differences. You can copy from this material, but to allow the grader to assess your understanding, you are expected to present a logical, self-contained answer to this question.

In the solution to that homework problem, it was stated that further analysis of the orbits in a Schwarzschild geometry shows that the smallest *stable* circular orbit occurs for $r = 3R_S$. Circular orbits are possible for $\frac{3}{2}R_S < r < 3R_S$, but they are not stable. In this problem we will explore the calculations behind this statement.

We will consider a body which undergoes small oscillations about a circular orbit at $r(t) = r_0$, $\theta = \pi/2$, where r_0 is a constant. The coordinate θ will therefore be fixed, but all the other coordinates will vary as the body follows its orbit.

(a) *(12 points)* The first step, since $r(\tau)$ will not be a constant in this solution, will be to derive the equation of motion for $r(\tau)$. That is, for the Schwarzschild metric

$$ds^2 = -c^2 d\tau^2 = -h(r)c^2 dt^2 + h(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ , \qquad (1)$$

where

$$h(r) \equiv 1 - \frac{R_S}{r} \ ,$$

work out the explicit form of the geodesic equation

$$\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \ , \qquad (2)$$

for the case $\mu = r$. You should use this result to find an explicit expression for

$$\frac{d^2 r}{d\tau^2} \ .$$

You may allow your answer to contain $h(r)$, its derivative $h'(r)$ with respect to r, and the derivative with respect to τ of any coordinate, including $dt/d\tau$.

— *Problem 4 Continues* —

(b) *(6 points)* It is useful to consider r and ϕ to be the independent variables, while treating t as a dependent variable. Find an expression for

$$\left(\frac{dt}{d\tau}\right)^2$$

in terms of r, $dr/d\tau$, $d\phi/d\tau$, $h(r)$, and c. Use this equation to simplify the expression for $d^2r/d\tau^2$ obtained in part (a). The goal is to obtain an expression of the form

$$\frac{d^2r}{d\tau^2} = f_0(r) + f_1(r)\left(\frac{d\phi}{d\tau}\right)^2 . \tag{3}$$

where the functions $f_0(r)$ and $f_1(r)$ might depend on R_S or c, and might be positive, negative, or zero. Note that the intermediate steps in the calculation involve a term proportional to $(dr/d\tau)^2$, but the net coefficient for this term vanishes.

(c) *(7 points)* To understand the orbit we will also need the equation of motion for ϕ. Evaluate the geodesic equation (2) for $\mu = \phi$, and write the result in terms of the quantity L, defined by

$$L \equiv r^2 \frac{d\phi}{d\tau} . \tag{4}$$

(d) *(5 points)* Finally, we come to the question of stability. Substituting Eq. (4) into Eq. (3), the equation of motion for r can be written as

$$\frac{d^2r}{d\tau^2} = f_0(r) + f_1(r)\frac{L^2}{r^4} .$$

Now consider a small perturbation about the circular orbit at $r = r_0$, and write an equation that determines the stability of the orbit. (That is, if some external force gives the orbiting body a small kick in the radial direction, how can you determine whether the perturbation will lead to stable oscillations, or whether it will start to grow?) You should express the stability requirement in terms of the unspecified functions $f_0(r)$ and $f_1(r)$. You are NOT asked to carry out the algebra of inserting the explicit forms that you have found for these functions.

USEFUL INFORMATION:

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta\ell_0/c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) , \quad \vec{p} = \gamma m_0 \vec{v} , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0c^2\right)^2 + |\vec{p}|^2 c^2} ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 .$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} , \quad \ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R ,$$

$$\rho_m(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho_m(t_i) \ \text{(matter)}, \quad \rho_r(t) = \frac{R^4(t_i)}{R^4(t)}\,\rho_r(t_i) \ \text{(radiation)}.$$

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat ($k = 0$): $R(t) \propto t^{2/3}$ (matter-dominated) ,

$R(t) \propto t^{1/2}$ (radiation-dominated) ,

$\Omega = 1$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed ($k > 0$):

$$ct = \alpha(\theta - \sin\theta) \ , \quad \frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$

$$\Omega = \frac{2}{1 + \cos\theta} > 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$):

$$ct = \alpha\left(\sinh\theta - \theta\right) \ , \quad \frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$

$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{\kappa}}\right)^3 \ ,$$

$$\kappa \equiv -k > 0 \ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\frac{k^4T^3}{(\hbar c)^3} \ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions} \ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 \ .$$

$$g_\gamma = g^*_\gamma = 2 \ ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} \ ,$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2} \ ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \ ,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3 \ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

HORIZON DISTANCE:

$$\ell_{p,\text{horizon}}(t) = R(t)\int_0^t \frac{c}{R(t')}dt'$$

$$= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).} \end{cases}$$

PHYSICAL CONSTANTS:

$$G = 6.673 \times 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16} \text{ erg/K}$$

$$= 8.617 \times 10^{-5} \text{ eV/K}$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27} \text{ erg-s}$$

$$= 6.582 \times 10^{-16} \text{ eV-s}$$

$$c = 2.998 \times 10^{10} \text{ cm/s}$$

$$1 \text{ yr} = 3.156 \times 10^7 \text{ s}$$

$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg}$$

$$1 \text{ GeV} = 10^9 \text{ eV} = 1.783 \times 10^{-24} \text{ gram (where } c \equiv 1) .$$