MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 27, 2009
Prof. Alan Guth

REVIEW PROBLEMS FOR QUIZ 3

QUIZ DATE: Thursday, December 3, 2009, during the normal class time.

COVERAGE: Lecture Notes 7, 8, and 10, Problem Sets 7, 8, and 9; Steven Weinberg, *The First Three Minutes,* Chapter 8 and Afterword; Barbara Ryden, *Introduction to Cosmology*, Chapters 8 *(Dark Matter)* and 9 *(The Cosmic Microwave Background)*; Alan Guth, *Inflation and the New Era of High-Precision Cosmology,*

http://web.mit.edu/physics/alumniandfriends/physicsjournal_fall_02_cosmology.pdf.

One of the problems on the quiz will be taken verbatim (or at least *almost* verbatim) from either the homework assignments, or from the starred problems from this set of Review Problems. The starred problems are the ones that I recommend that you review most carefully: Problems 2, 4, 5, 6, 7, 9, 10, and 11.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They come mainly from quizzes in previous years. In some cases the number of points assigned to the problem on the quiz is listed — in all such cases it is based on 100 points for the full quiz.

In addition to this set of problems, you will find on the course web page the actual quizzes that were given in 1994, 1996, 1998, 2000, 2002, 2004, and 2007. The relevant problems from those quizzes have mostly been incorporated into these review problems, but you still may be interested in looking at the quizzes, just to see how much material has been included in each quiz. The coverage of the upcoming quiz will not necessarily match the coverage of any of the quizzes from previous years. The coverage for each quiz in recent years is usually described at the start of the review problems, as I did here.

REVIEW SESSION AND OFFICE HOURS: A review session and special office hours will be held to help you study for the quiz. Details will follow.

INFORMATION TO BE GIVEN ON QUIZ:

SPEED OF LIGHT IN COMOVING COORDINATES:

$$v_{\rm coord} = \frac{c}{a(t)} .$$

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c)$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\rm observed}}{\lambda_{\rm emitted}} = \frac{a(t_{\rm observed})}{a(t_{\rm emitted})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

Energy-Momentum Four-Vector:

$$p^\mu = \left(\frac{E}{c}, \vec p\right) , \quad \vec p = \gamma m_0 \vec v , \quad E = \gamma m_0 c^2 = \sqrt{\left(m_0 c^2\right)^2 + |\vec p|^2 c^2} ,$$

$$p^2 \equiv |\vec p|^2 - \left(p^0\right)^2 = |\vec p|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 .$$

COSMOLOGICAL EVOLUTION:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} , \quad \ddot a = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a ,$$

$$\rho_m(t) = \frac{a^3(t_i)}{a^3(t)}\, \rho_m(t_i) \ \text{(matter)}, \quad \rho_r(t) = \frac{a^4(t_i)}{a^4(t)}\, \rho_r(t_i) \ \text{(radiation)}.$$

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) , \quad \Omega \equiv \rho/\rho_c , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} .$$

Flat $(k = 0)$: $a(t) \propto t^{2/3}$ (matter-dominated) ,

$a(t) \propto t^{1/2}$ (radiation-dominated) ,

$\Omega = 1$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

Closed $(k > 0)$: $ct = \alpha(\theta - \sin\theta)$, $\dfrac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)$,

$$\Omega = \frac{2}{1+\cos\theta} > 1 ,$$

where $\alpha \equiv \dfrac{4\pi}{3}\dfrac{G\rho}{c^2}\left(\dfrac{a}{\sqrt{k}}\right)^3$.

Open $(k < 0)$: $ct = \alpha(\sinh\theta - \theta)$, $\dfrac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1)$,

$$\Omega = \frac{2}{1+\cosh\theta} < 1 ,$$

where $\alpha \equiv \dfrac{4\pi}{3}\dfrac{G\rho}{c^2}\left(\dfrac{a}{\sqrt{\kappa}}\right)^3$,

$$\kappa \equiv -k > 0 .$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{g_{ij}\frac{dx^j}{ds}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{ds}\frac{dx^\ell}{ds}$$

or: $$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \qquad \text{(energy density)}$$

$$p = \frac{1}{3}u \qquad \rho = u/c^2 \qquad \text{(pressure, mass density)}$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} \qquad \text{(number density)}$$

$$s = g\frac{2\pi^2}{45}\frac{k^4T^3}{(\hbar c)^3} , \qquad \text{(entropy density)}$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions ,} \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 .$$

$$g_\gamma = g_\gamma^* = 2 ,$$

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\\nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} ,$$

$$g_\nu^* = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\\nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2} ,$$

$$g_{e^+e^-} = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2}\,,$$

$$g^*_{e^+e^-} = \underbrace{\frac{3}{4}}_{\substack{\text{Fermion}\\ \text{factor}}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\substack{\text{Particle/}\\ \text{antiparticle}}} \times \underbrace{2}_{\text{Spin states}} = 3\,.$$

CHEMICAL EQUILIBRIUM:

Ideal Gas of Classical Nonrelativistic Particles:

$$n_i = g_i \frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3}\, e^{(\mu_i - m_i c^2)/kT}\,.$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$\rho = \frac{3}{32\pi G t^2}$$

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}} \left(\frac{10.75}{g}\right)^{1/4}$$

After the freeze-out of electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}\,.$$

HORIZON DISTANCE:

$$\ell_{p,\text{horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt' = \begin{cases} 3ct & \text{(flat, matter-dominated)},\\ 2ct & \text{(flat, radiation-dominated)}. \end{cases}$$

COSMOLOGICAL CONSTANT:

$$u_{\text{vac}} = \rho_{\text{vac}} c^2 = \frac{\Lambda c^4}{8\pi G}\,,$$

$$p_{\text{vac}} = -\rho_{\text{vac}} c^2 = -\frac{\Lambda c^4}{8\pi G}\,.$$

GENERALIZED COSMOLOGICAL EVOLUTION:

$$x\frac{dx}{dt} = H_0\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2}\,,$$

where

$$x \equiv \frac{a(t)}{a(t_0)} \equiv \frac{1}{1+z}\,,$$

$$\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0}\,.$$

Age of universe:

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{x\,dx}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \Omega_{k,0}x^2}}$$
$$= \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}}\,.$$

Look-back time:

$$t_{\text{look-back}}(z) = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\text{rad},0}(1+z')^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z')^2}}\,.$$

PHYSICAL CONSTANTS:

$$G = 6.674 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}} = 6.674 \times 10^{-8}\ \mathrm{cm^3 \cdot g^{-1} \cdot s^{-2}}$$

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381 \times 10^{-23}\,\mathrm{joule/K} \\ &= 1.381 \times 10^{-16}\,\mathrm{erg/K} \\ &= 8.617 \times 10^{-5}\,\mathrm{eV/K} \end{aligned}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055 \times 10^{-34}\ \mathrm{joule \cdot s} \\ &= 1.055 \times 10^{-27}\ \mathrm{erg \cdot s} \\ &= 6.582 \times 10^{-16}\ \mathrm{eV \cdot s} \end{aligned}$$

$$\begin{aligned} c &= 2.998 \times 10^{8}\ \mathrm{m/s} \\ &= 2.998 \times 10^{10}\ \mathrm{cm/s} \end{aligned}$$

$$\hbar c = 197.3\ \text{MeV-fm}, \quad 1\ \mathrm{fm} = 10^{-15}\ \mathrm{m}$$

$$1\ \mathrm{yr} = 3.156 \times 10^{7}\ \mathrm{s}$$

$$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{joule} = 1.602 \times 10^{-12}\ \mathrm{erg}$$

$$\begin{aligned} 1\ \mathrm{GeV} = 10^9\ \mathrm{eV} &= 1.783 \times 10^{-27}\ \ \mathrm{kg}\ (\text{where } c \equiv 1) \\ &= 1.783 \times 10^{-24}\ \mathrm{g}\ . \end{aligned}$$

Planck Units: The Planck length ℓ_P, the Planck time t_P, the Planck mass m_P, and the Planck energy E_p are given by

$$\begin{aligned} \ell_P = \sqrt{\frac{G\hbar}{c^3}} &= 1.616 \times 10^{-35}\ \mathrm{m}\ , \\ &= 1.616 \times 10^{-33}\ \mathrm{cm}\ , \\ t_P = \sqrt{\frac{\hbar G}{c^5}} &= 5.391 \times 10^{-44}\ \mathrm{s}\ , \\ m_P = \sqrt{\frac{\hbar c}{G}} &= 2.177 \times 10^{-8}\ \mathrm{kg}\ , \\ &= 2.177 \times 10^{-5}\ \mathrm{g}\ , \\ E_P = \sqrt{\frac{\hbar c^5}{G}} &= 1.221 \times 10^{19}\ \mathrm{GeV}\ . \end{aligned}$$

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following problem was Problem 1, Quiz 3, in 2007. Each part was worth 5 points.

(a) (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

(ii) After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

(b) (CMB experiments) The current mean energy per CMB photon, about 6×10^{-4} eV, is comparable to the energy of vibration or rotation for a small molecule such as H_2O. Thus microwaves with wavelengths shorter than $\lambda \sim 3$ cm are strongly absorbed by water molecules in the atmosphere. To measure the CMB at $\lambda < 3$ cm, which one of the following methods is *not* a feasible solution to this problem?

(i) Measure CMB from high-altitude balloons, e.g. MAXIMA.

(ii) Measure CMB from the South Pole, e.g. DASI.

(iii) Measure CMB from the North Pole, e.g. BOOMERANG.

(iv) Measure CMB from a satellite above the atmosphere of the Earth, e.g. COBE, WMAP and PLANCK.

(c) (Temperature fluctuations) The creation of temperature fluctuations in CMB by variations in the gravitational potential is known as the Sachs-Wolfe effect. Which one of the following statements is *not* correct concerning this effect?

(i) A CMB photon is redshifted when climbing out of a gravitational potential well, and is blueshifted when falling down a potential hill.

(ii) At the time of last scattering, the nonbaryonic dark matter dominated the energy density, and hence the gravitational potential, of the universe.

(iii) The large-scale fluctuations in CMB temperatures arise from the gravitational effect of primordial density fluctuations in the distribution of nonbaryonic dark matter.

(iv) The peaks in the plot of temperature fluctuation Δ_T vs. multipole l are due to variations in the density of nonbaryonic dark matter, while the contributions from baryons alone would not show such peaks.

(d) (Dark matter candidates) Which one of the following is *not* a candidate of nonbaryonic dark matter?

(i) massive neutrinos

(ii) axions

(iii) matter made of top quarks (a type of quarks with heavy mass of about 171 GeV).

(iv) WIMPs (Weakly Interacting Massive Particles)

(v) primordial black holes

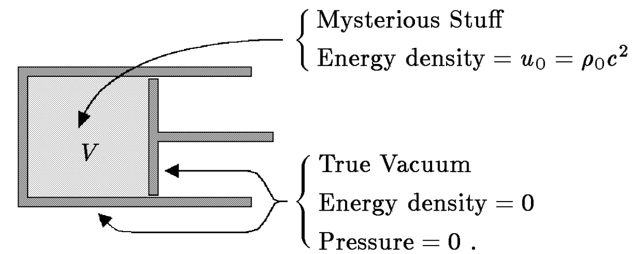
(e) (Signatures of dark matter) By what methods can signatures of dark matter be detected? List two methods. (Grading: 3 points for one correct answer, 5 points for two correct answers. If you give more than two answers, your score will be based on the number of right answers minus the number of wrong answers, with a lower bound of zero.)

*PROBLEM 2: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

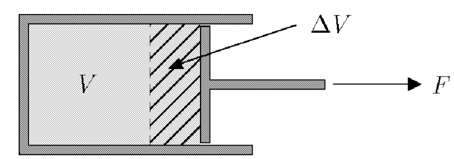
The following problem was Problem 3, Quiz 3, 2002. Although it is couched in the language of Lecture Notes 13, the physics is really the same as the pressure calculations in Lecture Notes 7, so a modified form of this problem would be fair for the coming quiz.

In Lecture Notes 13, a thought experiment involving a piston was used to show that $p = -\rho c^2$ for any substance for which the energy density remains constant under expansion. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

PROBLEM 3: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

Today the temperature of the cosmic microwave background radiation is 2.7°K. Calculate the number density of photons in this radiation. What is the number density of thermal neutrinos left over from the big bang?

∗PROBLEM 4: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

The following problem was Problem 4, Quiz 3, 1998.

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\zeta(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?

∗PROBLEM 5: A NEW SPECIES OF LEPTON

The following problem was Problem 2, Quiz 3, 1992, worth 25 points.

Suppose the calculations describing the early universe were modified by including an additional, hypothetical lepton, called an 8.286ion. The 8.286ion has roughly the same properties as an electron, except that its mass is given by $mc^2 = 0.750$ MeV.

Parts (a)-(c) of this question require numerical answers, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included. (For example, if you were asked how many meters a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$.)

a) *(5 points)* What would be the number density of 8.286ions, in particles per cubic meter, when the temperature T was given by $kT = 3$ MeV?

b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the

value of the mass density at $t = .01$ sec? You may assume that 0.75 MeV $\ll kT \ll 100$ MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, e^+-e^- pairs, and 8.286ion-anti8286ion pairs. Express your answer in the units of g/cm^3.

c) *(5 points)* Under the same assumptions as in (b), what would be the value of kT, in MeV, at $t = .01$ sec?

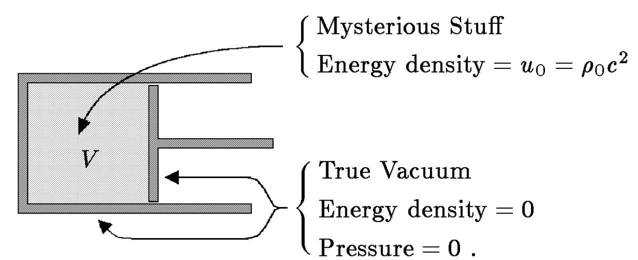
d) *(5 points)* When nucleosynthesis calculations are modified to include the effect of the 8.286ion, is the production of helium increased or decreased? Explain your answer in a few sentences.

e) *(5 points)* Suppose the neutrinos decouple while $kT \gg 0.75$ MeV. If the 8.286ions are included, what does one predict for the value of T_ν/T_γ today? (Here T_ν denotes the temperature of the neutrinos, and T_γ denotes the temperature of the cosmic background radiation photons.)

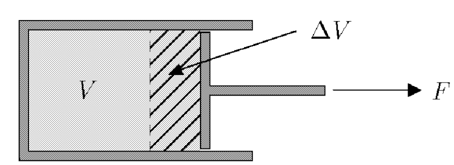
∗PROBLEM 6: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

In Lecture Notes 13, a thought experiment involving a piston was used to show that $p = -\rho c^2$ for any substance for which the energy density remains constant under expansion. In this problem you will apply the same technique to calculate the pressure of ***mysterious stuff***, which has the property that the energy density falls off in proportion to $1/\sqrt{V}$ as the volume V is increased.

If the initial energy density of the mysterious stuff is $u_0 = \rho_0 c^2$, then the initial configuration of the piston can be drawn as



The piston is then pulled outward, so that its initial volume V is increased to $V + \Delta V$. You may consider ΔV to be infinitesimal, so ΔV^2 can be neglected.



(a) *(15 points)* Using the fact that the energy density of mysterious stuff falls off as $1/\sqrt{V}$, find the amount ΔU by which the energy inside the piston changes when the volume is enlarged by ΔV. Define ΔU to be positive if the energy increases.

(b) *(5 points)* If the (unknown) pressure of the mysterious stuff is called p, how much work ΔW is done by the agent that pulls out the piston?

(c) *(5 points)* Use your results from (a) and (b) to express the pressure p of the mysterious stuff in terms of its energy density u. (If you did not answer parts (a) and/or (b), explain as best you can how you would determine the pressure if you knew the answers to these two questions.)

∗ PROBLEM 7: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

Consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and the same ***mysterious stuff*** that was introduced in the previous problem. Since the mass density of mysterious stuff falls off as $1/\sqrt{V}$, where V is the volume, it follows that in an expanding universe the mass density of mysterious stuff falls off as $1/a^{3/2}(t)$.

Suppose that you are given the present value of the Hubble parameter H_0, and also the present values of the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\text{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe t_0 in terms of these quantities.

(a) *(8 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{a(t)}{a(t_0)}$$

for an arbitrary time t. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(b) *(7 points)* Write an integral expression for the age of the universe t_0. The expression should depend only on H_0 and the various contributions to Ω_0 listed above ($\Omega_{m,0}$, $\Omega_{r,0}$, etc.), but it might include x as a variable of integration.

Extra Credit for Super-Sharpies (no partial credit): For 5 points extra credit, you can calculate the angular diameter $\Delta\theta$ of the image of a spherical object at redshift z which had a physical diameter w at the time of emission. You should assume that $\Omega_{\text{tot}} < 1$, and also that $\Delta\theta \ll 1$. The calculation is to be carried out for the same model universe described above.

PROBLEM 8: TIME SCALES IN COSMOLOGY

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-37} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the beginning of the processes involved in big bang nucleosynthesis;

(b) the end of the processes involved in big bang nucleosynthesis;

(c) the time of the phase transition predicted by grand unified theories, which takes place when $kT \approx 10^{16}$ GeV;

(d) "recombination", the time at which the matter in the universe converted from a plasma to a gas of neutral atoms;

(e) the phase transition at which the quarks became confined, believed to occur when $kT \approx 300$ MeV.

Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give **ONLY ONE** of the acceptable answers.

∗ PROBLEM 9: EVOLUTION OF FLATNESS *(15 points)*

The following problem was Problem 3, Quiz 3, 2004.

The "flatness problem" is related to the fact that during the evolution of the standard cosmological model, Ω is always driven away from 1.

(a) *(9 points)* During a period in which the universe is matter-dominated (meaning that the only relevant component is nonrelativistic matter), the quantity

$$\frac{\Omega - 1}{\Omega}$$

grows as a power of t. Show that this is true, and derive the power. (Stating the right power without a derivation will be worth 3 points.)

(b) *(6 points)* During a period in which the universe is radiation-dominated, the same quantity will grow like a different power of t. Show that this is true, and derive the power. (Stating the right power without a derivation will again be worth 3 points.)

In each part, you may assume that the universe was *always* dominated by the specified form of matter.

*PROBLEM 10: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

The following problem was Problem 4, Quiz 3, 2004.

On April 13, 2000, the Sloan Digital Sky Survey announced the discovery of what was then the most distant object known in the universe: a quasar at $z = 5.82$. To explain to the public how this object fits into the universe, the SDSS posted on their website an article by Michael Turner and Craig Wiegert titled "How Can An Object We See Today be 27 Billion Light Years Away If the Universe is only 14 Billion Years Old?" Using a model with $H_0 = 65$ km-s^{-1}-Mpc^{-1}, $\Omega_m = 0.35$, and $\Omega_\Lambda = 0.65$, they claimed

(a) that the age of the universe is 13.9 billion years.

(b) that the light that we now see was emitted when the universe was 0.95 billion years old.

(c) that the distance to the quasar, as it would be measured by a ruler today, is 27 billion light-years.

(d) that the distance to the quasar, at the time the light was emitted, was 4.0 billion light-years.

(e) that the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing, is 1.8 times the velocity of light.

The goal of this problem is to check all of these conclusions, although you are of course not expected to actually work out the numbers. Your answers can be expressed in terms of H_0, Ω_m, Ω_Λ, and z. Definite integrals need not be evaluated.

Note that Ω_m represents the present density of nonrelativistic matter, expressed as a fraction of the critical density; and Ω_Λ represents the present density of vacuum energy, expressed as a fraction of the critical density. In answering each of the following questions, you may consider the answer to any previous part — whether you answered it or not — as a given piece of information, which can be used in your answer.

(a) *(15 points)* Write an expression for the age t_0 of this model universe?

(b) *(5 points)* Write an expression for the time t_e at which the light which we now receive from the distant quasar was emitted.

(c) *(10 points)* Write an expression for the present physical distance $\ell_{\text{phys},0}$ to the quasar.

(d) *(5 points)* Write an expression for the physical distance $\ell_{\text{phys,e}}$ between us and the quasar at the time that the light was emitted.

(e) *(5 points)* Write an expression for the present speed of the quasar, defined as the rate at which the distance between us and the quasar is increasing.

*PROBLEM 11: NEUTRINO NUMBER AND THE NEUTRON/ PROTON EQUILIBRIUM *(35 points)*

The following problem was 1998 Quiz 4, Problem 4.

In the standard treatment of big bang nucleosynthesis it is assumed that at early times the ratio of neutrons to protons is given by the Boltzmann formula,

$$\frac{n_n}{n_p} = e^{-\Delta E/kT} , \tag{1}$$

where k is Boltzmann's constant, T is the temperature, and $\Delta E = 1.29$ MeV is the proton-neutron mass-energy difference. This formula is believed to be very accurate, but it assumes that the chemical potential for neutrons μ_n is the same as the chemical potential for protons μ_p.

(a) *(10 points)* Give the correct version of Eq. (1), allowing for the possibility that $\mu_n \neq \mu_p$.

The equilibrium between protons and neutrons in the early universe is sustained mainly by the following reactions:

$$e^+ + n \longleftrightarrow p + \bar{\nu}_e$$
$$\nu_e + n \longleftrightarrow p + e^- .$$

Let μ_e and μ_ν denote the chemical potentials for the electrons (e^-) and the electron neutrinos (ν_e) respectively. The chemical potentials for the positrons (e^+) and the anti-electron neutrinos ($\bar{\nu}_e$) are then $-\mu_e$ and $-\mu_\nu$, respectively, since the chemical potential of a particle is always the negative of the chemical potential for the antiparticle.*

(b) *(10 points)* Express the neutron/proton chemical potential difference $\mu_n - \mu_p$ in terms of μ_e and μ_ν.

The black-body radiation formulas at the beginning of the quiz did not allow for the possibility of a chemical potential, but they can easily be generalized. For example, the formula for the number density n_i (of particles of type i) becomes

$$n_i = g_i^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{\mu_i/kT} .$$

(c) *(10 points)* Suppose that the density of anti-electron neutrinos $\bar{n}_\nu$ in the early universe was higher than the density of electron neutrinos n_ν. Express the thermal equilibrium value of the ratio n_n/n_p in terms of ΔE, T, and either the ratio $\bar{n}_\nu/n_\nu$ or the antineutrino excess $\Delta n = \bar{n}_\nu - n_\nu$. (Your answer may also contain fundamental constants, such as k, $\hbar$, and c.)

(d) *(5 points)* Would an excess of anti-electron neutrinos, as considered in part (c), increase or decrease the amount of helium that would be produced in the early universe? Explain your answer.

PROBLEM 12: SECOND HUBBLE CROSSING *(40 points)*

This problem was Problem 3, Quiz 3, 2007. In 2009 we have not yet talked about Hubble crossings and the evolution of density perturbations, so this problem would not be fair as worded. Actually, however, you have learned how to do these calculations, so the problem would be fair if it described in more detail what needs to be calculated.

In Problem Set 9 we calculated the time $t_{H1}(\lambda)$ of the first Hubble crossing for a mode specified by its (physical) wavelength λ at the present time. In this problem we will calculate the time $t_{H2}(\lambda)$ of the second Hubble crossing, the time at which the growing Hubble length $cH^{-1}(t)$ catches up to the physical wavelength, which is also growing. At the time of the second Hubble crossing for the wavelengths of

* This fact is a consequence of the principle that the chemical potential of a particle is the sum of the chemical potentials associated with its conserved quantities, while particle and antiparticle always have the opposite values of all conserved quantities.

interest, the universe can be described very simply: it is a radiation-dominated flat universe. However, since λ is defined as the present value of the wavelength, the evolution of the universe between $t_{H2}(\lambda)$ and the present will also be relevant to the problem. We will need to use methods, therefore, that allow for both the matter-dominated era and the onset of the dark-energy-dominated era. As in Problem Set 9, the model universe that we consider will be described by the WMAP 3-year best fit parameters:

Hubble expansion rate	H_0	$= 73.5 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$
Nonrelativistic mass density	Ω_m	$= 0.237$
Vacuum mass density	Ω_{vac}	$= 0.763$
CMB temperature	$T_{\gamma,0}$	$= 2.725$ K

The mass densities are defined as contributions to Ω, and hence describe the mass density of each constituent relative to the critical density. Note that the model is exactly flat, so you need not worry about spatial curvature. Here you are not expected to give a numerical answer, so the above list will serve only to define the symbols that can appear in your answers, along with λ and the physical constants G, $\hbar$, c, and k.

(a) *(5 points)* For a radiation-dominated flat universe, what is the Hubble length $\ell_H(t) \equiv cH^{-1}(t)$ as a function of time t?

(b) *(10 points)* The second Hubble crossing will occur during the interval

$$30 \text{ sec} \ll t \ll 50,000 \text{ years},$$

when the mass density of the universe is dominated by photons and neutrinos. During this era the neutrinos are a little colder than the photons, with $T_\nu = (4/11)^{1/3} T_\gamma$. The total energy density of the photons and neutrinos together can be written as

$$u_{\text{tot}} = g_1 \frac{\pi^2}{30} \frac{(kT_\gamma)^4}{(\hbar c)^3} .$$

What is the value of g_1? (For the following parts you can treat g_1 as a given variable that can be left in your answers, whether or not you found it.)

(c) *(10 points)* For times in the range described in part (b), what is the photon temperature $T_\gamma(t)$ as a function of t?

(d) *(15 points)* Finally, we are ready to find the time $t_{H2}(\lambda)$ of the second Hubble crossing, for a given value of the physical wavelength λ today. Making use of the previous results, you should be able to determine $t_{H2}(\lambda)$. If you were not able to answer some of the previous parts, you may leave the symbols $\ell_H(t)$, g_1, and/or $T_\gamma(t)$ in your answer.

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following parts are each worth 5 points.

(a) (CMB basic facts) Which one of the following statements about CMB is *not* correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is $\langle T \rangle = 2.725K$.

[(ii)] After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}$.

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

Explanation: After subtracting the dipole contribution, the temperature fluctuation is about 1.1×10^{-5}.

(b) (CMB experiments) The current mean energy per CMB photon, about 6×10^{-4} eV, is comparable to the energy of vibration or rotation for a small molecule such as H_2O. Thus microwaves with wavelengths shorter than $\lambda \sim 3$ cm are strongly absorbed by water molecules in the atmosphere. To measure the CMB at $\lambda < 3$ cm, which one of the following methods is *not* a feasible solution to this problem?

(i) Measure CMB from high-altitude balloons, e.g. MAXIMA.

(ii) Measure CMB from the South Pole, e.g. DASI.

[(iii)] Measure CMB from the North Pole, e.g. BOOMERANG.

(iv) Measure CMB from a satellite above the atmosphere of the Earth, e.g. COBE, WMAP and PLANCK.

Explanation: The North Pole is at sea level. In contrast, the South Pole is nearly 3 kilometers above sea level. BOOMERANG is a balloon-borne experiment launched from Antarctica.

(c) (Temperature fluctuations) The creation of temperature fluctuations in CMB by variations in the gravitational potential is known as the Sachs-Wolfe effect. Which one of the following statements is *not* correct concerning this effect?

(i) A CMB photon is redshifted when climbing out of a gravitational potential well, and is blueshifted when falling down a potential hill.

(ii) At the time of last scattering, the nonbaryonic dark matter dominated the energy density, and hence the gravitational potential, of the universe.

(iii) The large-scale fluctuations in CMB temperatures arise from the gravitational effect of primordial density fluctuations in the distribution of nonbaryonic dark matter.

[(iv)] The peaks in the plot of temperature fluctuation Δ_T vs. multipole l are due to variations in the density of nonbaryonic dark matter, while the contributions from baryons alone would not show such peaks.

Explanation: These peaks are due to the acoustic oscillations in the photon-baryon fluid.

(d) (Dark matter candidates) Which one of the following is *not* a candidate of nonbaryonic dark matter?

(i) massive neutrinos

(ii) axions

[(iii)] matter made of top quarks (a type of quarks with heavy mass of about 171 GeV).

(iv) WIMPs (Weakly Interacting Massive Particles)

(v) primordial black holes

Explanation: Matter made of top quarks is so unstable that it is seen only fleetingly as a product in high energy particle collisions.

(e) (Signatures of dark matter) By what methods can signatures of dark matter be detected? List two methods. (Grading: 3 points for one correct answer, 5 points for two correct answers. If you give more than two answers, your score will be based on the number of right answers minus the number of wrong answers, with a lower bound of zero.)

Answers:

(i) Galaxy rotation curves. (I.e., measurements of the orbital speed of stars in spiral galaxies as a function of radius R show that these curves remain flat at radii far beyond the visible stellar disk. If most of the matter were contained in the disk, then these velocities should fall off as $1/\sqrt{R}$.*)*

(ii) *Use the virial theorem to estimate the mass of a galaxy cluster. (For example, the virial analysis shows that only 2% of the mass of the Coma cluster consists of stars, and only 10% consists of hot intracluster gas.*

(iii) *Gravitational lensing. (For example, the mass of a cluster can be estimated from the distortion of the shapes of the galaxies behind the cluster.)*

(iv) *CMB temperature fluctuations. (I.e., the analysis of the intensity of the fluctuations as a function of multipole number shows that* $\Omega_{\rm tot} \approx 1$*, and that dark energy contributes* $\Omega_\Lambda \approx 0.7$*, baryonic matter contributes* $\Omega_{\rm bary} \approx 0.04$*, and dark matter contributes* $\Omega_{\rm dark\ matter} \approx 0.26$*.)*

There are other possible answers as well, but these are the ones discussed by Ryden in Chapters 8 and 9.

PROBLEM 2: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} \ .$$

(The above expression is proportional to $1/\sqrt{V + \Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1 + \frac{\Delta V}{V}}} = \frac{u_0}{1 + \frac{1}{2}\frac{\Delta V}{V}} = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) \ .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) \ ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2} u_0 \ .}$$

PROBLEM 3: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

In general, the number density of a particle in the black-body radiation is given by

$$n = g^* \frac{\xi(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3$$

For photons, one has $g^* = 2$. Then

$$\left.\begin{aligned} k &= 1.381 \times 10^{-16} \text{erg}/^\circ\text{K} \\ T &= 2.7\,^\circ\text{K} \\ \hbar &= 1.055 \times 10^{-27} \text{erg-sec} \\ c &= 2.998 \times 10^{10} \text{cm/sec} \end{aligned}\right\} \implies \left(\frac{kT}{\hbar c}\right)^3 = 1.638 \times 10^3 \text{cm}^{-3} \ .$$

Then using $\xi(3) \simeq 1.202$, one finds

$$\boxed{n_\gamma = 399/\text{cm}^3 \ .}$$

For the neutrinos,

$$g_\nu^* = 2 \times \frac{3}{4} = \frac{3}{2} \quad \text{per species.}$$

The factor of 2 is to account for ν and $\bar{\nu}$, and the factor of $3/4$ arises from the Pauli exclusion principle. So for three species of neutrinos one has

$$g_\nu^* = \frac{9}{2} \ .$$

Using the result

$$T_\nu^3 = \frac{4}{11} T_\gamma^3$$

from Problem 8 of Problem Set 3 (2000), one finds

$$n_\nu = \left(\frac{g^*_\nu}{g^*_\gamma}\right)\left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma$$
$$= \left(\frac{9}{4}\right)\left(\frac{4}{11}\right) 399\text{cm}^{-3}$$
$$\Longrightarrow \boxed{n_\nu = 326/\text{cm}^3 \text{ (for all three species combined).}}$$

PROBLEM 4: PROPERTIES OF BLACK-BODY RADIATION

(a) The average energy per photon is found by dividing the energy density by the number density. The photon is a boson with two spin states, so $g = g^* = 2$. Using the formulas on the front of the exam,

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{\pi^4}{30\zeta(3)} kT\ .}$$

You were not expected to evaluate this numerically, but it is interesting to know that

$$E = 2.701\, kT\ .$$

Note that the average energy per photon is significantly more than kT, which is often used as a rough estimate.

(b) The method is the same as above, except this time we use the formula for the entropy density:

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{2\pi^4}{45\zeta(3)} k\ .}$$

Numerically, this gives $3.602\, k$, where k is the Boltzmann constant.

(c) In this case we would have $g = g^* = 1$. The average energy per particle and the average entropy particle depends only on the ratio g/g^*, so there would be **no difference from the answers given in parts (a) and (b).**

(d) For a fermion, g is $7/8$ times the number of spin states, and g^* is $3/4$ times the number of spin states. So the average energy per particle is

$$E = \frac{g\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \frac{\dfrac{7}{8}\dfrac{\pi^2}{30}\dfrac{(kT)^4}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{7\pi^4}{180\zeta(3)} kT\ .}$$

Numerically, $E = 3.1514\, kT$.

> Warning: the Mathematician General has determined that the memorization of this number may adversely affect your ability to remember the value of π.

If one takes into account both neutrinos and antineutrinos, the average energy per particle is unaffected — the energy density and the total number density are both doubled, but their ratio is unchanged.

Note that the energy per particle is higher for fermions than it is for bosons. This result can be understood as a natural consequence of the fact that fermions must obey the exclusion principle, while bosons do not. Large numbers of bosons can therefore collect in the lowest energy levels. In fermion systems, on the other hand, the low-lying levels can accommodate at most one particle, and then additional particles are forced to higher energy levels.

(e) The values of g and g^* are again 7/8 and 3/4 respectively, so

$$S = \frac{g\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{g^*\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \frac{\dfrac{7}{8}\dfrac{2\pi^2}{45}\dfrac{k^4T^3}{(\hbar c)^3}}{\dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2}\dfrac{(kT)^3}{(\hbar c)^3}} = \boxed{\frac{7\pi^4}{135\zeta(3)}k} \ .$$

Numerically, this gives $S = 4.202\,k$.

PROBLEM 5: A NEW SPECIES OF LEPTON

a) The number density is given by the formula at the start of the exam,

$$n = g^*\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} \ .$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$n = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\ \cancel{\text{MeV}}}{6.582\times 10^{-16}\ \cancel{\text{eV-sec}} \times 2.998\times 10^{10}\ \cancel{\text{cm-sec}^{-1}}}\right)^3 \times \left(\frac{10^6\ \cancel{\text{eV}}}{1\ \cancel{\text{MeV}}}\right)^3 \times \left(\frac{10^2\ \cancel{\text{cm}}}{1\ \text{m}}\right)^3$$

$$= 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\times 10^6 \times 10^2}{6.582\times 10^{-16} \times 2.998\times 10^{10}}\right)^3 \text{m}^{-3} \ .$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\times 10^6 \times 10^2}{6.582\times 10^{-16} \times 2.998\times 10^{10}}\right)^3} \ .$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho \ .$$

During the radiation-dominated era $a(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot a}{a} = \frac{1}{2t} \ .$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3}G\rho \ .$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \ .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}$ cm^3 sec^{-2} g^{-1}, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732\times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9$ g/cm^3.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \ .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion − anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\text{tot}} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2}\frac{1}{g_{\text{tot}}}\hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/a$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\text{rest}} T^3(t) a^3(t)$$

where $g_{\text{rest}} = g_{\text{tot}} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 a^3(t) = g_{\text{rest}} T^3(t) a^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\text{rest}}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 6: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} \; .$$

(The above expression is proportional to $1/\sqrt{V + \Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1 + \frac{\Delta V}{V}}} = \frac{u_0}{1 + \frac{1}{2}\frac{\Delta V}{V}} = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) \; .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) \; ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \; .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V}U_0 .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2}u_0 .}$$

PROBLEM 7: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF

(a) The critical density ρ_c is defined as that density for which $k = 0$, where the Friedmann equation from the front of the exam implies that

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} .$$

Thus the critical density today is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} .$$

The mass density today of any species X is then related to $\Omega_{X,0}$ by

$$\rho_{X,0} = \rho_c\Omega_{X,0} = \frac{3H_0^2\Omega_{X,0}}{8\pi G} .$$

The total mass density today is then expressed in terms of its four components as

$$\rho_0 = \frac{3H_0^2}{8\pi G}\left[\Omega_{m,0} + \Omega_{r,0} + \Omega_{v,0} + \Omega_{\text{ms},0}\right] .$$

But we also know how each of these contributions to the mass density scales with $x(t)$: $\rho_m \propto 1/x^3$, $\rho_r \propto 1/x^4$, $\rho_v \propto 1$, and $\rho_{\text{ms}} \propto 1/\sqrt{V} \propto 1/x^{3/2}$. Inserting these factors,

$$\rho(t) = \frac{3H_0^2}{8\pi G}\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}}\right] .$$

(b) The Friedmann equation then becomes

$$\left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G}{3}\frac{3H_0^2}{8\pi G}\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}}\right] - \frac{kc^2}{a^2} .$$

Defining

$$H_0^2\Omega_{k,0} = -\frac{kc^2}{a^2(t_0)} ,$$

so

$$-\frac{kc^2}{a^2(t)} = -\frac{kc^2}{a^2(t_0)}\frac{1}{x^2} = \frac{H_0^2\Omega_{k,0}}{x^2} ,$$

and then the Friedmann equation becomes

$$\left(\frac{\dot{x}}{x}\right)^2 = H_0^2\left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}\right] .$$

Applying this equation today, when $\dot{x}/x = H_0$, one finds that

$$\boxed{\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{v,0} - \Omega_{\text{ms},0} .}$$

Rearranging the equation for $(\dot{x}/x)^2$ above,

$$H_0\,\mathrm{d}t = \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} .$$

The age of the universe is found by integrating over the full range of x, which starts from 0 when the universe is born, and is equal to 1 today. So

$$\boxed{t_0 = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} .}$$

Extra Credit for Super-Sharpies (no partial credit):

Since $\Omega_{\text{tot}} < 1$, we use the Robertson-Walker open universe form

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1+r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} ,$$

where I have started with the general form from the front of the exam, and replaced k by -1. To discuss the radial transmission of light rays it is useful to define a new radial coordinate

$$r = \sinh\psi ,$$

so

$$\mathrm{d}r = \cosh\psi\,\mathrm{d}\psi = \sqrt{1+r^2}\,\mathrm{d}\psi ,$$

where I used the identity that $\cosh^2\psi = 1 + \sinh^2\psi$. The metric can then be rewritten as

$$\mathrm{d}s^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} .$$

Light rays then travel with $\mathrm{d}\tau^2 = 0$, so

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{c}{a(t)} .$$

If a light ray leaves the object at time t_e and arrives at Earth today, then it will travel an interval of ψ given by

$$\psi = \int_{t_e}^{t_0} \frac{c}{a(t')}\,\mathrm{d}t' .$$

We will need to know ψ, but we don't know either t_e or $a(t)$. So we need to manipulate the right-hand side of the above equation to express it in terms of things that we do know. Changing integration variables from t' to x, where $x = a(t')/a(t_0)$, one finds $\mathrm{d}x = \dot{x}\,\mathrm{d}t'$, and then

$$\psi = \int_{x_e}^{1} \frac{c}{a(t_0)}\frac{1}{x}\frac{\mathrm{d}x}{\dot{x}} .$$

Using $H = \dot{x}/x$,

$$\psi = \frac{c}{a(t_0)}\int_{x_e}^{1}\frac{\mathrm{d}x}{x^2 H} .$$

Now use the formula for $H = \dot{x}/x$ from part (b), so

$$\psi = \frac{c}{a(t_0)H_0}\int_{x_e}^{1}\frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} .$$

Here

$$\boxed{x_e = \frac{a(t_e)}{a(t_0)} = \frac{1}{1+z}} ,$$

and the coefficient in front of the integral can be evaluated using the Friedman equation for $k = -1$:

$$H_0^2 = \frac{8\pi}{3}G\rho_0 + \frac{c^2}{a^2(t_0)} = H_0^2\Omega_0 + \frac{c^2}{a^2(t_0)} ,$$

so

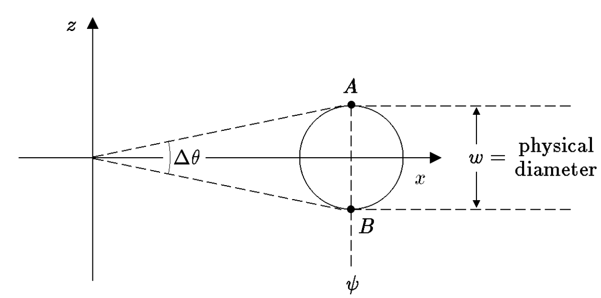
$$\frac{c^2}{a^2(t_0)\,H_0^2} = 1 - \Omega_0 = \Omega_{k,0} .$$

Finally, then, the expression for ψ can be written

$$\boxed{\psi = \sqrt{\Omega_{k,0}}\int_{x_e}^{1}\frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}}} ,$$

where x_e is given by the boxed equation above.

Once we know ψ, the rest is straightforward. We draw a picture in comoving coordinates of the light rays leaving the object and arriving at Earth:



In this picture $\Delta\theta$ is the angular size that would be measured. Using the $\mathrm{d}\theta^2$ part of the metric,

$$\mathrm{d}s^2 = a^2(t)\sinh^2\psi\,\mathrm{d}\theta^2 ,$$

we can relate w, the physical size of the object at the time of emission, to $\Delta\theta$:

$$w = a(t_e)\sinh\psi\,\Delta\theta .$$

To evaluate $a(t_e)$ we can use

$$a(t_e) = x_e a(t_0) = \frac{x_e c}{H_0 \sqrt{\Omega_{k,0}}} .$$

Finally, then,

$$\boxed{\Delta\theta = \frac{w H_0 \sqrt{\Omega_{k,0}}}{x_e c \sinh\psi}} ,$$

where ψ is given by the boxed equation above.

PROBLEM 8: TIME SCALES IN COSMOLOGY

(a) 1 sec. *[This is the time at which the weak interactions begin to "freeze out", so that free neutron decay becomes the only mechanism that can interchange protons and neutrons. From this time onward, the relative number of protons and neutrons is no longer controlled by thermal equilibrium considerations.]*

(b) 4 mins. *[By this time the universe has become so cool that nuclear reactions are no longer initiated.]*

(c) 10^{-37} sec. *[We learned in Lecture Notes 7 that kT was about 1 MeV at $t = 1$ sec. Since 1 GeV = 1000 MeV, the value of kT that we want is 10^{19} times higher. In the radiation-dominated era $T \propto a^{-1} \propto t^{-1/2}$, so we get 10^{-38} sec.]*

(d) 10,000 – 1,000,000 years. *[This number was estimated in Lecture Notes 7 as 200,000 years.]*

(e) 10^{-5} sec. *[As in (c), we can use $t \propto T^{-2}$, with $kT \approx 1$ MeV at $t = 1$ sec.]*

PROBLEM 9: EVOLUTION OF FLATNESS *(15 points)*

(a) We start with the Friedmann equation from the formula sheet on the quiz:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} .$$

The critical density is the value of ρ corresponding to $k = 0$, so

$$H^2 = \frac{8\pi}{3} G\rho_c .$$

Using this expression to replace H^2 on the left-hand side of the Friedmann equation, and then dividing by $8\pi G/3$, one finds

$$\rho_c = \rho - \frac{3kc^2}{8\pi G a^2} .$$

Rearranging,

$$\frac{\rho - \rho_c}{\rho} = \frac{3kc^2}{8\pi G a^2 \rho} .$$

On the left-hand side we can divide the numerator and denominator by ρ_c, and then use the definition $\Omega \equiv \rho/\rho_c$ to obtain

$$\frac{\Omega - 1}{\Omega} = \frac{3kc^2}{8\pi G a^2 \rho} . \qquad (1)$$

For a matter-dominated universe we know that $\rho \propto 1/a^3(t)$, and so

$$\frac{\Omega - 1}{\Omega} \propto a(t) .$$

If the universe is nearly flat we know that $a(t) \propto t^{2/3}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t^{2/3}} .$$

(b) Eq. (1) above is still true, so our only task is to re-evaluate the right-hand side. For a radiation-dominated universe we know that $\rho \propto 1/a^4(t)$, so

$$\frac{\Omega - 1}{\Omega} \propto a^2(t) .$$

If the universe is nearly flat then $a(t) \propto t^{1/2}$, so

$$\boxed{\frac{\Omega - 1}{\Omega} \propto t} .$$

PROBLEM 10: THE SLOAN DIGITAL SKY SURVEY $z = 5.82$ QUASAR *(40 points)*

(a) Since $\Omega_m + \Omega_\Lambda = 0.35 + 0.65 = 1$, the universe is flat. It therefore obeys a simple form of the Friedmann equation,

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G(\rho_m + \rho_\Lambda) ,$$

where the overdot indicates a derivative with respect to t, and the term proportional to k has been dropped. Using the fact that $\rho_m \propto 1/a^3(t)$ and $\rho_\Lambda =$ const, the energy densities on the right-hand side can be expressed in terms of their present values $\rho_{m,0}$ and $\rho_\Lambda \equiv \rho_{\Lambda,0}$. Defining

$$x(t) \equiv \frac{a(t)}{a(t_0)} \ ,$$

one has

$$\begin{aligned} \left(\frac{\dot{x}}{x}\right)^2 &= \frac{8\pi}{3} G \left(\frac{\rho_{m,0}}{x^3} + \rho_\Lambda\right) \\ &= \frac{8\pi}{3} G\rho_{c,0} \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \\ &= H_0^2 \left(\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}\right) \ . \end{aligned}$$

Here we used the facts that

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}}; \qquad \Omega_{\Lambda,0} \equiv \frac{\rho_\Lambda}{\rho_{c,0}} \ ,$$

and

$$H_0^2 = \frac{8\pi}{3} G\rho_{c,0} \ .$$

The equation above for $(\dot{x}/x)^2$ implies that

$$\dot{x} = H_0 \, x \sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}} \ ,$$

which in turn implies that

$$dt = \frac{1}{H_0} \frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .$$

Using the fact that x changes from 0 to 1 over the life of the universe, this relation can be integrated to give

$$t_0 = \int_0^{t_0} dt = \boxed{\frac{1}{H_0} \int_0^1 \frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} \ .}$$

The answer can also be written as

$$\boxed{t_0 = \frac{1}{H_0} \int_0^1 \frac{x\, dx}{\sqrt{\Omega_{m,0} x + \Omega_{\Lambda,0} x^4}}}$$

or

$$\boxed{t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}} \ ,}$$

where in the last answer I changed the variable of integration using

$$x = \frac{1}{1+z} \ ; \qquad dx = -\frac{dz}{(1+z)^2} \ .$$

Note that the minus sign in the expression for dx is canceled by the interchange of the limits of integration: $x = 0$ corresponds to $z = \infty$, and $x = 1$ corresponds to $z = 0$.

Your answer should look like one of the above boxed answers. You were not expected to complete the numerical calculation, but for pedagogical purposes I will continue. The integral can actually be carried out analytically, giving

$$\int_0^1 \frac{x\, dx}{\sqrt{\Omega_{m,0} x + \Omega_{\Lambda,0} x^4}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0}} + \sqrt{\Omega_{\Lambda,0}}}{\sqrt{\Omega_m}}\right) \ .$$

Using

$$\frac{1}{H_0} = \frac{9.778 \times 10^9}{h_0} \text{ yr} \ ,$$

where $H_0 = 100\, h_0$ km-sec^{-1}-Mpc^{-1}, one finds for $h_0 = 0.65$ that

$$\frac{1}{H_0} = 15.043 \times 10^9 \text{ yr} \ .$$

Then using $\Omega_m = 0.35$ and $\Omega_{\Lambda,0} = 0.65$, one finds

$$\boxed{t_0 = 13.88 \times 10^9 \text{ yr} \ .}$$

So the SDSS people were right on target.

(b) Having done part (a), this part is very easy. The dynamics of the universe is of course the same, and the question is only slightly different. In part (a) we found the amount of time that it took for x to change from 0 to 1. The light from the quasar that we now receive was emitted when

$$x = \frac{1}{1+z} ,$$

since the cosmological redshift is given by

$$1 + z = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})} .$$

Using the expression for $\mathrm{d}t$ from part (a), the amount of time that it took the universe to expand from $x = 0$ to $x = 1/(1+z)$ is given by

$$t_e = \int_0^{t_e} \mathrm{d}t = \boxed{\frac{1}{H_0} \int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Again one could write the answer other ways, including

$$\boxed{t_0 = \frac{1}{H_0} \int_z^{\infty} \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0}}} .}$$

Again you were expected to stop with an expression like the one above. Continuing, however, the integral can again be done analytically:

$$\int_0^{x_{\max}} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{\sqrt{\Omega_m + \Omega_{\Lambda,0} x_{\max}^3} + \sqrt{\Omega_{\Lambda,0}}\, x_{\max}^{3/2}}{\sqrt{\Omega_m}} \right) .$$

Using $x_{\max} = 1/(1+5.82) = .1466$ and the other values as before, one finds

$$\boxed{t_e = \frac{0.06321}{H_0} = 0.9509 \times 10^9 \text{ yr} .}$$

So again the SDSS people were right.

(c) To find the physical distance to the quasar, we need to figure out how far light can travel from $z = 5.82$ to the present. Since we want the present distance, we multiply the coordinate distance by $a(t_0)$. For the flat metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + a^2(t)\left\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right\} ,$$

the coordinate velocity of light (in the radial direction) is found by setting $\mathrm{d}s^2 = 0$, giving

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{a(t)} .$$

So the total coordinate distance that light can travel from t_e to t_0 is

$$\ell_c = \int_{t_e}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t .$$

This is not the final answer, however, because we don't explicitly know $a(t)$. We can, however, change variables of integration from t to x, using

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \frac{\mathrm{d}x}{\dot{x}} .$$

So

$$\ell_c = \frac{c}{a(t_0)} \int_{x_e}^{1} \frac{\mathrm{d}x}{x\,\dot{x}} ,$$

where x_e is the value of x at the time of emission, so $x_e = 1/(1+z)$. Using the equation for $\dot{x}$ from part (a), this integral can be rewritten as

$$\ell_c = \frac{c}{H_0 a(t_0)} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .$$

Finally, then

$$\boxed{\ell_{\text{phys},0} = a(t_0)\,\ell_c = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} .}$$

Alternatively, this result can be written as

$$\boxed{\ell_{\text{phys},0} = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_{m,0}\,x + \Omega_{\Lambda,0}\,x^4}} ,}$$

or by changing variables of integration to obtain

$$\boxed{\ell_{\rm phys,0} = \frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}\,(1+z')^3+\Omega_{\Lambda,0}}}}\ .$$

Continuing for pedagogical purposes, this time the integral has no analytic form, so far as I know. Integrating numerically,

$$\int_0^{5.82} \frac{dz'}{\sqrt{0.35\,(1+z')^3+0.65}} = 1.8099\ ,$$

and then using the value of $1/H_0$ from part (a),

$$\boxed{\ell_{\rm phys,0} = 27.23 \text{ light-yr}}\ .$$

Right again.

(d) $\ell_{{\rm phys},e} = a(t_e)\ell_c$, so

$$\ell_{{\rm phys},e} = \frac{a(t_e)}{a(t_0)}\,\ell_{\rm phys,0} = \boxed{\frac{\ell_{\rm phys,0}}{1+z}}\ .$$

Numerically this gives

$$\boxed{\ell_{{\rm phys},e} = 3.992\times 10^9 \text{ light-yr}}\ .$$

The SDSS announcement is still okay.

(e) The speed defined in this way obeys the Hubble law exactly, so

$$v = H_0\,\ell_{\rm phys,0} = c\int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}\,(1+z')^3+\Omega_{\Lambda,0}}}\ .$$

Then

$$\boxed{\frac{v}{c} = \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}\,(1+z')^3+\Omega_{\Lambda,0}}}}\ .$$

Numerically, we have already found that this integral has the value

$$\boxed{\frac{v}{c} = 1.8099}\ .$$

The SDSS people get an A.

PROBLEM 11: NEUTRINO NUMBER AND THE NEUTRON/PROTON EQUILIBRIUM

(a) From the chemical equilibrium equation on the front of the exam, the number densities of neutrons and protons can be written as

$$n_n = g_n \frac{(2\pi m_n kT)^{3/2}}{(2\pi\hbar)^3}\, e^{(\mu_n - m_n c^2)/kT}$$

$$n_p = g_p \frac{(2\pi m_p kT)^{3/2}}{(2\pi\hbar)^3}\, e^{(\mu_p - m_p c^2)/kT}\ ,$$

where $g_n = g_p = 2$. Dividing,

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(\Delta E + \mu_p - \mu_n)/kT}\ ,$$

where $\Delta E = (m_n - m_p)c^2$ is the proton-neutron mass-energy difference. Approximating $m_n/m_p \approx 1$, one has

$$\boxed{\frac{n_n}{n_p} = e^{-(\Delta E + \mu_p - \mu_n)/kT}}\ .$$

The approximation $m_n/m_p \approx 1$ is very accurate (0.14%), but is clearly not necessary. Full credit was given whether or not this approximation was used.

(b) For any allowed chemical reaction, the sum of the chemical potentials on the two sides must be equal. So, from

$$e^+ + n \longleftrightarrow p + \bar\nu_e\ ,$$

we can infer that

$$-\mu_e + \mu_n = \mu_p - \mu_\nu\ ,$$

which implies that

$$\boxed{\mu_n - \mu_p = \mu_e - \mu_\nu}\ .$$

(c) Applying the formula given in the problem to the number densities of electron neutrinos and the corresponding antineutrinos,

$$n_\nu = g_\nu^* \frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}\, e^{\mu_\nu/kT}$$

$$\bar n_\nu = g_\nu^* \frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}\, e^{-\mu_\nu/kT}\ ,$$

since the chemical potential for the antineutrinos ($\bar{\nu}$) is the negative of the chemical potential for neutrinos. A neutrino has only one spin state, so $g_\nu = 3/4$, where the factor of 3/4 arises because neutrinos are fermions. Setting

$$x \equiv e^{-\mu_\nu/kT}$$

and

$$A \equiv \frac{3}{4}\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} ,$$

the number density equations can be written compactly as

$$n_\nu = \frac{A}{x} , \qquad \bar{n}_\nu = xA .$$

To express x in terms of the ratio $\bar{n}_\nu/n_\nu$, divide the second equation by the first to obtain

$$\frac{\bar{n}_\nu}{n_\nu} = x^2 \quad \Longrightarrow \quad x = \sqrt{\frac{\bar{n}_\nu}{n_\nu}} .$$

Alternatively, x can be expressed in terms of the difference in number densities $\bar{n}_\nu - n_\nu$ by starting with

$$\Delta n = \bar{n}_\nu - n_\nu = xA - \frac{A}{x} .$$

Rewriting the above formula as an explicit quadratic,

$$Ax^2 - \Delta n\, x - A = 0 ,$$

one finds

$$x = \frac{\Delta n \pm \sqrt{\Delta n^2 + 4A^2}}{2A} .$$

Since the definition of x implies $x > 0$, only the positive root is relevant. Since the number of electrons is still assumed to be equal to the number of positrons, $\mu_e = 0$, so the answer to (b) reduces to $\mu_n - \mu_p = -\mu_\nu$. From (a),

$$\begin{aligned} \frac{n_n}{n_p} &= e^{-(\Delta E + \mu_p - \mu_n)/kT} \\ &= e^{-(\Delta E + \mu_\nu)/kT} \\ &= x e^{-\Delta E/kT} \\ &= \boxed{\sqrt{\frac{\bar{n}_\nu}{n_\nu}}\, e^{-\Delta E/kT} .} \end{aligned}$$

Alternatively, one can write the answer as

$$\boxed{\frac{n_n}{n_p} = \frac{\sqrt{\Delta n^2 + 4A^2} + \Delta n}{2A}\, e^{-\Delta E/kT} ,}$$

where

$$\boxed{A \equiv \frac{3}{4}\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} .}$$

(d) For $\Delta n > 0$, the answer to (c) implies that the ratio n_n/n_p would be larger than in the usual case ($\Delta n = 0$). This is consistent with the expectation that an excess of antineutrinos will tend to cause p's to turn into n's according to the reaction

$$p + \bar{\nu}_e \longrightarrow e^+ + n .$$

Since the amount of helium produced is proportional to the number of neutrons that survive until the breaking of the deuterium bottleneck, starting with a higher equilibrium abundance of neutrons will $\boxed{\text{increase}}$ the production of helium.

PROBLEM 12: SECOND HUBBLE CROSSING *(40 points)*

(a) From the formula sheets, we know that for a flat radiation-dominated universe,

$$a(t) \propto t^{1/2} .$$

Since

$$H = \frac{\dot{a}}{a} ,$$

(which is also on the formula sheets),

$$H = \frac{1}{2t} .$$

Then

$$\ell_H(t) \equiv cH^{-1}(t) = \boxed{2ct .}$$

(b) We are told that the energy density is dominated by photons and neutrinos, so we need to add together these two contributions to the energy density. For photons, the formula sheet reminds us that $g_\gamma = 2$, so

$$u_\gamma = 2\,\frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3} .$$

For neutrinos the formula sheet reminds us that

$$g_\nu = \underbrace{\frac{7}{8}}_{\substack{\text{Fermion}\\\text{factor}}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\substack{\text{Particle/}\\\text{antiparticle}}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4} ,$$

so

$$u_\nu = \frac{21}{4}\frac{\pi^2}{30}\frac{(kT_\nu)^4}{(\hbar c)^3} .$$

Combining these two expressions and using $T_\nu = (4/11)^{1/3} T_\gamma$, one has

$$u = u_\gamma + u_\nu = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3} ,$$

so finally

$$\boxed{g_1 = 2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3} .}$$

(c) The Friedmann equation tells us that, for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho ,$$

where in this case $H = 1/(2t)$ and

$$\rho = \frac{u}{c^2} = g_1 \frac{\pi^2}{30}\frac{(kT_\gamma)^4}{\hbar^3 c^5} .$$

Thus

$$\left(\frac{1}{2t}\right)^2 = \frac{8\pi G}{3} g_1 \frac{\pi^2}{30}\frac{(kT_\gamma)^4}{\hbar^3 c^5} .$$

Solving for T_γ,

$$\boxed{T_\gamma = \frac{1}{k}\left(\frac{45\hbar^3 c^5}{16\pi^3 g_1 G}\right)^{1/4}\frac{1}{\sqrt{t}} .}$$

(d) The condition for Hubble crossing is

$$\lambda(t) = cH^{-1}(t) ,$$

and the first Hubble crossing always occurs during the inflationary era. Thus any Hubble crossing during the radiation-dominated era must be the second Hubble crossing.

If λ is the present physical wavelength of the density perturbations under discussion, the wavelength at time t is scaled by the scale factor $a(t)$:

$$\lambda(t) = \frac{a(t)}{a(t_0)}\lambda .$$

Between the second Hubble crossing and now, there have been no freeze-outs of particle species. Today the entropy of the universe is still dominated by photons and neutrinos, so the conservation of entropy implies that aT_γ has remained essentially constant between then and now. Thus,

$$\lambda(t) = \frac{T_{\gamma,0}}{T_\gamma(t)}\lambda .$$

Using the previous results for $cH^{-1}(t)$ and for $T_\gamma(t)$, the condition $\lambda(t) = cH^{-1}(t)$ can be rewritten as

$$kT_{\gamma,0}\left(\frac{16\pi^3 g_1 G}{45\hbar^3 c^5}\right)^{1/4}\sqrt{t}\,\lambda = 2ct .$$

Solving for t, the time of second Hubble crossing is found to be

$$\boxed{t_{H2}(\lambda) = (kT_{\gamma,0}\lambda)^2\left(\frac{\pi^3 g_1 G}{45\hbar^3 c^9}\right)^{1/2} .}$$

Extension: You were not asked to insert numbers, but it is of course interesting to know where the above formula leads. If we take $\lambda = 10^6$ lt-yr, it gives

$$t_{H2}(10^6 \text{ lt-yr}) = 1.04\times 10^7 \text{ s} = 0.330 \text{ year} .$$

For $\lambda = 1$ Mpc,

$$t_{H2}(1 \text{ Mpc}) = 1.11\times 10^8 \text{ s} = 3.51 \text{ year} .$$

Taking $\lambda = 2.5\times 10^6$ lt-yr, the distance to Andromeda, the nearest spiral galaxy,

$$t_{H2}(2.5\times 10^6 \text{ lt-yr}) = 6.50\times 10^7 \text{ sec} = 2.06 \text{ year} .$$