Physics Department

Physics 8.286: The Early Universe

September 30, 2009

Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 1

QUIZ DATE: Tuesday, October 6, 2009, during the normal class time.

QUIZ COVERAGE: Lecture Notes 1 (sections on the Doppler shift only); Lecture Notes 3 and 4; Problem Sets 1, 2, and 3; Weinberg, Chapters 1-3, Ryden, Chapters 1 and 2, and Section 3.1. (The starred problems are intended to help you understand the lecture material. There will be no quiz questions based specifically on this material.)

One of the problems on the quiz will be taken verbatim (or at least almost verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems. The starred problems are the ones that I recommend that you review most carefully: Problems 2, 4, 7, 12, 14, 16, 18, and 20. The starred problems do not include any reading questions, but parts of the reading questions in these Review Problems may also recur on the upcoming quiz.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. Except for a few parts which are clearly marked, I would consider any problem that you solve on your own to be fair game. If you solved a problem that I marked with an asterisk, I would not consider you to have cheated, even if subsequent problems show that the same solution was correct. The asterisked problems are those whose solutions are similar to those on the quiz and whose solutions I have given due consideration. The points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, and 2007. The relevant problems from those quizzes may also be fair game, though the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, 2005, and 2007 are not intended to be fair. Problem 4 of the 2007 Quiz 1, for example, would not be a fair problem this year.

REVIEW SESSION: To help you study for the quiz, Leo Stein will hold a review session on Sunday, October 4, at 7:30 p.m. The location will be announced on the course web site.

FUTURE QUIZZES: The other quiz dates this term will be Thursday November 5, and Thursday December 3.

INFORMATION TO BE GIVEN ON QUIZ:

Each quiz in this course will have a section of "useful information" at the beginning. For the first quiz, this useful information will be the following:

DOPPLER SHIFT (For motion along a line):

\[
\frac{\nu}{\nu_{\text{source}}} = \frac{1}{1 - \frac{v}{c}}, \quad \text{(nonrelativistic, observer moving)}
\]

\[
\frac{\nu}{\nu_{\text{source}}} = \frac{1}{1 - \frac{v}{c} \sqrt{1 - \frac{v^2}{c^2}}}, \quad \text{(special relativity)}
\]

COSMOLOGICAL REDSHIFT:

\[
1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}
\]

SPECIAL RELATIVITY:

Time Dilation Factor:

\[
\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta \equiv \frac{v}{c}
\]

Lorentz-Fitzgerald Contraction Factor:

\[
\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}
\]

Relativity of Simultaneity:

Trailing clock reads later by an amount \(\beta \triangleq \frac{c}{\gamma c} \),

E VOLUTION OF A M A TTER-DOMINATED UNIVERSE:

\[
H^2 = \frac{\dot{a}}{a^2} = \frac{8\pi G \rho_{\text{matter}}}{3}, \quad \rho_{\text{matter}}(t) = \frac{a(t)^3}{a(t_i)^3} \rho_{\text{matter}}(t_i)
\]

\[
\Omega \equiv \frac{\rho_{\text{matter}}}{\rho_c}, \quad \rho_c \equiv \frac{3H^2}{8\pi G}.
\]

Flat (\(k = 0\)): \(a(t) \propto t^{2/3}\), \(\Omega = 1\).
PROBLEM 1: DID YOU DO THE READING?

(35 points)

The following problem was Problem 1, Quiz 1, 2000. The parts were each worth 5

(a) 10 points) The Doppler effect proposes that the Hubble constant, which

is a number that represents the rate of expansion of the universe,

(b) 3 points) The Doppler effect for both sound and light waves is

(c) 2 points) The Doppler effect for light waves is

(d) 1 point) the Doppler effect for light waves is

(e) 1 point) the Doppler effect for light waves is

(f) 1 point) the Doppler effect for light waves is

(g) 1 point) the Doppler effect for light waves is

(h) 1 point) the Doppler effect for light waves is

(i) 1 point) the Doppler effect for light waves is

(j) 1 point) the Doppler effect for light waves is

(k) 1 point) the Doppler effect for light waves is

(l) 1 point) the Doppler effect for light waves is

(m) 1 point) the Doppler effect for light waves is

(n) 1 point) the Doppler effect for light waves is

(o) 1 point) the Doppler effect for light waves is

(p) 1 point) the Doppler effect for light waves is

(q) 1 point) the Doppler effect for light waves is

(r) 1 point) the Doppler effect for light waves is

(s) 1 point) the Doppler effect for light waves is

(t) 1 point) the Doppler effect for light waves is

(u) 1 point) the Doppler effect for light waves is

(v) 1 point) the Doppler effect for light waves is

(w) 1 point) the Doppler effect for light waves is

(x) 1 point) the Doppler effect for light waves is

(y) 1 point) the Doppler effect for light waves is

(z) 1 point) the Doppler effect for light waves is

(1) 1 point) the Doppler effect for light waves is

(2) 1 point) the Doppler effect for light waves is

(3) 1 point) the Doppler effect for light waves is

(4) 1 point) the Doppler effect for light waves is

(5) 1 point) the Doppler effect for light waves is

(6) 1 point) the Doppler effect for light waves is

(7) 1 point) the Doppler effect for light waves is

(8) 1 point) the Doppler effect for light waves is

(9) 1 point) the Doppler effect for light waves is

(10) 1 point) the Doppler effect for light waves is

1. 35 points)

2. 25 points)

3. 20 points)

4. 15 points)

5. 10 points)

6. 5 points)

7. 2 points)

8. 1 point)

9. 1 point)

10. 1 point)
Problem 3: Did you do the reading?

(25 points)

The following problem was Problem 1 on Quiz 1, 2007, where each of the five questions was worth 5 points:

(a) In the 1940’s, three astrophysicists proposed a “steady state” theory of cosmology, in which the universe has always looked about the same as it does now. State the last name of at least one of these authors. (Bonus points: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)

(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:

(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.
(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.
(iii) published a catalog, Nebulae and Star Clusters, listing 103 objects that astronomers should avoid when looking for comets.
(iv) published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.
(v) discovered that the orbital periods of the planets are proportional to the 3/2 power of the semi-major axis of their elliptical orbits.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted by a group of physicists at a neighboring institution as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were not part of the story behind this spectacular discovery:

(i) Bell Telephone Laboratory (ii) MIT (iii) Princeton University (iv) pigeons (v) groundhogs (vi) Hubble’s constant

(vii) liquid helium (viii) 7.35 cm

(Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer.)

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for rotation about the Sun) and by the behavior of the Foucault pendulum (which showed that the Earth rotates). These discoveries were made

(i) during Copernicus’ lifetime.
(ii) approximately two and three decades after Copernicus’ death, respectively.
(iii) about one hundred years after Copernicus’ death.
(iv) approximately two and three centuries after Copernicus’ death, respectively.

(e) If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this is true?

(i) 1 AU (1 AU = 1.496 × 10^11 m).
(ii) 100 kpc (1 kpc = 1000 pc, 1 pc = 3.086 × 10^16 m = 3.262 light-year).
(iii) 1 Mpc (1 Mpc = 10^6 pc).
(iv) 10 Mpc.
(v) 100 Mpc.
(vi) 1000 Mpc.

*Problem 4: An exponentially expanding universe

(20 points)

The following problem was Problem 2, Quiz 2, 1994, and had also appeared on the 1994 Review Problems. As is the case this year, it was announced that one of the problems on the quiz would come from either the homework or the Review Problems.

Consider a flat (i.e., a k = 0, or a Euclidean) universe with scale factor given by

a(t) = a_0 e^{\chi t},

where we note the origin of the story behind the spectacular discovery:

that were not part of the story behind the Copernican theory; clear the two lines on the Copernican List of Important Predictions of the Copernican Theory were confirmed by the discovery. It must not include the question for any topic that was not included on the 1994 Review Problems. Your answer will consist of the universe’s name and the line that corresponds to it.

In part (a), describe the idea of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (b), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (c), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (d), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (e), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (f), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (g), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (h), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (i), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (j), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (k), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (l), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (m), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (n), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (o), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (p), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter. In part (q), describe the results of which we must start with the Copernican List of Important Predictions of the Copernican Theory, which was later discovered by the famous astronomer named William de Sitter.
where $a_0$ and $\chi$ are constants.

(a) (5 points) Find the Hubble constant $H$ at an arbitrary time $t$.

(b) (5 points) Let $(x,y,z,t)$ be the coordinates of a comoving coordinate system.
Suppose that at $t = 0$ a galaxy located at the origin of this system emits a light pulse along the positive $x$-axis. Find the trajectory $x(t)$ which the light pulse follows.

(c) (5 points) Suppose that we are living on a galaxy along the positive $x$-axis, and that we receive this light pulse at some later time. We analyze the spectrum of the pulse and determine the redshift $z$. Express the time $t$ at which we receive the pulse in terms of $z$, $\chi$, and any relevant physical constants.

(d) (5 points) At the time of reception, what is the physical distance between our galaxy and the galaxy which emitted the pulse? Express your answer in terms of $z$, $\chi$, and any relevant physical constants.

PROBLEM 5: "DID YOU DO THE READING?"

(a) The assumptions of homogeneity and isotropy greatly simplify the description of our universe. We find that there are three possibilities for a homogeneous and isotropic universe: an open universe, a flat universe, and a closed universe. What quantity or condition distinguishes between these three cases: the temperature of the microwave background, the value of $\Omega = \rho/\rho_c$, matter vs. radiation domination, or redshift?

(b) What is the temperature, in Kelvin, of the cosmic microwave background today?

(c) Which of the following supports the hypothesis that the universe is isotropic: the distances to nearby clusters, observations of the cosmic microwave background, clustering of galaxies on large scales, or the age and distribution of globular clusters?

(d) Is the distance to the Andromeda Nebula (roughly) 10 kpc, 5 billion light years, 2 million light years, or 3 light years?

(e) Did Hubble discover the law which bears his name in 1862, 1880, 1906, 1929, or 1948?

(f) When Hubble measured the value of his constant, he found $H^{-1} \approx 100$ million years $2billion$ years, $10 billion years, or $2 \times 10^{10}$ billion years?

(g) Cepheid variables are important to cosmology because they can be used to estimate the distances to the nearby galaxies. What property of Cepheid variables makes them useful for this purpose, and how are they used?

(h) Cepheid variable stars can be used as estimators of distance for distances up to about 100 light-years, $10^4$ light-years, $10^7$ light years, or $10^{10}$ light-years?

[Note for 2009: this question was based on the reading from Joseph Silk's The Big Bang, and therefore would not be a fair question for this year.]

The following problem was Problem 3, Quiz 2, 1989.

PROBLEM 6: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

Consider a flat universe filled with a new and peculiar form of matter, with a Robertson–Walker scale factor that behaves as $a(t) = bt^{1/3}$. Here $b$ denotes a constant.

(a) If a light pulse is emitted at time $t_e$ and observed at time $t_o$, find the physical separation $\ell_p(t_o)$ between the emitter and the observer, at the time of observation.

(b) Again assuming that $t_e$ and $t_o$ are given, find the observed redshift $z$.

(c) Find the physical distance $\ell_p(t_o)$ which separates the emitter and observer at the time of observation, expressed in terms of $c$, $t_o$, and $z$ (i.e., without $t_e$ appearing).

(d) At an arbitrary time $t$ in the interval $t_e < t < t_o$, find the physical distance $\ell_p(t)$ between the light pulse and the observer. Express your answer in terms of $c$, $t_o$, and $z$.
Problem 8: Did you do the READING?

The following problem was Problem 1, Quiz 1, 1996:

\( a = b \)

The following problem was Problem 1, Quiz 3, 2000:

\( a = b \)
Consider a flat universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as 

\[ a(t) = \frac{t}{t_a} \]

where \( t_a \) is a constant. From Problem 3, Quiz 1, 1996:

1. What is the speed of B relative to A at the time \( t \)?
2. Suppose a light pulse leaves galaxy A at time \( t \). At what time does this second pulse arrive at galaxy A?
3. What is the physical horizon distance at time \( t \)?
4. What is the physical separation between galaxy A and galaxy B at the time \( t \)?
5. What is the coordinate distance between these two galaxies?
6. Suppose a light pulse leaves galaxy A at time \( t \). At what time is the light pulse equidistant from the two galaxies?

The following was Problem 3, Quiz 1, 1998:

a) Suppose a message is transmitted by radio signal (traveling at the speed of light) from galaxy A to galaxy B. The message is sent at cosmic speed of light \( c \) with total power \( P \). What is the power per area received at galaxy B?

b) Suppose the radiation from galaxy A is emitted at an arbitrary time \( t \). At what time is the photon detected at galaxy B?

c) When the light pulse is received by galaxy B, a pulse is immediately sent back toward galaxy A. At what time does this second pulse arrive at galaxy A?

d) When the light pulse is recorded by galaxy A, \( P \) is diminished by half. What is the power per area recorded at galaxy B?

PROBLEM 11: ANOTHER FLAT UNIVERSE WITH A NON-CONSTANT COEFFICIENT

Consider a flat universe which is filled with some peculiar form of matter, so

\[ H(t) = \frac{1}{t} \]

where \( t \) is the cosmic time. A between the galaxies is \( \ell \). The message is sent at the speed of light \( c \), and the coordinate distance between galaxies A and B is \( d \). What is the power per area received at galaxy B?

PROBLEM 10: DID YOU DO THE READING?

Consider a flat universe which is filled with some peculiar form of matter, so

\[ H(t) = \frac{1}{t} \]

where \( t \) is the cosmic time. A between the galaxies is \( \ell \). The message is sent at the speed of light \( c \), and the coordinate distance between galaxies A and B is \( d \). What is the power per area received at galaxy B?

PROBLEM 9: A FLAT UNIVERSE WITH A COSMOLOGICAL CONSTANT

Consider a flat universe which is filled with some peculiar form of matter, so

\[ H(t) = \frac{1}{t} \]

where \( t \) is the cosmic time. A between the galaxies is \( \ell \). The message is sent at the speed of light \( c \), and the coordinate distance between galaxies A and B is \( d \). What is the power per area received at galaxy B?

PROBLEM 8: FLAT FLUX UNIVERSE WITH A NON-CONSTANT COEFFICIENT

Consider a flat universe which is filled with some peculiar form of matter, so

\[ H(t) = \frac{1}{t} \]

where \( t \) is the cosmic time. A between the galaxies is \( \ell \). The message is sent at the speed of light \( c \), and the coordinate distance between galaxies A and B is \( d \). What is the power per area received at galaxy B?
The following problem was Problem 2, Quiz 2, 1992, where it counted 10 points out of 100.

**Law of Gravitation (55 points)**

**Problem 14: A Possible Modification of Newton's Law of Gravity**

In Lecture Notes 4 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density \( \rho_0 \). We denoted the radius at time \( t \) as \( r(t) \), and the expansion parameter \( \dot{a} \) is proportional to \( t^{3/2} \). How does \( a(t) \) behave for a photon-dominated universe? Do we have a matter-dominated homogeneous and isotropic universe?

**Problem 13: A Radiation- Dominated Planar Universe**

In this section, we developed a Newtonian model of cosmology by considering a uniform sphere of mass, centered at the origin, with initial mass density \( \rho_0 \). We denoted the radius at time \( t \) as \( r(t) \), and the expansion parameter \( \dot{a} \) is proportional to \( t^{3/2} \). How does \( a(t) \) behave for a photon-dominated universe? Do we have a matter-dominated homogeneous and isotropic universe?
Consider the Doppler shift of radio waves, for a case in which both the source and the observer are moving. Suppose the source is a spaceship moving with a speed $v$ relative to the source spaceship as it would be measured by the observer spaceship. (Recall that radio waves are electromagnetic waves, just like light except that the wavelength is longer.)

The following problem was taken from Problem 2, Quiz 1, 2004, where it counted 8 points.

(a) Find the total mass contained initially in the region $r < r_i$.

(b) Suppose that the law of gravity is modified to contain a new, repulsive term, $\frac{\gamma}{r^2}$, with $\gamma > 0$. Show that the differential equation found in (a) remains valid for this new physically possible situation, with $\gamma > 0$. What does this say about the fate of the universe?

(c) If all is going well, then you have learned that for a certain value of the power $n$, the differential equation found in (a), uniquely determines the function $\gamma(r)$. What is the cosmological interpretation of this value of $n$?

(d) The Planck length is of the order of $10^{-35}$ m, the shortest length that is independent of the mass. That is, suppose $\ell_P$ is given by

$$\ell_P = \frac{h}{m c}$$

where $m$ is the mass of the gravitating object $m$. The gravitational force is zero if $r > \ell_P$. This suggests that the total mass contained initially in the region $r > \ell_P$ is

$$M(\ell_P) = \int_{\ell_P}^{\infty} \rho(r) 4\pi r^2 dr$$

where $\rho(r)$ denotes the total mass contained initially in the region $r$. What is the significance of this result?
PROBLEM 17: DID YOU DO THE READING?

(a) (4 points) What was the first external galaxy that was shown to be at a distance significantly greater than the most distant known objects in our galaxy? How was the distance estimated?

(b) (5 points) What is recombination? Did galaxies begin to form before or after recombination? Why?

(c) (4 points) In Chapter IV of his book, Weinberg develops a “recipe for a hot universe,” in which the matter of the universe is described as a gas in thermal equilibrium at a very high temperature, in the vicinity of \(10^9\) K (several thousand million degrees Kelvin). Such a thermal equilibrium gas is completely described by specifying its temperature and the density of the conserved quantities. Which of the following is on this list of conserved quantities? Circle as many as apply.

(i) baryon number
(ii) energy per particle
(iii) proton number
(iv) electric charge
(v) pressure

(d) (4 points) The wavelength corresponding to the mean energy of a CMB (cosmic microwave background) photon today is approximately equal to which of the following quantities? (You may wish to look up the values of various physical constants at the end of the quiz.)

(i) \(2\times10^{-15}\) m
(ii) \(2\times10^{-16}\) m
(iii) \(2\times10^{-3}\) m
(iv) 2 m

(e) (4 points) What is the equivalence principle?

(f) (4 points) Why is it difficult for Earth-based experiments to look at the small wavelength portion of the graph of CMB energy density per wavelength vs. wavelength?

PROBLEM 18: TRACING A LIGHT PULSE THROUGH A RADIATION-DOMINATED UNIVERSE

Consider a flat universe that expands with a scale factor:

\[ a(t) = bt^{1/2}, \]

where \(b\) is a constant. We will learn later that this is the behavior of the scale factor for a radiation-dominated universe.

(a) (5 points) At an arbitrary time \(t_f\), what is the physical horizon distance? (By “physical,” I mean the physical distance of the photon from the origin at that time, not the distance as measured by a sequence of rulers, each of which is at rest with respect to some reference frame, such as a distant galaxy or a black hole.)

(b) (3 points) Suppose that a photon arrives at the origin at time \(t_f\), which is the horizon distance at time \(t_f\). What is the time \(t_e\) at which the photon was emitted?

(c) (2 points) What is the coordinate distance from the origin to the point from which the photon was emitted?

(d) (10 points) For an arbitrary time \(t\) in the interval \(0 \leq t \leq t_f\), while the photon is traveling, what is the physical distance \(\ell_p(t)\) from the origin to the location of the photon?

(e) (5 points) At what time \(t_{max}\) is the physical distance of the photon from the origin at its largest value?

PROBLEM 19: TRANSVERSE DOPPLER SHIFTS

Consider a light pulse that expands with a scale factor:

\[ a(t) = \frac{t}{t_f}. \]

The following problem was taken from Problem 9, Quiz I, Fall 2005, where it counted 25 points.

PROBLEM 19: TRANSVERSE DOPPLER SHIFTS

Consider a light pulse that expands with a scale factor:

\[ a(t) = \frac{t}{t_f}. \]

The following problem was taken from Problem 9, Quiz I, Fall 2005, where it counted 25 points.
(a) (8 points) Suppose that a spaceship \( \text{Xanthu} \) is at rest at location \((x=0, y=a, z=0)\) in a Cartesian coordinate system. (We assume that the space is Euclidean, and that the distance scales in the problem are small enough so that the expansion of the universe can be neglected.)

The spaceship \( \text{Emmerac} \) is moving at speed \( v_0 \) along the \( x \)-axis in the positive direction, as shown in the diagram, where \( v_0 \) is comparable to the speed of light. As the \( \text{Emmerac} \) crosses the origin, it receives a radio signal that had been sent some time earlier from the \( \text{Xanthu} \). Is the radiation received redshifted or blueshifted? What is the redshift \( z \) (where negative values of \( z \) can be used to describe blueshifts)?

(b) (7 points) Now suppose that the \( \text{Emmerac} \) is at the origin, while the \( \text{Xanthu} \) is moving in the negative \( x \)-direction, at \( y=a \) and \( z=0 \), as shown in the diagram. That is, the trajectory of the \( \text{Xanthu} \) can be taken as \((x=-v_0 t, y=a, z=0)\).

At \( t=0 \) the \( \text{Xanthu} \) crosses the \( y \)-axis, and at that instant it emits a radio signal along the \( y \)-axis, directed at the origin. The radiation is received some time later by the \( \text{Emmerac} \). In this case, is the radiation received redshifted or blueshifted? What is the redshift \( z \) (where again negative values of \( z \) can be used to describe blueshifts)?

(c) (5 points) Is the sequence of events described in (b) physically distinct from the sequence described in (a), or is it really the same sequence of events described in a reference frame that is moving relative to the reference frame used in part (a)? Explain your reasoning in a sentence or two.

(Hint: note that there are three objects in the problem: \( \text{Xanthu}, \text{Emmerac}, \) and the photons of the radio signal.)

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\[ x = 0 \]

\[ (0) \]

\[ y = 0, v = 0 \]

\[ a = a \]

\[ z = z \]

\[ \text{Emmerac} \]

\[ \text{Xanthu} \]

\[ \text{hub} \]

\[ \text{radius} \]

\[ \text{speed} \]

\[ \text{redshift} \]

\[ \text{blue shift} \]

\[ \text{coordinate system} \]

\[ \text{Euclidean} \]

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\[ 8.286 \text{ QUIZ 1 REVIEW PROBLEMS, FALL 2009 p. 19} \]

\[ 8.286 \text{ QUIZ 1 REVIEW PROBLEMS, FALL 2009 p. 20} \]
Solutions

Problem 1: Did you do the Reading?

(a) Doppler predicted the Doppler effect in 1842.

(b) Most of the stars of our galaxy, including our sun, lie in a flat disk. We therefore see much more light when we look out from Earth along the plane of the disk than when we look in any other direction.

(c) Hubble's original paper on the expansion of the universe was based on a study of only 18 galaxies. For my own book I made a copy of Hubble's original graph, which seems to show 24 black dots, each of which represents a galaxy, as reproduced below. The vertical axis shows the recession velocity, in kilometers per second. The solid line shows the best fit to the black dots, each of which represents a galaxy. Each open circle represents a group of the galaxies shown as black dots, selected by their proximity in direction and distance; the broken line is the best fit to these points. The cross shows a statistical analysis of 22 galaxies for which individual distance measurements were not available. I am not sure why Weinberg refers to 18 galaxies, but it is possible that the text of Hubble's article indicated that 18 of these galaxies were measured with more reliability than the rest.

(d) During a time interval in which the linear size of the universe grows by 1%, the horizon distance grows by more than 1%.

(e) During a time interval in which the linear size of the universe grows by 1%, the horizon distance grows by more than 1%.

(f) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

(g) (i) the average distance between photons: proportional to the size of the universe (Photons are neither created nor destroyed, so the only effect is that the average distance between them is stretched with the expansion. Since the universe expands uniformly, all distances grow by the same factor.)

(ii) the typical wavelength of the radiation: proportional to the size of the universe (See Lecture Notes 3.)

(iii) the number density of photons in the radiation: inversely proportional to the square of the size of the universe (From (i), the average distance between photons grows in proportion to the size of the universe. Since the volume of a sphere is proportional to the square of the radius, the average distance between photons is inversely proportional to the square of the size of the universe.)

(iv) the energy density of the radiation: inversely proportional to the size of the universe (The energy of each photon is inversely proportional to its wavelength, which is proportional to its frequency, which is proportional to its speed times its mass. So, from (iii), the energy density of the radiation is inversely proportional to the size of the universe.)

(c) The distance of a galaxy from us is proportional to the size of the universe. The number density of a photon is inversely proportional to the size of the universe (Try! The universe is expanding, so the number density of photons in the radiation is inversely proportional to the size of the universe.)

(d) The number density of photons in the radiation is inversely proportional to the size of the universe.
The temperature of the radiation is inversely proportional to the size of the universe. (The temperature is directly proportional to the average energy of a photon, which according to (iv) is inversely proportional to the size of the universe.)

**Problem 2: The Steady-State Universe Theory**

(a) Given that the temperature is inversely proportional to the size of the universe,

\[ T \propto \frac{1}{a(t)} \]

(b) Integrating, we have

\[ \ln a = H_0 t + c \]

where \( c \) is a constant of integration. Exponentiating, we get

\[ a = b e^{H_0 t} \]

where \( b = e^c \) is an arbitrary constant.

In the absence of matter creation, the total mass within a comoving volume

\[ \frac{d^3 \rho (t_c) \delta v}{dt^3} (t_c) = 0 \]

So in this case

\[ \frac{d^3 \rho (t_c) \delta v}{dt^3} (t_c) = 0 \]

Since the mass density is fixed at \( \rho_0 \), the total mass inside this cube is any

\[ \frac{d^3 \rho (t_c) \delta v}{dt^3} (t_c) = 0 \]

is

Therefore, the physical length of each side is then \( a(t) \), so the physical volume

\[ V(t) = a(t)^3 \]

and the total mass inside this cube is any

\[ M(t) = a(t)^3 \]

In the absence of matter creation, the total mass within a comoving volume

\[ M(t) = a(t)^3 \]

must be attributed to matter creation. The rate of matter creation per unit

\[ \frac{d^3 \rho (t_c) \delta v}{dt^3} (t_c) = 0 \]

You were not asked to insert numbers, but it is worthwhile to consider the

\[ \frac{d^3 \rho (t_c) \delta v}{dt^3} (t_c) = 0 \]

The rate of matter production required for the steady-state universe theory can then

\[ \frac{d^3 \rho (t_c) \delta v}{dt^3} (t_c) = 0 \]

be expressed as a function of \( \frac{d^3 \rho (t_c) \delta v}{dt^3} \), which must be integrated from the future to the present. In the absence of matter creation, the total mass within a comoving volume

\[ M(t) = a(t)^3 \]

would not change, so the increase in mass described by the above equation

\[ M(t) = a(t)^3 \]

is

\[ \frac{d^3 \rho (t_c) \delta v}{dt^3} (t_c) = 0 \]

You were not asked to insert numbers, but it is worthwhile to consider the

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\[ M(t) = a(t)^3 \]

would not change, so the increase in mass described by the above equation

\[ M(t) = a(t)^3 \]

is

\[ \frac{d^3 \rho (t_c) \delta v}{dt^3} (t_c) = 0 \]
PROBLEM 3: DID YOU DO THE READING? (25 points)

(a) In the 1940's, three astrophysicists proposed a "steady state" theory of cosmology, in which the universe has always looked about the same as it does now. State the last name of at least one of these authors. (Bonus points: you can earn 1 point each for naming the other two authors, and hence up to 2 additional points, but 1 point will be taken off for each incorrect answer.)


(b) In 1917, a Dutch astronomer named Willem de Sitter did which one of the following accomplishments:

(i) measured the size of the Milky Way galaxy, finding it to be about one billion light-years in diameter.
(ii) resolved Cepheid variable stars in Andromeda and thereby obtained persuasive evidence that Andromeda is not within our own galaxy, but is apparently another galaxy like our own.
(iii) published a catalog, *Nebulae and Star Clusters*, listing 103 objects that astronomers should avoid when looking for comets.
(iv) published a model for the universe, based on general relativity, which appeared to be static but which produced a redshift proportional to the distance.
(v) discovered that the orbital periods of the planets are proportional to the $3/2$ power of the semi-major axis of their elliptical orbits.

Discussion: (i) is false in part because de Sitter was not involved in the measurement of the size of the Milky Way, but the most obvious error is in the size of the Milky Way. Its actual diameter is about 100,000 light-years, although it is supposed to be about twice that. (ii) is the work of Charles Messier in 1781 (Weinberg, p. 17). (v) is of course one of Kepler's laws of planetary motion.

(c) In 1964–65, Arno A. Penzias and Robert W. Wilson observed a flux of microwave radiation coming from all directions in the sky, which was interpreted by a group of physicists at a neighboring institution as the cosmic background radiation left over from the big bang. Circle the two items on the following list that were not part of the story behind this spectacular discovery:

(i) Bell Telephone Laboratory (ii) MIT (iii) Princeton University (iv) pigeons (v) ground hogs (vi) Hubble's constant (vii) liquid helium (viii) 7.35 cm

Grading: 3 pts for 1 correct answer, 5 for 2 correct answers, and -2 for each incorrect answer, but the minimum score is zero.

Discussion: The discovery of the cosmic background radiation was described in some detail by Weinberg in Chapter 3. The observation was done at Bell Telephone Laboratories, in Holmdel, New Jersey. The detector was cooled with liquid helium to minimize electrical noise, and the measurements were made at a wavelength of 7.35 cm. During the course of the experiment the astronomers had to eject a pair of pigeons who were roosting in the antenna. Penzias and Wilson were not initially aware that the radiation they discovered might have come from the big bang, but Bernard Burke of MIT put them in touch with a group at Princeton University (Robert Dicke, James Peebles, P.G. Roll, and David Wilkinson) who were actively working on this hypothesis.

(d) Important predictions of the Copernican theory were confirmed by the discovery of the aberration of starlight (which showed that the velocity of the Earth has the time-dependence expected for rotation about the Sun) and by the behavior of the Foucault pendulum (which showed that the velocity of the Earth is constant). The discovery of the aberration of starlight (i) during Copernicus' lifetime. (ii) approximately two and three decades after Copernicus' death, respectively. (iii) approximately two and three centuries after Copernicus' death, respectively. (iv) approximately two and three centuries after Copernicus' death, respectively. Ryden discusses this on p. 5. The aberration of starlight was discovered in 1728, while the Foucault pendulum was invented in 1851.

(e) If one averages over sufficiently large scales, the universe appears to be homogeneous and isotropic. How large must the averaging scale be before this statement is true? (i) 1 AU (1 AU = 1.496 × 10^11 m). (ii) 100 kpc (1 kpc = 1000 pc, 1 pc = 3.086 × 10^16 m = 3.262 light-years). (iii) 1 Mpc (1 Mpc = 10^6 pc). (iv) 10 Mpc. (v) 100 Mpc. (vi) 1000 Mpc. This issue is discussed in Ryden's book on p. 11.

The following 2 questions are each worth 5 points:

PROBLEM 3: DID YOU DO THE READING?
Problem 4: An Exponentially Expanding Universe

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by
\[ H = \frac{\dot{a}}{a} \]

So
\[ H = \chi_0 e^{\chi t} \]

(b) According to Eq. (3.8), the coordinate velocity of light is given by
\[ \frac{dx}{dt} = c a(t) = c a_0 e^{-\chi t} \]

Integrating,
\[ x(t) = c a_0 \int_0^t e^{-\chi t'} dt' = c a_0 \left[ -\frac{1}{\chi} e^{-\chi t'} \right]_0^t = c a_0 \left[ 1 - e^{-\chi t} \right] \]

(c) From Eq. (3.11), or from the front of the quiz, one has
\[ 1 + z = \frac{a(t_r)}{a(t_e)} \]

Here \( t_e = 0 \), so
\[ 1 + z = a_0 e^{\chi t_r} a_0 \]

\[ \Rightarrow e^{\chi t_r} = 1 + z \]

\[ \Rightarrow t_r = \frac{1}{\chi} \ln(1 + z) \]

(d) The coordinate distance is
\[ x(t_r), \text{ where } x(t) \text{ is the function found in part (b), and } t_r \text{ is the time found in part (c).} \]

\[ e^{\chi t_r} = 1 + z, \quad \text{and} \quad x(t_r) = c \chi a_0 \left[ 1 - e^{-\chi t_r} \right] = c \chi a_0 \frac{z}{1 + z} \]

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so
\[ \frac{(z + 1)\dot{a}}{a} = \frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} \]

Integration yields
\[ x = \frac{\chi a_0}{1 + z} \]

Problem 5: "Did you do the reading?"

(a) The distinguishing quantity is \( \Omega \equiv \frac{\rho}{\rho_c} \). The universe is open if \( \Omega < 1 \), flat if \( \Omega = 1 \), or closed if \( \Omega > 1 \).

(b) The temperature of the microwave background today is about 3 Kelvin. (The best determination to date was made by the COBE satellite, which measured the temperature as \( 2.728 \pm 0.004 \) Kelvin. The error here is quoted with a 95% confidence limit, which means that the experimenters believe that the probability that the true value lies outside this range is only 5%.)

(c) The cosmic microwave background is observed to be highly isotropic.

(d) The distance to the Andromeda nebula is roughly 2 million light years. 

(e) 1929.

(f) 2 billion years. Hubble's value for Hubble's constant was high by modern standards, by a factor of 5 to 10.

(g) The absolute luminosity (i.e., the total light output) of a Cepheid variable appears to be highly correlated with the period of its pulsations. This correlation can be used to estimate the distance to a Cepheid variable. For example, the period of the Cepheid variable is given by
\[ \frac{P}{\pi} = \frac{2.07}{\chi} \]

(h) \( 10^7 \) light-years.

(i) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories. 

(j) Princeton University.

By combining the answers to (a) and (b), one has

\[ \frac{r(z+1)}{1} - 1 \mathcal{P} \frac{cz}{\mathcal{P}} = (\mathcal{P})^{d_f} \]

\( q(n) \) and one has

\[ 1 - \left( \frac{r_{\mathcal{P}}}{t_{\mathcal{P}}} \right) = z \]

\[ 1 - \left( \frac{r_{\mathcal{P}}}{t_{\mathcal{P}}} \right) = z + 1 \]

So, the comonomial redshift is given by the usual form:

\[ \left[ \frac{r_{\mathcal{P}}}{t_{\mathcal{P}}} - \frac{r_{\mathcal{P}}}{t_{\mathcal{P}}} \right] \mathcal{P} \frac{cz}{\mathcal{P}} = \]

\[ \left[ \frac{r_{\mathcal{P}}}{t_{\mathcal{P}}} - \frac{r_{\mathcal{P}}}{t_{\mathcal{P}}} \right] \mathcal{P} \frac{cz}{\mathcal{P}} \]

\[ \left[ \frac{(\mathcal{P})^{d_f}}{t_{\mathcal{P}}} \right] - \left( \frac{(\mathcal{P})^{d_f}}{t_{\mathcal{P}}} \right) \]
(10 points) A nearly identical problem was worked through in Problem Set 1. The energy of the observed photons will be redshifted by a factor of \((1+z)\), in addition to the rate of arrival of photons. The observed rate, then, is another factor of \((1+z)\). Consequently, the observed power will be redshifted by two factors of \((1+z)\).

Finally, then,

\[
\ell^0(\ell) = \frac{c\ell^0}{1+z} \left[ 1 - (1+z)^{-1} \right].
\]

This can be rewritten in terms of \(z\) and \(H_0\) using the result of part (a) as well as,

\[
H_0 = \frac{d^2a}{dt^2} = \frac{c}{H_0} = \frac{c}{\gamma}. \tag{1}
\]

Calculating this integral gives

\[
\frac{\ell^0(\ell)}{\ell^0(0)} = \frac{1}{1 - (1+z)^{-1}}.
\]

Therefore,

\[
\frac{t}{t_0} = \frac{t_0}{t} = (1+z)^{-1}.
\]

(10 points) The present value of the physical distance to the object, \(d(0)\), is found from

\[
\int_{t_0}^{t} \frac{dt}{1 - (1+z)^{-1}} = \frac{1}{1 - (1+z)^{-1}}.
\]

Factoring \(1-z^{-1}\) out of the parentheses gives

\[
\frac{1}{1 - (1+z)^{-1}} = \frac{1}{z}.
\]

Using the result of part (c) to eliminate \(t/t_0\) in favor of \(z\). From (6),

\[
\frac{t}{t_0} = \frac{t_0}{t} = (1+z)^{-1}.
\]

Therefore, we can use the result of part (a) to eliminate \(t/t_0\) in favor of \(z\). From (6),

\[
\frac{t}{t_0} = \frac{t_0}{t} = (1+z)^{-1}.
\]

Since the object radiates uniformly in all directions, the patch will intercept a fraction \((A_0/A)\) of the photons passing through the sphere. Therefore, the radiation energy flux \(J\), which is the received power per area, reaching the earth, is then given by

\[
J = \frac{4\pi \ell^0(\ell)}{1-1} \left[ \left( \frac{\ell^0(\ell)}{\ell^0(0)} \right)^2 \frac{P}{P} \right].
\]

Now consider the photons passing through radius equal to the observing distance \(c\). Imagine a hypothetical sphere, centered in the observing coordinates, as drawn above. Consider the photons passing through radius equal to the observing distance \(c\). Now consider the photons passing through radius equal to the observing distance \(c\). Since the object radiates uniformly in all directions, the patch will intercept a fraction \((A_0/A)\) of the photons passing through the sphere. This area. A. The object radiates uniformly in all directions. The patch will intercept a fraction \((A_0/A)\) of the photons passing through the sphere.
The population of this low-lying state is therefore small. The state to a low-lying excited state, a state in which the C and N atoms are just a source at rest in the comoving coordinate system, so that a source at rest in the comoving coordinate system. So the special relativistic Doppler shift formula, which says that the redshift of a source is due to the expansion of the universe is equal to, where \( \Delta t \) is the time interval between the emission and the reception of the signal, is given by:

\[
\Delta t = \frac{c}{v} \Delta \tau,
\]

where \( \Delta \tau \) is the proper time interval between the emission and the reception of the signal. The special relativistic Doppler shift formula is given by:

\[
\Delta \tau = \frac{\Delta \tau_{\text{observed}}}{1 + \frac{v}{c}}
\]

and is not modified by general relativistic effects. The special relativistic Doppler shift formula is given by:

\[
\Delta \tau_{\text{observed}} = \Delta \tau_{\text{proper}} (1 - \frac{v}{c})^{1/2}
\]

Thus, if you measure the time interval between the emission and the reception of the signal in the proper frame, you will see the signal blue-shifted. The special relativistic Doppler shift formula is given by:

\[
\Delta \tau_{\text{observed}} = \Delta \tau_{\text{proper}} (1 - \frac{v}{c})^{1/2}
\]

where \( \Delta \tau_{\text{observed}} \) is the time interval between the emission and the reception of the signal, and \( \Delta \tau_{\text{proper}} \) is the proper time interval between the emission and the reception of the signal. The special relativistic Doppler shift formula is given by:

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\]

where \( \Delta \tau \) is the proper time interval between the emission and the reception of the signal, is given by:

\[
\Delta \tau = \frac{c}{v} \Delta \tau_{\text{observed}}
\]
\[
\left[ 1 - \frac{V_t}{g/c} \right] \frac{c}{g} \left[ \frac{g}{c} \right] = \left[ 1 - \frac{V_t}{g/c} \right] \frac{V_p g}{c} = (V_t)g (V_t)H = a
\]

According to Hubble’s law, the speed is equal to Hubble’s constant times the physical distance. By combining the answers to parts (a) and (d), one has

\[
\frac{c}{g} \left( \frac{V_t}{g/c} - \frac{g}{c} \right) \frac{\partial P}{\partial g} = (V_t)g = \frac{\partial (\pi d)}{\partial c}
\]

Solving for \( b \cdot c \cdot \left( \frac{V_t}{g/c} - \frac{g}{c} \right) \frac{\partial P}{\partial g} \), one finds that the coordinate speed \( c \), the time \( t \), and the coordinate distance \( d \) are given by

\[
\frac{c}{g} \left( \frac{V_t}{g/c} - \frac{g}{c} \right) \frac{\partial P}{\partial g} = (V_t)d_g (V_t)H = a
\]

In this case one has

\[
\int (V_t)d_g (V_t) = t = \int (V_t)dt (V_t)H = a
\]

\[
\frac{\partial (\pi d)}{\partial c} = \frac{\partial (\pi d)}{\partial c} = \frac{\partial (\pi d)}{\partial c} = \frac{\partial (\pi d)}{\partial c}
\]

Problem 9: A Flat Universe with Hubble’s Law

PROBLEM 9: A FLAT UNIVERSE WITH HHUBBLE’S LAW

In this case one has

\[
\int (V_t)d_g (V_t) = t = \int (V_t)dt (V_t)H = a
\]

In general, the horizon distance is given by

\[
(\pi d)_H = \int (V_t)d_g (V_t)H = a
\]

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\[
(\pi d)_H = \int (V_t)d_g (V_t)H = a
\]
The redshift $z$ of the light pulse received at galaxy B is given by

$$z = \frac{V_f}{c} \frac{v_f}{V_f} = z + 1$$

The energy received by the detector is

$$\Delta E = \frac{\Delta E}{V}$$

As the expression approaches $z \rightarrow \infty$, the horizon distance $\ell_p$ is

$$\ell_p = \frac{5}{2} c t \left[ 1 - \left( \frac{t_B}{t_A} \right)^{2/3} \right]$$

The redshift $z$ of the light pulse observed at galaxy B is

$$1 + z = \frac{\alpha(t_B)}{\alpha(t_A)} = \left( \frac{t_B}{t_A} \right)^{3/5}$$

Next consider the rate at which photons leave the sphere. In the comoving coordinate picture, the detector will receive photons from a sphere centered on galaxy A with physical area $A$. The rate of arrival of photons at the detector will be slower than the rate of emission by a redshift factor $1 + z = \frac{\alpha(t_B)}{\alpha(t_A)}$. The energy flux is reduced by this factor, and each photon is redshifted in frequency by $1 + z$. Therefore, the energy flux reaches the detector

$$\Delta E = \Delta E \left[ \frac{1}{V} \right]$$

Substituting the expression for $\Delta E$, one has

$$\Delta E = \Delta E \left[ \frac{1}{V} \right]$$

The energy from the quasar that enters the detector will be

$$\Delta E = \Delta E \left[ \frac{1}{V} \right]$$

Thus, the energy flux reaches the detector with

$$\Delta E = \Delta E \left[ \frac{1}{V} \right]$$
Using once more the expression for $\ell_P(t_B)$ from part (d), one has

$$J = \text{Power hitting detector} \quad A = \frac{P(t_A/t_B)}{25\pi c^2 t_B^2 \left[1 - \left(\frac{t_A}{t_B}\right)^2/5\right]^2}. $$

The problem is worded so that $t_A$, and not $z$, is the given variable that determines how far galaxy A is from galaxy B. In practice, however, it is usually more useful to express the answer in terms of the redshift $z$ of the received radiation. One can do this by using the above expression for $1+z$ to eliminate $t_A$ in favor of $z$, finding

$$J = \frac{P}{25\pi c^2 t_B^2 (1+z)^2 \left[(1+z)^2/3 - 1\right]^2}. $$

\[\text{PROBLEM 10: DID YOU DO THE READING?}\]

\[a)\] Einstein believed that the universe was static, and the cosmological term was necessary to prevent a static universe from collapsing under the attractive force of normal gravity. [The repulsive effect of a cosmological constant grows linearly with distance, so if the coefficient is small it is important only when these separations are very large. Such a term can be important cosmologically while still being too small to be detected by observation of the solar system or even the early Milky Way. Indeed, evidence that the universe was static, and the cosmological term was necessary to prevent a static universe from collapsing under the attraction of normal gravity was not even needed to infer a static universe from observation of the redshifts of遥远 galaxies].

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\[b)\] At the time of its discovery, de Sitter's model was thought to be static although it was known that the model predicted a redshift which, at least for nearby galaxies, was proportional to the distance. From a modern perspective the model is thought to be cosmologically correct.

\[c)\] $n_1 = 3$, and $n_2 = 4$.

\[d)\] Above 3,000 K the universe was hot enough that the atoms were ionized, dissociated into nuclei and free electrons. At about this temperature, however, the nuclei and free electrons, although the temperature was so high that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.

\[e)\] Above 3,000 K the universe was hot enough that the atoms were ionized, dissociated into nuclei and free electrons. At about this temperature, however, the nuclei and free electrons, although the temperature was so high that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.

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\[g)\] Above 3,000 K the universe was so hot that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.

\[h)\] Above 3,000 K the universe was so hot that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.

\[i)\] Above 3,000 K the universe was so hot that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.

\[j)\] Above 3,000 K the universe was so hot that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.

\[k)\] Above 3,000 K the universe was so hot that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.

\[l)\] Above 3,000 K the universe was so hot that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.

\[m)\] Above 3,000 K the universe was so hot that the atoms were ionized, dissociated, were cool enough that the nuclei and free electrons could interact, allowing for the formation of neutral atoms.
Thus, the cosmic time interval between the receipt of the message and the
signal at rest will respect to the matter of the universe at the same location.

\[ t = \frac{1}{\sqrt{3}} \left( \frac{1}{\sqrt{x^2}} + 1 \right) = \varepsilon t \]

The above answer is perfectly acceptable, but one could also replace \( t \) by using

\[ t_{\text{rel}} = \frac{1}{\sqrt{3}} \left( \frac{1}{\sqrt{x^2}} + 1 \right) = \varepsilon t \]

Solving for \( \varepsilon \) gives

\[ \frac{1}{\sqrt{x^2}} = \left( \frac{t}{t_{\text{rel}}} \right)^2 = \frac{1}{\sqrt{3}} \]

which leads to

\[ \frac{1}{\sqrt{x^2}} = \frac{t}{t_{\text{rel}}} = \varepsilon t \]

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which leads to

\[ \frac{1}{\sqrt{x^2}} = \frac{t}{t_{\text{rel}}} = \varepsilon t \]

The above answer is perfectly acceptable, but one could also replace \( t \) by using

\[ t_{\text{rel}} = \frac{1}{\sqrt{3}} \left( \frac{1}{\sqrt{x^2}} + 1 \right) = \varepsilon t \]

Solving for \( \varepsilon \) gives

\[ \frac{1}{\sqrt{x^2}} = \left( \frac{t}{t_{\text{rel}}} \right)^2 = \frac{1}{\sqrt{3}} \]

which leads to

\[ \frac{1}{\sqrt{x^2}} = \frac{t}{t_{\text{rel}}} = \varepsilon t \]
In agreement with the previous answer,
\[ (\varepsilon t^4) \mathcal{O} + B \left( \frac{t}{\varepsilon} \right) = \varepsilon t - t \]

Putting this back into the Taylor series gives
\[ \left( \frac{t}{\varepsilon} \right) \mathcal{O} + \left( \frac{t}{\varepsilon} \right) = 0 = \mathcal{O} \left( \frac{t}{\varepsilon} \right) \]

which when specialized to \( t \), the answer to part (c), can be simplified to
\[ \left( \frac{t}{\varepsilon} \right) \mathcal{O} + \left( \frac{t}{\varepsilon} \right) \mathcal{O} = \mathcal{O} \left( \frac{t}{\varepsilon} \right) \]

Evaluating the necessary derivatives gives
\[ (\mathcal{O} t^4) \mathcal{O} + B \left( \frac{t}{\varepsilon} \right) = \varepsilon t - t \]

Since the answer contains an explicit factor of \( \mathcal{O} \), the other factors can be

\[ (\varepsilon t^4) \mathcal{O} + i \mathcal{O} \left( \frac{t}{\varepsilon} \right) = \varepsilon t - t \]

Therefore, the answer to part (d) can be replaced by \( \varepsilon t \), so
\[ (\mathcal{O} t^4) \mathcal{O} + i \mathcal{O} \left( \frac{t}{\varepsilon} \right) = \varepsilon t - t \]

The response is therefore sent at a time that is increased by \( \varepsilon \) (i) in order to change significantly.

\[ (\mathcal{O} t^4) \mathcal{O} + i \mathcal{O} \left( \frac{t}{\varepsilon} \right) = \varepsilon t - t \]
PROBLEM 12: THE DECELERATION PARAMETER

From the front of the exam, we are reminded that
\[ \ddot{a} = -\frac{4}{3} \pi G \rho a. \]

and
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8}{3} \pi G \rho - \frac{k c^2}{a^2}, \]

where a dot denotes a derivative with respect to time \( t \).

The critical mass density \( \rho_c \) is defined to be the mass density that corresponds to a flat (\( k = 0 \)) universe, so from the equation above it follows that
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8}{3} \pi G \rho_c. \]

Substituting into the definition of \( q \), we find
\[ q = -\frac{\ddot{a}}{a} \left( \frac{\dot{a}}{a} \right) \frac{a}{\dot{a}} = -\frac{\ddot{a}}{a} \frac{a}{\dot{a}}. \]

\[ = -\frac{\ddot{a}}{a} \frac{a}{\dot{a}} \]

Substituting into the definition of \( \dot{a} \), this becomes
\[ \frac{\dot{a}}{a^2} \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) + \frac{\dot{a}}{a^2} \frac{d^2}{dt^2} \left( \frac{\dot{a}}{a} \right) = -\frac{\ddot{a}}{a} \frac{a}{\dot{a}}. \]

Dividing both sides of the equation by \( \dot{a} \), one has
\[ \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) = -\frac{\ddot{a}}{a} \frac{a}{\dot{a}}. \]

Substituting the equation for \( \dot{a} \) in the last term, which is proportional to \( q \), the only dependence on \( t \) occurs in the last term, which is proportional to \( \dot{a} \).

PROBLEM 13: A RADIATION-DOMINATED FLAT UNIVERSE

The flatness of the model universe means that \( k = 0 \), so
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8}{3} \pi G \rho_c. \]

Since
\[ \rho(t) \propto \frac{1}{a^4(t)}, \]

it follows that
\[ \frac{d}{dt} \frac{\dot{a}}{a} = \text{const}. \]

Rewriting this as
\[ d \frac{\dot{a}}{a} = \text{const} \, dt, \]

the indefinite integral becomes
\[ \frac{1}{2} a^2 = (\text{const}) \, t + c', \]

where \( c' \) is a constant of integration. Different choices for \( c' \) correspond to different choices for the definition of \( t = 0 \). We will follow the standard convention of setting \( c' = 0 \), which sets \( t = 0 \) to be the time when \( a = 0 \). Thus the above equation implies that \( \rho \propto t^{-4} \), and therefore
\[ \rho + \text{const} = \frac{v}{t}. \]

The indefinite integral becomes
\[ \frac{v}{t} \propto (t) \frac{d}{dt} \]

PROBLEM 14: A POSSIBLE MODIFICATION OF NEWTON'S LAW

(a) Substituting the equation for \( M(r_i) \), given on the quiz, into the differential equation for \( r \), also given on the quiz, one finds:
\[ \ddot{r} = -\frac{4}{3} \pi G r_i^3 \rho_i r^2 + \gamma r_n. \]

Dividing both sides of the equation by \( r_i \), one has
\[ \frac{\ddot{r}}{r_i} + \frac{\dot{r}}{r_i} \frac{d}{dt} \frac{\dot{r}}{r_i} - r_n \frac{1}{r_i} = \gamma. \]

Substituting \( u = r/r_i \), this becomes
\[ \ddot{u} = -\frac{4}{3} \pi G \rho_i u^2 + \gamma u_n. \]

(b) The only dependence on \( r_i \) occurs in the last term, which is proportional to \( q \).

(c) This is exactly the same as the case discussed in the lecture notes, since the only dependence on \( r_i \) occurs in the last term, which is proportional to \( q \).

PROBLEM 15: A POSSIBLE MODIFICATION OF NEWTON'S LAW

For a photon-dominated universe, we have
\[ \frac{v}{t} \propto (t) \frac{d}{dt} \]

Where a dot denotes a derivative with respect to time \( t \). The critical mass density
\[ \rho + \text{const} = \frac{v}{t}. \]

That \( \rho \propto t^{-4} \), and therefore
\[ \rho + \text{const} = \frac{v}{t}. \]

The indefinite integral becomes
\[ \frac{v}{t} \propto (t) \frac{d}{dt} \]
consistent with it?

(c) What is the Cosmological Principle? Is the Hubble expansion of the universe

\[ \dot{a} \] consistent with its rate of expansion?

\[ \dot{a} = \frac{3}{2} \]


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Note that it guarantees that $v \leq \frac{c}{\lambda}$ so that as long as $c/\lambda \leq v$.

$$\frac{c^2}{\alpha^2} + 1 = \tan \theta$$

Then the equation above determines $\tan \theta$ in terms of $c/\lambda$ and $v$, so the rest is just algebra. To simplify the notation, let $\alpha \equiv c/v$, $\beta \equiv 1/v$, and $\gamma \equiv 1/\lambda$. Although we need the pressure of $\gamma$ in determining the redshift $z$ of the emission, we can simply define $\gamma \equiv 1/\lambda$.

$$\frac{c}{\alpha a} - I = \frac{c}{\alpha a} - I \frac{c}{\alpha a} - I$$

Combining Eqs. (18.3) and (18.4),

$$\frac{c/\alpha a - I}{\alpha a + I} \frac{c/\alpha a - I}{\alpha a + I} = z$$

Problem 1: Special Relativity Doppler Shift (20 points)

(a) The easiest way to solve this problem is by a double application of the standard special-relativity Doppler formula which was given on the front of the exam:

$$\frac{c}{\alpha a} - I = \frac{c/\alpha a - I}{\alpha a + I}$$

Weinberg, Chapter 1, Page 5)

(b) Although we used the presence of $\gamma$ in determining the redshift $z$ of the emission, we can simply define $\gamma \equiv 1/\lambda$.

$$\frac{c}{\alpha a} - I = \frac{c}{\alpha a} - I \frac{c}{\alpha a} - I$$

Combining Eqs. (18.3) and (18.4),

$$\frac{c/\alpha a - I}{\alpha a + I} \frac{c/\alpha a - I}{\alpha a + I} = z$$

So algebraically:

$$\frac{c/\alpha a - I}{\alpha a + I} \frac{c/\alpha a - I}{\alpha a + I} = z$$

By the observer can be written as

$$\frac{c}{\alpha a} - I = \frac{c/\alpha a - I}{\alpha a + I}$$

Alpha-7, the nearest relativistic Doppler shift formula, which was given on the front of the exam:

$$\frac{c}{\alpha a} - I = \frac{c/\alpha a - I}{\alpha a + I}$$

Weinberg, Chapter 2, Page 11)

(c) The easiest way to solve this problem is by a double application of the standard special-relativity Doppler formula which was given on the front of the exam:

$$\frac{c}{\alpha a} - I = \frac{c/\alpha a - I}{\alpha a + I}$$
What is the wavelength of the portion of the graph of CMB energy density vs. wavelength that is approximately equal to the mean energy of a CMB photon today? (4 points)

The wavelength corresponding to the mean energy of a CMB (cosmic microwave) photon today is approximately equal to which of the following quantities? Circle as many as apply.

- (i) baryon number
- (ii) energy per particle
- (iii) proton number
- (iv) electric charge
- (v) pressure

At this point, if we define $G = G' = \frac{\pi^2}{90}$, we have a gravitational theory with $G$ proportional to the period. A measurement of a particular Cepheid's period determines the star's absolute luminosity, which, compared to the measured luminosity, determines the distance to the star. (Hubble's initial measurement of the distance to Andromeda was an order of magnitude more distant than the most distant known objects in our galaxy.)

The wavelength of the photon is then $\lambda = \frac{h}{E} = \frac{h}{kT}$. This approximation gives $\lambda \approx \frac{3 \times 10^{-5}}{T}$ mm, which is not very accurate. The wavelength of the light from a distant galaxy is determined by the temperature of the cosmic background radiation, which is approximately 3 Kelvin. The typical photon energy is then on the order of $kT = \frac{1}{2} m_\gamma c^2 = 1.4 \times 10^{-20}$ Joules.

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Ans: (Weinberg, page 67) The Earth's atmosphere is increasingly opaque for wavelength shorter than \(0.3\) cm. Therefore, radiation at these wavelengths will be absorbed and rescattered by the Earth's atmosphere; observations of the cosmic microwave background at small wavelengths must be performed above the Earth's atmosphere.

**PROBLEM 18: TRACING A LIGHT PULSE THROUGH A RADIATION-DOMINATED UNIVERSE**

(a) The physical horizon distance is given in general by

\[
\ell_{\text{p, horizon}} = a(t) \int_{t_f}^{0} c a(t) \, dt
\]

so in this case

\[
\ell_{\text{p, horizon}} = b t_1^{1/2} \int_{t_f}^{0} c b t_1^{1/2} \, dt = 2 c t_f.
\]

(b) If the source is at the horizon distance, it means that a photon leaving the source at \(t = 0\) would just be reaching the origin at \(t_f\). So, \(t = 0\). Since \(t_f\) is the time of departure from the Xanthu at a minimum, the coordinate distance between the source and the origin is the coordinate horizon distance, given by

\[
\ell_{c, \text{horizon}} = \int_{t_f}^{0} c b t_1^{1/2} \, dt = 2 c t_1^{1/2} f b.
\]

(c) The photon starts at coordinate distance \(2 c \sqrt{t_f/b}\), and by time \(t_f\) it will have traveled a coordinate distance

\[
\int_{0}^{t_f} c b t_1^{1/2} \, dt = 2 c \sqrt{t_f b}.
\]

(d) The photon will be at a coordinate distance \(2 c \sqrt{t_f} \) toward the origin. Thus the photon will be at a physical distance

\[
\ell_{\text{p}}(t_f) = a(t_f) \ell_{c} = 2 c (\sqrt{t_f} - \sqrt{t_f - t_1}).
\]

(e) To find the maximum of \(\ell_{\text{p}}(t_f)\), we differentiate it and set the derivative to zero:

\[
\frac{d\ell_{\text{p}}}{dt} = (\sqrt{t_f t - t_1^2}) c, \quad \text{so the maximum occurs when}
\]

\[
\sqrt{t_f t - t_1^2} = \frac{\sqrt{t_f}}{2} \quad \text{or} \quad t = \frac{t_f}{2}.
\]

Thus, the horizon distance is given in general by

\[
\ell_{\text{p, horizon}} = 2 c t_1^{1/2} \int_{0}^{t_f} (t_f - t_1) \, dt = \frac{4}{3} c t_f.
\]

**PROBLEM 19: TRANSVERSE DOPPLER SHIFTS**

(a) Describing the events in the coordinate system shown, the Xanthu is at rest, so its clocks run at the same speed as the coordinate system time variable, \(t\). The emission of the wavecrests of the radio signal are therefore separated by a time interval equal to the time interval as measured by the source, the Xanthu:

\[
\Delta t = \Delta t_s.
\]

(b) Since the Emmerac is moving perpendicular to the path of the radio waves, at the moment of reception its distance from the Xanthu is at a minimum, and hence its rate of change is zero. Hence successive wavecrests will travel the same distance, as long as \(c \Delta t \ll a\). Since the wavecrests travel the same distance, the time of departure from the Xanthu is at a minimum, and hence the coordinate system shows the Xanthu is at rest.

(c) To determine the events in the coordinate system shown, the Xanthu is at rest.

\[
\ell_{\text{p}} = \sqrt{t_f}
\]

The time interval between wave crests as measured by the receiver, on the Emmerac, however, is

\[
\ell_{\text{r}} = \ell_{\text{p}} \gamma = \ell_{\text{p}} \sqrt{1 - v^2/c^2}.
\]

Thus, there is a blueshift.

The redshift parameter \(z\) is defined by

\[
\frac{\ell_{\text{r}}}{\ell_{\text{p}}} = \gamma
\]

\[
z = \frac{\ell_{\text{r}}}{\ell_{\text{p}}} - 1
\]

The redshift parameter \(z\) is defined by

\[
z = \frac{\ell_{\text{r}}}{\ell_{\text{p}}} - 1
\]

\[
\ell_{\text{r}} = \ell_{\text{p}} \gamma = \ell_{\text{p}} \sqrt{1 - v^2/c^2}.
\]

So, \(0 = z\). If the source is at the horizon distance, it means that a photon leaves the horizon distance, given by

\[
\ell_{\text{p}} = \frac{t_f}{c} \int_{0}^{t_f} (t_f - t_1) \, dt = \frac{t_f^2}{2}.
\]

So in this case

\[
\ell_{\text{r}} = \frac{t_f}{c} \int_{0}^{t_f} (t_f - t_1) \, dt = \frac{t_f^2}{2}.
\]

**PROBLEM 18: TRACING A LIGHT PULSE THROUGH A RADIATION-DOMINATED UNIVERSE**

The physical horizon distance is given in general by

\[
\ell_{\text{p}}(t_f) = a(t_f) \ell_{c} = 2 c (\sqrt{t_f} - \sqrt{t_f - t_1}).
\]
so \( \frac{z^2/a^2 - 1}{1} > 0 \), and the factor for the inner cars, \( \gamma = 1 + \frac{1}{\sqrt{1 - v^2/c^2}} \), for the outer cars.

Thus, if we let \( \Delta t_S \) denote the time between crests as measured by a clock on the source, the time between crests as measured by the clock of the source will be increased by a factor of \( \gamma \) (eq. (b))."
\[ \frac{2}{\pi + 1} \left( \frac{v}{c} - 1 \right) \lambda = I - \frac{2}{\pi} \frac{I + 1}{\lambda} = z \]

Again \( S \nabla (z + 1) \equiv 0 \nabla \nabla \)

Combining the formulas above,

\[ S \nabla \frac{2}{\pi + 1} \frac{I}{\lambda} = 0 \nabla \]

As in part (a), the time dilation implies that

\[ \gamma^2 \Delta t_S = \Delta t_{\text{Lab}} \]

\[ \gamma \Delta t_O = \Delta t_{\text{Lab}} O \]

To make sure that the \( \gamma \)-factors are on the right side of the equation, you should keep in mind that any time interval should be measured as shorter on the moving clocks than on the lab clocks, since these clocks appear to run slow. Putting together the equations above, one has immediately that

\[ \Delta t_O = \gamma^2 \gamma \Delta t_S \]

The redshift \( z \) is defined by

\[ \Delta t_O \equiv (1 + z) \Delta t_S \]

so

\[ z = \gamma^2 \gamma - 1 = \sqrt{1 - \frac{v^2}{c^2}} \]

(b) For this part of the problem is useful to imagine a relay station located just to the right of car 6 in the diagram, at rest in the laboratory frame. The relay station re-broadcasts the waves as it receives them, and hence has no effect on the frequency received by the observer, but serves the purpose of allowing us to clearly separate the problem into two parts.

The first part of the discussion concerns the redshift of the signal as measured by the relay station. This calculation would involve both the time dilation and the Doppler shift. The waves as they receive from the source, and hence the effect on the frequency received by the observer, but serves the purpose of allowing us to clearly separate the problem into two parts.

The second part of the discussion concerns the transmission from the relay station to car 6. The velocity of car 6 is perpendicular to the direction from which the pulse is being received, so this is a transverse Doppler shift. Any change in path length between successive pulses is second order in \( \Delta t \), so we can put it off. The only change is a change in the direction from the relay station to car 6, which is parallel to the direction from the source to the relay station, so

\[ \Delta t_{\text{Lab}} O = \Delta t_R \]

As in part (a), the time dilation implies that

\[ \gamma^2 \Delta t_O = \Delta t_{\text{Lab}} O \]

Combining the formulas above,

\[ \Delta O = \gamma^2 \sqrt{1 + \frac{u}{c}} \left( 1 - \frac{u}{c} \right) \Delta t_S \]

Again \( \Delta t_O \equiv (1 + z) \Delta t_S \), so

\[ z = \frac{1}{\gamma^2 \sqrt{1 + \frac{u}{c}} \left( 1 - \frac{u}{c} \right)} \]

\[ \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} \]