Solving for $\frac{\dot{a}}{a} \propto H$ was calculated above, we have $z(1 + \frac{\dot{a}}{a}) = (\frac{\dot{a}}{a})_0 = (\frac{\dot{a}}{a})_0 = (i) H$.

Thus the energy density falls off one power of $\gamma a$ since $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$.

(c) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

We have seen that, if $\gamma a \propto T$, then the Hubble parameter $\frac{\dot{a}}{a} \propto H$. Thus, $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$.

(d) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

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(e) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

We have seen that, if $\gamma a \propto T$, then the Hubble parameter $\frac{\dot{a}}{a} \propto H$. Thus, $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$.

(f) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

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(g) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

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(h) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

We have seen that, if $\gamma a \propto T$, then the Hubble parameter $\frac{\dot{a}}{a} \propto H$. Thus, $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$.

(i) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

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(j) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

We have seen that, if $\gamma a \propto T$, then the Hubble parameter $\frac{\dot{a}}{a} \propto H$. Thus, $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$.

(k) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

We have seen that, if $\gamma a \propto T$, then the Hubble parameter $\frac{\dot{a}}{a} \propto H$. Thus, $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$.

(l) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

We have seen that, if $\gamma a \propto T$, then the Hubble parameter $\frac{\dot{a}}{a} \propto H$. Thus, $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$.

(m) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

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(n) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

We have seen that, if $\gamma a \propto T$, then the Hubble parameter $\frac{\dot{a}}{a} \propto H$. Thus, $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$.

(o) Today, the temperature of the background radiation is approximately 10 Kelvin, but if we take the matter density from (a)-(vii), then the Hubble parameter is $\frac{\dot{a}}{a} \propto H$. We can write from (a) that the radiation was approximately 3000 Kelvin. From (a)-(vii), we can see that the radiation and the baryons dominated different epochs, so they are simply diluted by the increase in the volume.

We have seen that, if $\gamma a \propto T$, then the Hubble parameter $\frac{\dot{a}}{a} \propto H$. Thus, $\gamma a \propto T$. Solving for $T$ gives $T = \frac{1}{4 \pi G}$, whereas the energy density for relativistic matter goes as $n^{\frac{4}{3}}$. 

Please answer all of the above questions.
The horizon distance at time $t$ is the physical distance that light has been traveling between when light was emitted and time $t$. The general physical distance that light has traveled between time $t_i$ and $t_f$ is $l_{pf}(t_f) = \int_{t_i}^{t_f} c \, dt'$.

We normally choose $t_f = 0$ to denote the time when light vanishes, so the horizon distance is $d_h(t) = \int_{t}^{0} c \, dt'$.

Evaluating this with $a(t) = b t^2/3$, we find $d_h(t) = 3c t$. One evaluates this formula with the two answers from part (d). The horizon distance today would be $8.73G^2 c = 2.7 \times 10^{26} \text{m}$. At the time of last scattering, the horizon distance would have been $8.526 \times 10^{5} \text{lyr} = 8.526 \times 10^{21} \text{m}$.

**Problem 2: A Two-Level High-Speed Merry-Go-Round (15 points)**

(a) Since the relative positions of all the cars remain fixed as the merry-go-round rotates, each successive pulse from any given car to any other car takes the same amount of time to complete its trip. Thus there will be no Doppler shift caused by pulses taking different amounts of time; the only Doppler shift will come from time dilation.

We will describe the events from the point of view of an inertial reference frame at rest relative to the hub of the merry-go-round, which we will call the laboratory frame. This is the frame in which the problem is described, in which the inner cars are moving at speed $v$, and the outer cars are moving at speed $2v$.

The clocks on the merry-go-round cars are moving relative to the laboratory frame, so they will appear to be running slower by the factor $\gamma_1 = 1/\sqrt{1 - v^2/c^2}$ for the inner cars, and by the factor $\gamma_2 = 1/\sqrt{1 - 4v^2/c^2}$ for the outer cars. Thus, if we let $\Delta t_S$ denote the time between crests as measured by a clock on the source, and $\Delta t_O$ as the time between crests as measured by a clock moving with the observer, then these quantities are related to the laboratory frame times by

$$\gamma_2 \Delta t_S = \Delta t_{Lab}$$
$$\gamma_1 \Delta t_O = \Delta t_{Lab}$$

To make sure that the $\gamma$-factors are on the right side of the equation, you should keep in mind that any time interval should be measured as shorter on the moving clocks than on the lab clocks, since these clocks appear to run slow. Putting together the equations above, one has immediately that

$$\Delta t_O = \gamma_2 \gamma_1 \Delta t_S$$

The redshift $z$ is defined by

$$\Delta t_O \equiv (1 + z) \Delta t_S$$

so

$$z = \gamma_2 \gamma_1 - 1 = \sqrt{1 - v^2/c^2} \sqrt{1 - 4v^2/c^2} - 1$$

(b) For this part of the problem, it is useful to imagine a relay station located just to the right of car 6 in the diagram, at rest in the laboratory frame. The relay station would be a clock located in the laboratory frame, which would measure the time between crests as

$$\Delta t_{Lab} = \Delta t_S$$

The towers on the merry-go-round cars are moving relative to the laboratory frame, so they will appear to be running slower by the factor $\gamma$. The two crests reaching the observer from the towers on the merry-go-round cars are moving relative to the laboratory frame, so

$$\Delta t_{Lab} = \gamma \Delta t_S$$

The towers are moving at speed $v$ and the outer cars are moving at speed $2v$.

We will describe the events from the point of view of an inertial reference frame at rest relative to the hub of the merry-go-round, which we will call the laboratory frame. This is the frame in which the problem is described, in which light is emitted at rest relative to the hub of the merry-go-round, which we will call the laboratory frame. We will describe the events from the point of view of an inertial reference frame.

The towers on the merry-go-round cars are moving relative to the laboratory frame, so they will appear to be running slower by the factor $\gamma$. The two crests reaching the observer from the towers on the merry-go-round cars are moving relative to the laboratory frame, so

$$\Delta t_{Lab} = \gamma \Delta t_S$$

The towers are moving at speed $v$ and the outer cars are moving at speed $2v$.

Evaluating this with $a(t) = b t^2/3$, we find $d_h(t) = 3c t$. One evaluates this formula with the two answers from part (d). The horizon distance today would be $8.73G^2 c = 2.7 \times 10^{26} \text{m}$. At the time of last scattering, the horizon distance would have been $8.526 \times 10^{5} \text{lyr} = 8.526 \times 10^{21} \text{m}$.}
station rebroadcasts the waves as it receives them, and hence has no effect on the frequency received by the observer, but serves the purpose of allowing us to clearly separate the problem into two parts.

The first part of the discussion concerns the redshift of the signal as measured by the relay station. This calculation is straightforward and is described in Lecture Notes 1. The Doppler shift is given by Eq. (1.33), which was included in the formula sheet. Writing the formula for a recession speed $v$, it becomes

\[
(1 + z) \frac{\sqrt{1 - \frac{u}{c}}}{c} = \sqrt{1 - \frac{u}{c}} \Delta t_S.
\]

If we again use the symbol $\Delta t_S$ for the time between wave crests as measured by a clock on the source, then the time between the receipt of wave crests as measured by the relay station is

\[
\Delta t_R = \sqrt{1 + \frac{u}{c}} \Delta t_S.
\]

The second part of the discussion concerns the transmission from the relay station to car 6. The velocity of car 6 is perpendicular to the direction from which the pulse is being received, so this is a transverse Doppler shift. Any change in path length between successive pulses is second order in $\Delta t$, so it can be ignored. The only effect is therefore the time dilation. As described in the laboratory frame, the time separation between crests reaching the observer is the same as the time separation measured by the relay station.

\[
\Delta t_O = \Delta t_{Lab}.
\]

As in part (a), the time dilation implies that

\[
\gamma^2 \Delta t_O = \Delta t_{Lab}.
\]

Combining the formulas above,

\[
\gamma^2 \Delta t_O = \gamma^2 \Delta t_S.
\]

As in part (a), the time dilation implies that

\[
(1 + z) \frac{\sqrt{1 - \frac{u}{c}}}{c} = \sqrt{1 - \frac{u}{c}} \Delta t_S.
\]

Problem 3: Signal Propagation in a Flat Matter-Dominated Universe

(55 points)

(a)-(i) If we let $\ell_c(t)$ denote the coordinate distance of the light signal from the source, then we can make use of Eq. (3.8) from the lecture notes for the coordinate velocity of light:

\[\frac{d\ell_c}{dt} = \frac{c}{a(t)}.
\]

Integrating the velocity,

\[\ell_c(t) = \int_{t_1}^{t} \frac{c}{a(t')} dt' = \frac{c}{b} \left[\frac{t}{3} - \frac{t_1}{3}\right].
\]

The physical distance is then

\[\ell_p = \ell_c(t) = \frac{bt_2}{3} \left[\frac{t}{3} - \frac{t_1}{3}\right].
\]

This is the same as the time separation measured by the relay station.

(b) The only effect is therefore the time dilation. As described in the previous part, the time separation measured by the relay station is

\[\Delta t_R = \gamma^2 \Delta t_S.
\]

Combining the formulas above,

\[
\gamma^2 \Delta t_O = \gamma^2 \Delta t_S.
\]

As in part (a), the time dilation implies that

\[
(1 + z) \frac{\sqrt{1 - \frac{u}{c}}}{c} = \sqrt{1 - \frac{u}{c}} \Delta t_S.
\]

\[
\gamma^2 \Delta t_O = \gamma^2 \Delta t_S.
\]

Again $\Delta t_O \equiv (1 + z) \Delta t_S$, so

\[
\Delta t_O = \gamma^2 \Delta t_S.
\]

The physical distance is then

\[\ell_p = \ell_c(t) = \frac{bt_2}{3} \left[\frac{t}{3} - \frac{t_1}{3}\right].
\]

This is the same as the time separation measured by the relay station.

(b) The only effect is therefore the time dilation. As described in the previous part, the time separation measured by the relay station is

\[\Delta t_R = \gamma^2 \Delta t_S.
\]

Combining the formulas above,

\[
\gamma^2 \Delta t_O = \gamma^2 \Delta t_S.
\]

As in part (a), the time dilation implies that

\[
(1 + z) \frac{\sqrt{1 - \frac{u}{c}}}{c} = \sqrt{1 - \frac{u}{c}} \Delta t_S.
\]
\[
\begin{align*}
\left[ \frac{1}{v/1} - 1 \right] \frac{d \mathbf{r}}{d t} &= 0 \\
\Rightarrow &\quad \frac{d \mathbf{r}}{d t} = 0
\end{align*}
\]

The coordinate distance is the physical distance divided by the scale factor, so

\[
(3.8) \quad \frac{d \mathbf{r}}{c} = \frac{v/1}{\sqrt{1 - H^2}} = \frac{\gamma}{c} v/1, \quad (3.9) \quad \ell = \frac{3}{c}, \quad (3.10) \quad p, s_{B-A} = v/1, \quad (3.11) \quad p, s_{B-A} = v/1, \quad (3.12) \quad p, s_{B-A} = v/1.
\]

So the physical distance from \( A \) to \( B \) at time \( t \) is

\[
(3.13) \quad \ell = \frac{v/1}{\gamma} = \frac{(1/1) v}{(1/1 - H^2)} = v/1 - c
\]

Although this answer is larger than \( c \), but we are asked to find the speed of the signal is far from its origin it is carried by the expansion of the universe, and so

\[
(3.14) \quad \frac{d \mathbf{r}}{c} = \frac{v/1}{\sqrt{1 - H^2}} = \frac{\gamma}{c} v/1.
\]

As a check, one can verify that this expression vanishes for

\[
(3.15) \quad \frac{d \mathbf{r}}{c} = \frac{v/1}{\sqrt{1 - H^2}} = \frac{\gamma}{c} v/1.
\]

So that it equals \( (3.15) \). But we are asked to find the speed of the light signal in terms of the coordinate distance. The answer is

\[
(3.16) \quad \frac{d \mathbf{r}}{c} = \frac{v/1}{\sqrt{1 - H^2}} = \frac{\gamma}{c} v/1.
\]

We now need to differentiate, which is done most easily with the middle line

\[
(3.17) \quad \frac{d \mathbf{r}}{c} = \frac{v/1}{\sqrt{1 - H^2}} = \frac{\gamma}{c} v/1.
\]
Following the solution to Problem 6 of Problem Set 2, we draw a diagram in comoving coordinates, putting the source at the center of a sphere:

\[ \text{(3.22)} \]

Numerically, \( z = 0.627. \)

\[ \frac{3}{1} \frac{\sqrt{t}}{c} + \frac{\sqrt{t}}{c} = \text{ditto} \]

\[ \text{(3.23)} \]

\[ \frac{3}{1} \frac{\sqrt{t}}{c} - 1 \]

\[ \text{These are given by Eq. (3.9), the time of emission must satisfy} \]

\[ \text{(3.25)} \]

\[ \frac{3}{1} \frac{\sqrt{t}}{c} + \frac{\sqrt{t}}{c} = \text{ditto} \]

\[ \text{(3.26)} \]

\[ 1 - \frac{3}{1} \frac{\sqrt{t}}{c} - 1 = \text{ditto} \]

\[ \text{(3.27)} \]

\[ \frac{3}{1} \frac{\sqrt{t}}{c} - 1 \]

\[ \text{then} \]

\[ \text{(3.28)} \]

\[ 1 \frac{\sqrt{t}}{c} - 1 = \text{ditto} \]

\[ \text{(3.29)} \]

\[ 1 \frac{\sqrt{t}}{c} - 1 = \text{ditto} \]

\[ \text{(3.30)} \]

or \( t = 3 \text{ditto} \)

\[ \text{(3.31)} \]

\[ \text{Thus,} \]

\[ \text{(3.32)} \]

\[ \text{The redshift is given by} \]

\[ \text{(3.33)} \]

\[ \text{which can be solved to give} \]

\[ \text{(3.34)} \]

\[ 1 \frac{\sqrt{t}}{c} - 1 = \text{ditto} \]

\[ \text{(3.35)} \]

\[ 1 \frac{\sqrt{t}}{c} = \text{ditto} \]

\[ \text{(3.36)} \]

\[ 1 \frac{\sqrt{t}}{c} - 1 = \text{ditto} \]

\[ \text{(3.37)} \]

\[ 1 \frac{\sqrt{t}}{c} = \text{ditto} \]

\[ \text{(3.38)} \]

\[ 1 \frac{\sqrt{t}}{c} - 1 = \text{ditto} \]

\[ \text{(3.39)} \]

\[ 1 \frac{\sqrt{t}}{c} = \text{ditto} \]

\[ \text{(3.40)} \]

\[ 2 \frac{\sqrt{t}}{c} - 1 = \text{ditto} \]

\[ \text{(3.41)} \]

\[ 2 \frac{\sqrt{t}}{c} = \text{ditto} \]
The energy from galaxy $A$ will radiate uniformly over the sphere. If the detector has physical area $A_D$, then in the comoving coordinate picture it has coordinate area $A_D/a^2(t^2_2)$, since the detection occurs at time $t^2_2$. The full coordinate area of the sphere is $4\pi \ell^2 c$, so the fraction of photons that hit the detector is

$$\text{fraction} = \frac{A}{a^2(t^2_2)}.$$ (3.28)

As in Problem 6, the power hitting the detector is reduced by two factors of $(1+z)$: one factor because the energy of each photon is proportional to the frequency, and hence is reduced by the redshift, and one more factor because the rate of arrival of photons is also reduced by the redshift factor $(1+z)$. Thus,

$$\text{Power hitting detector} = P \left[ \frac{A}{a^2(t^2_2)} \right] \frac{4\pi \ell^2 c}{a^2}.$$ (3.29)

The energy flux is given by $J = \frac{\text{Power hitting detector}}{A}$, so

$$J = \frac{P}{4\pi \ell^2 c} \left[ \frac{a}{a(t^1_1)} \right] \left[ \frac{a(t^2_2)}{a(t^1_1)} \right]^2.$$ (3.30)

From here it is just algebra, using Eqs. (3.9) and (3.11), and $a(t) = bt^2/3$:

$$J = \frac{P}{4\pi \ell^2 c} \left[ \frac{3c^2 b}{t^1_1^{1/3}} \right]^2 b^2 t^{4/3}.$$ (3.31)

The energy flux in Problem 6, the power hitting the detector is reduced by two factors of $(1+z)$, so the fraction of photons is also reduced by the redshift factor $(1+z)$. The total energy flux $J$ is then

$$J = \frac{P}{4\pi \ell^2 c} \left[ \frac{3c^2 b}{t^1_1^{1/3}} \right]^2 b^2 t^{4/3}.$$ (3.32)

It is debatable which of the last two expressions is the simplest, so I have boxed both of them. One could also write

$$J = 1.38 \times 10^{-38} \times \frac{P}{4\pi \ell^2 c}.$$ (3.33)

Solutions to parts (b)-(e) written by Leo Stein; solution to part (a) by Alan Guth.

Solution written by Alan Guth.

Attributions added 10/23/09.