MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 22, 2009
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# QUIZ 1 SOLUTIONS

## Quiz Date: October 6, 2009

## PROBLEM 1: QUESTIONS BASED ON READING AND SHORT CALCULATIONS *(30 points)*†

*Please answer part (a) directly next to each part, but answer parts (b) and (c) on the blank page at the right.*

(a) *(7 points)* The following quantities all have a power law dependence on the cosmological scale factor, $a(t)$. State the dependences (in the form of $\propto a^n$):

(i) The number density of baryons: $\boxed{\textit{Ans: } n_B \propto a^{-3}}$
*Explanation: baryons are conserved, so they are simply diluted by the increase in the volume.*

(ii) The number density of photons: $\boxed{\textit{Ans: } n_\gamma \propto a^{-3}}$
*Explanation: photons are not rigorously conserved, but their number is essentially unchanged as the universe expands, so they are also diluted by the increase in the volume.*

(iii) The energy density of baryons (protons and neutrons): $\boxed{\textit{Ans: } u_B \propto a^{-3}}$
*Explanation: baryons are heavy and behave nonrelatistically during most of the evolution of the universe, so their energy is dominated by their rest energy, which is in turn proportional to their number density.*

(iv) The energy density of photons: $\boxed{\textit{Ans: } u_\gamma \propto a^{-4}}$
*Explanation: photons redshift as the universe expands, so the energy of each photon falls off proportionally to $1/a$. Thus the energy density falls off one power of $a(t)$ faster than the number density.*

(v) The pressure of photons: $\boxed{\textit{Ans: } p_\gamma \propto a^{-4}}$
*Explanation: the pressure of a photon gas is proportional to its energy density. More precisely, $p_\gamma = u_\gamma/3$.*

(vi) The wavelength of photons: $\boxed{\textit{Ans: } \lambda_\gamma \propto a}$
*Explanation: the wavelength of a photon is simply stretched as the universe expands.*

(vii) The temperature of a blackbody distribution of photons: $\boxed{\textit{Ans: } T_\gamma \propto a^{-1}}$
*Explanation: the temperature of blackbody radiation is proportional to the average energy of each photon, which in turn redshifts as $a^{-1}$.*

(b) The Friedman equation for the first derivative of the scale factor,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

depends on the mass density for non-relativistic matter, or, for relativistic matter, on the energy density. From (a)-(iii) above, the energy density for non-relativistic matter goes like $a^{-3}$, whereas the energy density for relativistic matter goes as $a^{-4}$. Although the radiation background has a low energy density today (three orders of magnitude lower than baryonic matter), it does exist. One may trace the energy densities backwards in time, to when $a$ was very small in the early universe. No matter what the prefactors, for some sufficiently small $a$, $a^{-4}$ dominated over $a^{-3}$. At that time, and before, the universe would not have been matter dominated; rather, it would have been radiation dominated, which means that the assumption $a \propto t^{2/3}$ would not have been valid.

(c) Today, the temperature of the background radiation is approximately 3 Kelvin, and when the universe first became transparent (know as "last scattering," "decoupling," or "recombination," which should rightfully be called "combination"), the radiation was approximately 3000 Kelvin. From (a)-(vii), we can write

$$\left(\frac{T_{ls}}{T_0}\right) = \left(\frac{a_o}{a_{ls}}\right) = 1 + z_{ls},$$

where a subscript of $ls$ means "last scattering." Evaluating this with the values above, we have $z \approx 1000$.

(d) If we take the matter dominated form $a(t) = bt^{2/3}$, then the Hubble parameter is

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\frac{2}{3}bt^{-1/3}}{bt^{2/3}} = \frac{2}{3t}.$$

Solving this for $t$ at some time, $t = \frac{2}{3H}$. To find the age of the universe today, one evaluates this formula with $H = H_0$, finding an age $t_0 \approx 9 \times 10^9$ years. To find the age at last scattering, one can use

$$1 + z_{ls} = \frac{a(t_0)}{a(t_l s)} = \frac{t_0^{2/3}}{t_{ls}^{2/3}}.$$

Solving for $t_{ls}$, one finds $t_{ls} = t_0(1+z)^{-3/2}$. The value $t_0$ was calculated above, and $z$ comes from part (c). Plugging these in, one finds $t_{ls} \approx 2.8 \times 10^5$ years.

(e) The horizon distance at time $t$ is the physical distance that light has been traveling between when $a$ was zero and time $t$. The general physical distance that light has traveled between time $t_i$ and $t_f$ is

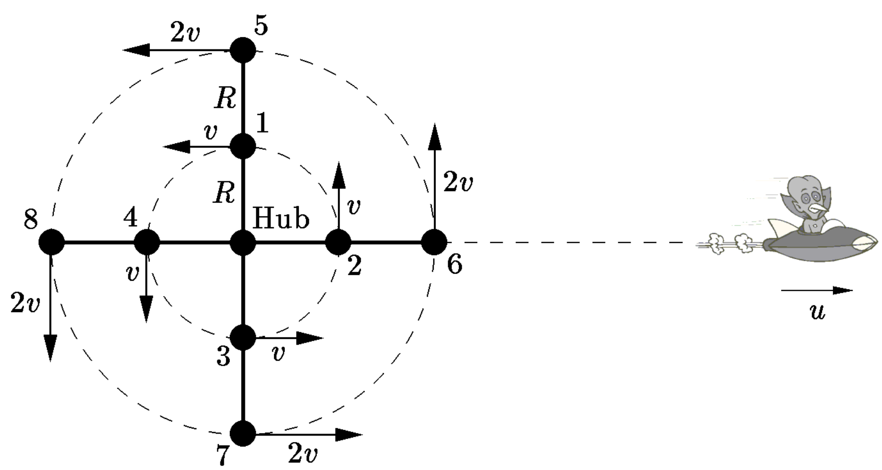
$$l_p(t_f) = a(t_f) \int_{t_i}^{t_f} \frac{c\,\mathrm{d}t'}{a(t')} .$$

We normally choose $t = 0$ to denote the time when $a$ vanishes, so the horizon distance is

$$d_h(t) = a(t) \int_{0}^{t} \frac{c\,\mathrm{d}t'}{a(t')} .$$

Evaluating this with $a(t) = bt^{2/3}$, we find $d_h(t) = 3ct$. One evaluates this formula with the two answers from part (d). The horizon distance today would be 8.3 Gpc $= 27 \times 10^9$ lyr $= 2.6 \times 10^{26}$ m. At the time of last scattering, the horizon distance would have been 0.26 Mpc $= 8.5 \times 10^5$ lyr $= 8 \times 10^{21}$ m.

## PROBLEM 2: A TWO-LEVEL HIGH-SPEED MERRY-GO-ROUND *(15 points)**




(a) Since the relative positions of all the cars remain fixed as the merry-go-round rotates, each successive pulse from any given car to any other car takes the same amount of time to complete its trip. Thus there will be no Doppler shift caused by pulses taking different amounts of time; the only Doppler shift will come from time dilation.

We will describe the events from the point of view of an inertial reference frame at rest relative to the hub of the merry-go-round, which we will call the laboratory frame. This is the frame in which the problem is described, in which the inner cars are moving at speed $v$, and the outer cars are moving at speed

$2v$. In the laboratory frame, the time interval between the wave crests emitted by the source $\Delta t_S^{\rm Lab}$ will be exactly equal to the time interval $\Delta t_O^{\rm Lab}$ between two crests reaching the observer:

$$\Delta t_O^{\rm Lab} = \Delta t_S^{\rm Lab} \ .$$

The clocks on the merry-go-round cars are moving relative to the laboratory frame, so they will appear to be running slowly by the factor

$$\gamma_1 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

for the inner cars, and by the factor

$$\gamma_2 = \frac{1}{\sqrt{1 - 4v^2/c^2}}$$

for the outer cars. Thus, if we let $\Delta t_S$ denote the time between crests as measured by a clock on the source, and $\Delta t_O$ as the time between crests as measured by a clock moving with the observer, then these quantities are related to the laboratory frame times by

$$\gamma_2 \Delta t_S = \Delta t_S^{\rm Lab} \quad \text{and} \quad \gamma_1 \Delta t_O = \Delta t_O^{\rm Lab} \ .$$

To make sure that the $\gamma$-factors are on the right side of the equation, you should keep in mind that any time interval should be measured as shorter on the moving clocks than on the lab clocks, since these clocks appear to run slowly. Putting together the equations above, one has immediately that

$$\Delta t_O = \frac{\gamma_2}{\gamma_1} \Delta t_S \ .$$

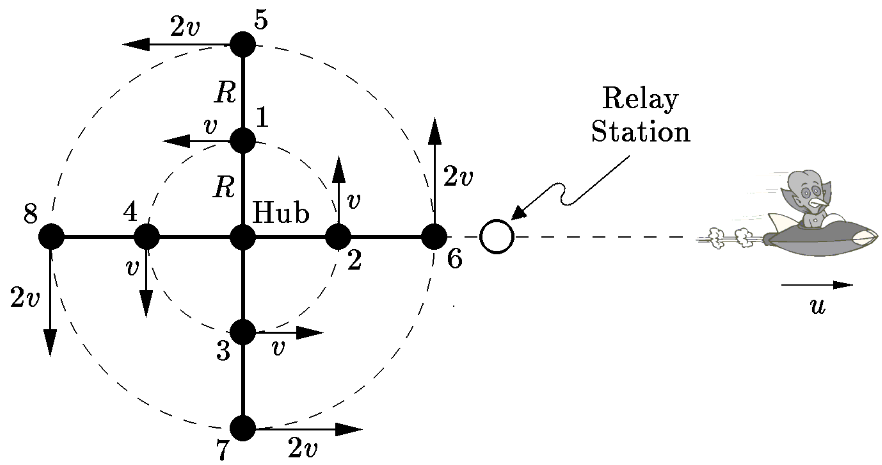
The redshift $z$ is defined by

$$\Delta t_O \equiv (1+z)\, \Delta t_S \ ,$$

so

$$\boxed{z = \frac{\gamma_2}{\gamma_1} - 1 = \sqrt{\frac{1 - \frac{v^2}{c^2}}{1 - \frac{4v^2}{c^2}}} - 1 \ .}$$

(b) For this part of the problem is useful to imagine a relay station located just to the right of car 6 in the diagram, at rest in the laboratory frame. The relay station rebroadcasts the waves as it receives them, and hence has no effect on the frequency received by the observer, but serves the purpose of allowing us to clearly separate the problem into two parts.




The first part of the discussion concerns the redshift of the signal as measured by the relay station. This calculation would involve both the time dilation and a change in path lengths between successive pulses, but we do not need to do it. It is the standard situation of a source and observer moving directly away from each other, as discussed at the end of Lecture Notes 1. The Doppler shift is given by Eq. (1.33), which was included in the formula sheet. Writing the formula for a recession speed $u$, it becomes

$$(1+z)\big|_{\text{relay}} = \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}} \ .$$

If we again use the symbol $\Delta t_S$ for the time between wave crests as measured by a clock on the source, then the time between the receipt of wave crests as measured by the relay station is

$$\Delta t_R = \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}\ \Delta t_S \ .$$

The second part of the discussion concerns the transmission from the relay station to car 6. The velocity of car 6 is perpendicular to the direction from which the pulse is being received, so this is a transverse Doppler shift. Any change in path length between successive pulses is second order in $\Delta t$, so it can be ignored. The only effect is therefore the time dilation. As described in the laboratory frame, the time separation between crests reaching the observer is the same as the time separation measured by the relay station:

$$\Delta t_O^{\text{Lab}} = \Delta t_R \ .$$

As in part (a), the time dilation implies that

$$\gamma_2 \Delta t_O = \Delta t_O^{\rm Lab} \ .$$

Combining the formulas above,

$$\Delta_O = \frac{1}{\gamma_2}\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}\,\Delta t_S \ .$$

Again $\Delta t_O \equiv (1+z)\,\Delta t_S$, so

$$\boxed{z = \frac{1}{\gamma_2}\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}} - 1 = \sqrt{\frac{\left(1-\frac{4v^2}{c^2}\right)\left(1+\frac{u}{c}\right)}{1-\frac{u}{c}}} - 1 \ .}$$

## PROBLEM 3: SIGNAL PROPAGATION IN A FLAT MATTER-DOMINATED UNIVERSE *(55 points)**

(a)-(i) If we let $\ell_c(t)$ denote the coordinate distance of the light signal from $A$, then we can make use of Eq. (3.8) from the lecture notes for the coordinate velocity of light:

$$\frac{{\rm d}\ell_c}{{\rm d}t} = \frac{c}{a(t)} \ . \tag{3.1}$$

Integrating the velocity,

$$\begin{aligned} \ell_c(t) &= \int_{t_1}^{t} \frac{c\,{\rm d}t'}{a(t')} = \frac{c}{b}\int_{t_1}^{t}\frac{{\rm d}t'}{t'^{2/3}} \\ &= \frac{3c}{b}\left[t^{1/3} - t_1^{1/3}\right] \ . \end{aligned} \tag{3.2}$$

The physical distance is then

$$\begin{aligned} \ell_{p,sA}(t) = a(t)\ell_c(t) &= bt^{2/3}\frac{3c}{b}\left[t^{1/3} - t_1^{1/3}\right] \\ &= 3c\left(t - t^{2/3}t_1^{1/3}\right) \\ &= 3ct\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] \ . \end{aligned} \tag{3.3}$$

We now need to differentiate, which is done most easily with the middle line of the above equation:

$$\boxed{\frac{\mathrm{d}\ell_{p,sA}}{\mathrm{d}t} = c\left[3 - 2\left(\frac{t_1}{t}\right)^{1/3}\right] .} \tag{3.4}$$

(ii) At $t = t_1$, the time of emission, the above formula gives

$$\boxed{\frac{\mathrm{d}\ell_{p,sA}}{\mathrm{d}t} = c .} \tag{3.5}$$

This is what should be expected, since the speed of separation of the light signal at the time of emission is really just a local measurement of the speed of light, which should always give the standard value $c$.

(iii) At arbitrarily late times, the second term in brackets in Eq. (3.4) becomes negligible, so

$$\boxed{\frac{\mathrm{d}\ell_{p,sA}}{\mathrm{d}t} \to 3c .} \tag{3.6}$$

Although this answer is larger than $c$, it does not violate relativity. Once the signal is far from its origin it is carried by the expansion of the universe, and relativity places no speed limit on the expansion of the universe.

(b) This part of the problem involves $H(t_1)$, so we can start by evaluating it:

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}(bt^{2/3})}{bt^{2/3}} = \frac{2}{3t} . \tag{3.7}$$

Thus, the physical distance from $A$ to $B$ at time $t_1$ is

$$\ell_{p,BA} = \frac{3}{2}ct_1 . \tag{3.8}$$

The coordinate distance is the physical distance divided by the scale factor, so

$$\ell_{c,BA} = \frac{cH^{-1}(t_1)}{a(t_1)} = \frac{\frac{3}{2}ct_1}{bt_1^{2/3}} = \frac{3c}{2b}\,t_1^{1/3} . \tag{3.9}$$

Since light travels at a coordinate speed $c/a(t)$, the light signal will reach galaxy $B$ at time $t_2$ if

$$\begin{aligned} \ell_{c,BA} &= \int_{t_1}^{t_2} \frac{c}{b t'^{2/3}}\, \mathrm{d}t' \\ &= \frac{3c}{b}\left[t_2^{1/3} - t_1^{1/3}\right] . \end{aligned} \tag{3.10}$$

Setting the expressions (3.9) and (3.10) for $\ell_{c,BA}$ equal to each other, one finds

$$\frac{1}{2}t_1^{1/3} = t_2^{1/3} - t_1^{1/3} \quad \Longrightarrow \quad t_2^{1/3} = \frac{3}{2}t_1^{1/3} \quad \Longrightarrow \quad \boxed{t_2 = \frac{27}{8}t_1 .} \tag{3.11}$$

(c)-(i) Physical distances are additive, so if one adds the distance from $A$ and the light signal to the distance from the light signal to $B$, one gets the distance from $A$ to $B$:

$$\ell_{p,sA} + \ell_{p,sB} = \ell_{p,BA} . \tag{3.12}$$

But $\ell_{p,BA}(t)$ is just the scale factor times the coordinate separation, $a(t)\ell_{c,BA}$. Using the previous relations (3.3) and (3.9) for $\ell_{p,sA}(t)$ and $\ell_{c,BA}$, we find

$$3ct\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] + \ell_{p,sB}(t) = \frac{3}{2}ct_1^{1/3}t^{2/3} , \tag{3.13}$$

so

$$\ell_{p,sB}(t) = \frac{9}{2}ct_1^{1/3}t^{2/3} - 3ct = 3ct\left[\frac{3}{2}\left(\frac{t_1}{t}\right)^{1/3} - 1\right] . \tag{3.14}$$

As a check, one can verify that this expression vanishes for $t = t_2 = (27/8)\,t_1$, and that it equals $(3/2)ct_1$ at $t = t_1$. But we are asked to find the speed of approach, the negative of the derivative of Eq. (3.14):

$$\begin{aligned} \text{Speed of approach} &= -\frac{\mathrm{d}\ell_{p,sB}}{\mathrm{d}t} \\ &= -3ct_1^{1/3}t^{-1/3} + 3c \\ &= \boxed{3c\left[1 - \left(\frac{t_1}{t}\right)^{1/3}\right] .} \end{aligned} \tag{3.15}$$

(ii) At the time of emission, $t = t_1$, Eq. (3.15) gives

$$\boxed{\text{Speed of approach} = 0 \ .} \tag{3.16}$$

This makes sense, since at $t = t_1$ galaxy $B$ is one Hubble length from galaxy $A$, which means that its recession velocity is exactly $c$. The recession velocity of the light signal leaving $A$ is also $c$, so the rate of change of the distance from the light signal to $B$ is initially zero.

(iii) At the time of reception, $t = t_2 = (27/8)\, t_1$, Eq. (3.15) gives

$$\boxed{\text{Speed of approach} = c \ ,} \tag{3.17}$$

which is exactly what is expected. As in part (a)-(ii), this is a local measurement of the speed of light.

(d) To find the redshift, we first find the time $t_{BA}$ at which a light pulse must be emitted from galaxy $B$ so that it arrives at galaxy $A$ at time $t_1$. Using the coordinate distance given by Eq. (3.9), the time of emission must satisfy

$$\frac{3c}{2b}\, t_1^{1/3} = \int_{t_{BA}}^{t_1} \frac{c}{bt'^{2/3}}\, \mathrm{d}t' = \frac{3c}{b}\left(t_1^{1/3} - t_{BA}^{1/3}\right) \ , \tag{3.18}$$

which can be solved to give

$$t_{BA} = \frac{1}{8}\, t_1 \ . \tag{3.19}$$

The redshift is given by

$$1 + z_{BA} = \frac{a(t_1)}{a(t_{BA})} = \left(\frac{t_1}{t_{BA}}\right)^{2/3} = 4 \ . \tag{3.20}$$

Thus,

$$\boxed{z_{BA} = 3 \ .} \tag{3.21}$$

(e) Applying Euclidean geometry to the triangle $C$-$A$-$B$ shows that the physical distance from $C$ to $B$, at time $t_1$, is $\sqrt{2}cH^{-1}$. The coordinate distance is also larger than the $A$-$B$ separation by a factor of $\sqrt{2}$. Thus,

$$\ell_{c,BC} = \frac{3\sqrt{2}c}{2b}\, t_1^{1/3} \ . \tag{3.22}$$

If we let $t_{BC}$ be the time at which a light pulse must be emitted from galaxy $B$ so that it arrives at galaxy $C$ at time $t_1$, we find

$$\frac{3\sqrt{2}c}{2b}\,t_1^{1/3} = \int_{t_{BC}}^{t_1} \frac{c}{bt'^{2/3}}\,\mathrm{d}t' = \frac{3c}{b}\left(t_1^{1/3} - t_{BC}^{1/3}\right) \;, \tag{3.23}$$

which can be solved to find

$$t_{BC} = \left(1 - \frac{\sqrt{2}}{2}\right)^3 t_1 \;. \tag{3.24}$$

Then

$$1 + z_{BC} = \frac{a(t_1)}{a(t_{BC})} = \left(\frac{t_1}{t_{BC}}\right)^{2/3} = \frac{1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2} \;, \tag{3.25}$$

and

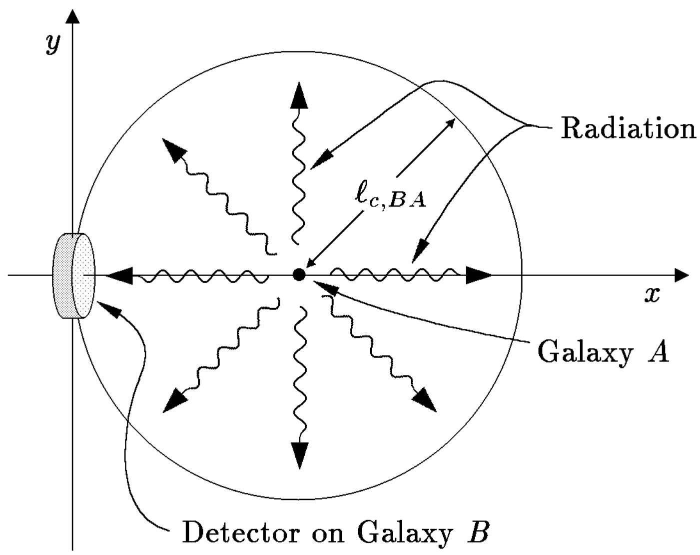
$$\boxed{z_{BC} = \frac{1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2} - 1 \;.} \tag{3.26}$$

Full credit will be given for the answer in the form above, but it can be simplified by rationalizing the fraction:

$$\begin{aligned} z_{BC} &= \frac{1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2}\,\frac{\left(1 + \frac{\sqrt{2}}{2}\right)^2}{\left(1 + \frac{\sqrt{2}}{2}\right)^2} - 1 \\ &= \frac{1 + \sqrt{2} + \frac{1}{2}}{\frac{1}{4}} - 1 \\ &= \boxed{5 + 4\sqrt{2} \;.} \end{aligned} \tag{3.27}$$

Numerically, $z_{BC} = 10.657$.

(f) Following the solution to Problem 6 of Problem Set 2, we draw a diagram in comoving coordinates, putting the source at the center of a sphere:




The energy from galaxy $A$ will radiate uniformly over the sphere. If the detector has physical area $A_D$, then in the comoving coordinate picture it has coordinate area $A_D/a^2(t_2)$, since the detection occurs at time $t_2$ The full coordinate area of the sphere is $4\pi\ell_{c,BA}^2$, so the fraction of photons that hit the detector is

$$\text{fraction} = \frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2} \ . \tag{3.28}$$

As in Problem 6, the power hitting the detector is reduced by two factors of $(1+z)$: one factor because the energy of each photon is proportional to the frequency, and hence is reduced by the redshift, and one more factor because the rate of arrival of photons is also reduced by the redshift factor $(1+z)$. Thus,

$$\begin{aligned}\text{Power hitting detector} &= P\frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2}\frac{1}{(1+z)^2} \\ &= P\frac{\left[A/a(t_2)^2\right]}{4\pi\ell_{c,BA}^2}\left[\frac{a(t_1)}{a(t_2)}\right]^2 \\ &= P\frac{A}{4\pi\ell_{c,BA}^2}\frac{a^2(t_1)}{a^4(t_2)} \ .\end{aligned} \tag{3.29}$$

The energy flux is given by

$$J = \frac{\text{Power hitting detector}}{A} \ , \tag{3.30}$$

so

$$J = \frac{P}{4\pi\ell_{c,BA}^2}\frac{a^2(t_1)}{a^4(t_2)} \ . \tag{3.31}$$

From here it is just algebra, using Eqs. (3.9) and (3.11), and $a(t) = bt^{2/3}$:

$$\begin{aligned}
J &= \frac{P}{4\pi\left[\frac{3c}{2b}t_1^{1/3}\right]^2}\frac{b^2 t_1^{4/3}}{b^4 t_2^{8/3}} \\
&= \frac{P}{4\pi\left[\frac{3c}{2b}t_1^{1/3}\right]^2}\frac{b^2 t_1^{4/3}}{\left(\frac{27}{8}\right)^{8/3} b^4 t_1^{8/3}} \\
&= \frac{P}{4\pi\left[\frac{3c}{2}t_1^{1/3}\right]^2}\frac{t_1^{4/3}}{\left(\frac{3}{2}\right)^{8} t_1^{8/3}} \qquad (3.32)\\
&= \boxed{\frac{2^8}{3^{10}\pi}\frac{P}{c^2 t_1^2}} \\
&= \boxed{\frac{256}{59,049\pi}\frac{P}{c^2 t_1^2}\ .}
\end{aligned}$$

It is debatable which of the last two expressions is the simplest, so I have boxed both of them. One could also write

$$J = 1.380 \times 10^{-3}\frac{P}{c^2 t_1^2}\ . \qquad (3.33)$$

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†Solutions to parts (b)-(e) written by Leo Stein; solution to part (a) by Alan Guth.

*Solution written by Alan Guth.

(Attributions added 10/23/09.)