MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 6, 2007

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QUIZ 3

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BLANK PAGES AND A SUMMARY OF USEFUL INFORMATION ARE AT THE END OF THE EXAM.

Your Name

Problem	Maximum	Score
1	25	
2	35	
3	40	
TOTAL	100	

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following parts are each worth 5 points.

(a) (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

(ii) After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

(b) (CMB experiments) The current mean energy per CMB photon, about 6×10^{-4} eV, is comparable to the energy of vibration or rotation for a small molecule such as H_2O. Thus microwaves with wavelengths shorter than $\lambda \sim 3$ cm are strongly absorbed by water molecules in the atmosphere. To measure the CMB at $\lambda < 3$ cm, which one of the following methods is *not* a feasible solution to this problem?

(i) Measure CMB from high-altitude balloons, e.g. MAXIMA.

(ii) Measure CMB from the South Pole, e.g. DASI.

(iii) Measure CMB from the North Pole, e.g. BOOMERANG.

(iv) Measure CMB from a satellite above the atmosphere of the Earth, e.g. COBE, WMAP and PLANCK.

(c) (Temperature fluctuations) The creation of temperature fluctuations in CMB by variations in the gravitational potential is known as the Sachs-Wolfe effect. Which one of the following statements is *not* correct concerning this effect?

(i) A CMB photon is redshifted when climbing out of a gravitational potential well, and is blueshifted when falling down a potential hill.

(ii) At the time of last scattering, the nonbaryonic dark matter dominated the energy density, and hence the gravitational potential, of the universe.

(iii) The large-scale fluctuations in CMB temperatures arise from the gravitational effect of primordial density fluctuations in the distribution of nonbaryonic dark matter.

(iv) The peaks in the plot of temperature fluctuation Δ_T vs. multipole l are due to variations in the density of nonbaryonic dark matter, while the contributions from baryons alone would not show such peaks.

(d) (Dark matter candidates) Which one of the following is *not* a candidate of nonbaryonic dark matter?

(i) massive neutrinos

(ii) axions

(iii) matter made of top quarks (a type of quarks with heavy mass of about 171 GeV).

(iv) WIMPs (Weakly Interacting Massive Particles)

(v) primordial black holes

(e) (Signatures of dark matter) By what methods can signatures of dark matter be detected? List two methods. (Grading: 3 points for one correct answer, 5 points for two correct answers. If you give more than two answers, your score will be based on the number of right answers minus the number of wrong answers, with a lower bound of zero.)

PROBLEM 2: NEUTRINO NUMBER AND THE NEUTRON/PROTON EQUILIBRIUM *(35 points)*

The following problem was Problem 7 of Review Problems for Quiz 3.

In the standard treatment of big bang nucleosynthesis it is assumed that at early times the ratio of neutrons to protons is given by the Boltzmann formula,

$$\frac{n_n}{n_p} = e^{-\Delta E/kT} \ , \tag{1}$$

where k is Boltzmann's constant, T is the temperature, and $\Delta E = 1.29$ MeV is the proton-neutron mass-energy difference. This formula is believed to be very accurate, but it assumes that the chemical potential for neutrons μ_n is the same as the chemical potential for protons μ_p.

(a) *(10 points)* Give the correct version of Eq. (1), allowing for the possibility that $\mu_n \neq \mu_p$.

The equilibrium between protons and neutrons in the early universe is sustained mainly by the following reactions:

$$e^+ + n \longleftrightarrow p + \bar{\nu}_e$$
$$\nu_e + n \longleftrightarrow p + e^- \ .$$

Let μ_e and μ_ν denote the chemical potentials for the electrons (e^-) and the electron neutrinos (ν_e) respectively. The chemical potentials for the positrons (e^+) and the anti-electron neutrinos ($\bar{\nu}_e$) are then $-\mu_e$ and $-\mu_\nu$, respectively, since the chemical potential of a particle is always the negative of the chemical potential for the antiparticle.*

(b) *(10 points)* Express the neutron/proton chemical potential difference $\mu_n - \mu_p$ in terms of μ_e and μ_ν.

The black-body radiation formulas on the formula sheet do not allow for the possibility of a chemical potential, but they can easily be generalized. For example, the formula for the number density n_i (of particles of type i) becomes

$$n_i = g_i^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{\mu_i/kT} \ .$$

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* This fact is a consequence of the principle that the chemical potential of a particle is the sum of the chemical potentials associated with its conserved quantities, while particle and antiparticle always have the opposite values of all conserved quantities.

(c) *(10 points)* Suppose that the density of anti-electron neutrinos $\bar{n}_\nu$ in the early universe was higher than the density of electron neutrinos n_ν. Express the thermal equilibrium value of the ratio n_n/n_p in terms of ΔE, T, and the antineutrino to neutrino ratio $\bar{n}_\nu/n_\nu$. Assume that the number density of positrons is equal to that of electrons. (Your answer may also contain fundamental constants, such as k, $\hbar$, and c. In the Review Problems for Quiz 3 you were asked to express the answer in terms of the antineutrino excess $\Delta n = \bar{n}_\nu - n_\nu$. It is easier to express the answer in terms of the ratio $\bar{n}_\nu/n_\nu$, but if you prefer to express your answer in terms of Δn, that would also be acceptable.)

(d) *(5 points)* Would an excess of anti-electron neutrinos, as considered in part (c), increase or decrease the amount of helium that would be produced in the early universe? Explain your answer.

PROBLEM 3: SECOND HUBBLE CROSSING *(40 points)*

In Problem Set 9 we calculated the time $t_{H1}(\lambda)$ of the first Hubble crossing for a mode specified by its (physical) wavelength λ at the present time. In this problem we will calculate the time $t_{H2}(\lambda)$ of the second Hubble crossing, the time at which the growing Hubble length $cH^{-1}(t)$ catches up to the physical wavelength, which is also growing. At the time of the second Hubble crossing for the wavelengths of interest, the universe can be described very simply: it is a radiation-dominated flat universe. However, since λ is defined as the present value of the wavelength, the evolution of the universe between $t_{H2}(\lambda)$ and the present will also be relevant to the problem. We will need to use methods, therefore, that allow for both the matter-dominated era and the onset of the dark-energy-dominated era. As in Problem Set 9, the model universe that we consider will be described by the WMAP 3-year best fit parameters:

Hubble expansion rate	H_0	$= 73.5 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$
Nonrelativistic mass density	Ω_m	$= 0.237$
Vacuum mass density	Ω_{vac}	$= 0.763$
CMB temperature	$T_{\gamma,0}$	$= 2.725$ K

The mass densities are defined as contributions to Ω, and hence describe the mass density of each constituent relative to the critical density. Note that the model is exactly flat, so you need not worry about spatial curvature. Here you are not expected to give a numerical answer, so the above list will serve only to define the symbols that can appear in your answers, along with λ and the physical constants G, $\hbar$, c, and k.

(a) *(5 points)* For a radiation-dominated flat universe, what is the Hubble length $\ell_H(t) \equiv cH^{-1}(t)$ as a function of time t?

(b) *(10 points)* The second Hubble crossing will occur during the interval

$$1 \text{ sec} \ll t \ll 50{,}000 \text{ years},$$

when the mass density of the universe is dominated by photons and neutrinos. During this era the neutrinos are a little colder than the photons, with $T_\nu = (4/11)^{1/3} T_\gamma$. The total energy density of the photons and neutrinos together can be written as

$$u_{\text{tot}} = g_1 \frac{\pi^2}{30} \frac{(kT_\gamma)^4}{(\hbar c)^3} \ .$$

What is the value of g_1? (For the following parts you can treat g_1 as a given variable that can be left in your answers, whether or not you found it.)

(c) *(10 points)* For times in the range described in part (b), what is the photon temperature $T_\gamma(t)$ as a function of t?

(d) *(15 points)* Finally, we are ready to find the time $t_{H2}(\lambda)$ of the second Hubble crossing, for a given value of the physical wavelength λ today. Making use of the previous results, you should be able to determine $t_{H2}(\lambda)$. If you were not able to answer some of the previous parts, you may leave the symbols $\ell_H(t)$, g_1, and/or $T_\gamma(t)$ in your answer.

USEFUL INFORMATION:

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) , \quad \vec{p} = \gamma m_0 \vec{v} , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec{p}|^2 c^2} ,$$

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 .$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} , \quad \ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R ,$$

$$\rho_m(t) = \frac{R^3(t_i)}{R^3(t)} \rho_m(t_i) \ \text{(matter)}, \quad \rho_r(t) = \frac{R^4(t_i)}{R^4(t)} \rho_r(t_i) \ \text{(radiation)}.$$

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \ , \quad \Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat ($k = 0$):

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)} \ ,$$
$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)} \ ,$$
$$\Omega = 1 \ .$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed ($k > 0$):

$$ct = \alpha(\theta - \sin\theta) \ , \quad \frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) \ ,$$
$$\Omega = \frac{2}{1 + \cos\theta} > 1 \ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{k}}\right)^3 \ .$$

Open ($k < 0$):

$$ct = \alpha\left(\sinh\theta - \theta\right) \ , \quad \frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$
$$\Omega = \frac{2}{1 + \cosh\theta} < 1 \ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho}{c^2}\left(\frac{R}{\sqrt{\kappa}}\right)^3 \ ,$$
$$\kappa \equiv -k > 0 \ .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\,\frac{dx^\ell}{ds}$$

$$\text{or:}\qquad \frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

$$g_\gamma = g_\gamma^* = 2 \; ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} \; ,$$

$$g_\nu^* = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2} \; ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} \; ,$$

$$g_{e^+e^-}^* = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3 \; .$$

CHEMICAL EQUILIBRIUM:

Ideal Gas of Classical Nonrelativistic Particles:

$$n_i = g_i \frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} \, e^{(\mu_i - m_i c^2)/kT} \; .$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} \left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

HORIZON DISTANCE:

$$\ell_{p,\text{horizon}}(t) = R(t) \int_0^t \frac{c}{R(t')} dt'$$

$$= \begin{cases} 3ct & \text{(flat, matter-dominated),} \\ 2ct & \text{(flat, radiation-dominated).} \end{cases}$$

COSMOLOGICAL CONSTANT:

$$u_{\text{vac}} = \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} ,$$

$$p_{\text{vac}} = -\rho_{\text{vac}} c^2 = -\frac{\Lambda c^4}{8\pi G} .$$

GENERALIZED COSMOLOGICAL EVOLUTION:

$$x \frac{dx}{dt} = H_0 \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4 + \Omega_{k,0} x^2} ,$$

where

$$x \equiv \frac{R(t)}{R(t_0)} \equiv \frac{1}{1+z} ,$$

$$\Omega_{k,0} \equiv -\frac{kc^2}{R^2(t_0) H_0^2} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0} .$$

Age of universe:

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{x\,dx}{\sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4 + \Omega_{k,0} x^2}}$$

$$= \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} .$$

Look-back time:

$$t_{\text{look-back}}(z) =$$
$$\frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3+\Omega_{\text{rad},0}(1+z')^4+\Omega_{\text{vac},0}+\Omega_{k,0}(1+z')^2}} \; .$$

PHYSICAL CONSTANTS:

$$G = 6.673\times 10^{-8}\ \text{cm}^3\cdot\text{g}^{-1}\cdot\text{s}^{-2}$$
$$k = \text{Boltzmann's constant} = 1.381\times 10^{-16}\,\text{erg/K}$$
$$= 8.617\times 10^{-5}\,\text{eV/K}$$
$$\hbar = \frac{h}{2\pi} = 1.055\times 10^{-27}\ \text{erg-s}$$
$$= 6.582\times 10^{-16}\ \text{eV-s}$$
$$c = 2.998\times 10^{10}\ \text{cm/s}$$
$$\hbar c = 197.3\ \text{MeV-fm},\quad 1\ \text{fm} = 10^{-15}\ \text{m}$$
$$1\ \text{yr} = 3.156\times 10^{7}\ \text{s}$$
$$1\ \text{eV} = 1.602\times 10^{-12}\ \text{erg}$$
$$1\ \text{GeV} = 10^9\ \text{eV} = 1.783\times 10^{-24}\ \text{gram (where } c\equiv 1) \; .$$

Planck Units: The Planck length ℓ_P, the Planck time t_P, the Planck mass m_P, and the Planck energy E_p are given by

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.616\times 10^{-33}\,\text{cm}\,,$$
$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.391\times 10^{-44}\,\text{s}\,,$$
$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.177\times 10^{-5}\,\text{g}\,,$$
$$E_P = \sqrt{\frac{\hbar c^5}{G}} = 1.221\times 10^{19}\,\text{GeV}\,.$$