MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 13, 2009
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QUIZ 2 SOLUTIONS

Quiz Date: November 5, 2009

PROBLEM 1: DID YOU DO THE READING? *(32 points)*†

With the prevalence of Google today, a budding physicist may wonder "do I really need to memorize all these numbers?" This physicist has failed to consider the importance of impressing one's peers at cocktails or beer hour, when googling is generally considered a social faux pas.

(A) The following is a list of numbers you may need to have handy when doing a back-of-the-envelope calculation (BOTEC) in a bar (since these are going into BOTECs, knowing the answer to within an order of magnitude is good enough). For each of the following, one and only one answer is correct:

(i) *(2 points)* Rest mass of the proton:
(a) 9.38 eV (b) 93.8 keV (c) 93.8 MeV $\boxed{\text{(d) 0.938 GeV}}$ (e) 9.38×10^{13} eV.

The value of 1 GeV is often fine for order of magnitude calculations.

(ii) *(4 points)* Mass difference between the proton and neutron:
(a) .129 eV (b) 12.9 keV $\boxed{\text{(c) 1.29 MeV}}$ (d) 129 GeV (e) 1.29×10^{10} eV.
Circle the heavier species: proton or $\boxed{\text{neutron?}}$

If you have trouble remembering which species is heavier, just remember that it is the unstable species – since a decay can only produce a lighter product. Then, remember that most of the universe is Hydrogen, so a proton must be the stable species – and hence the lighter one!

(iii) *(2 points)* Rest mass of the electron:
(a) 0 eV (b) 0.511 eV $\boxed{\text{(c) 511 keV}}$ (d) 51.1 MeV (e) 5.11 GeV.

A convenient number to remember is that the proton to electron mass ratio is roughly 1800.

(iv) *(2 points)* Baryon to photon ratio (i.e, the ratio of number densities):
(a) 10^1 (b) 10^0 (c) 10^{-3} (d) 10^{-5} $\boxed{\text{(e) } 10^{-9}\text{.}}$

The WMAP 5-year value for $\eta = n_b/n_\gamma = (6.225 \pm 0.170) \times 10^{-10}$, *which to closest order of magnitude is* 10^{-9}.

(v) *(2 points)* Age of the universe when big bang nucleosynthesis is over:

(a) 1 nanosecond (b) 1 second $\boxed{\text{(c) 10 minutes}}$ (d) 380,000 years

(e) 1.37×10^{10} years.

The process ends gradually, so valid answers range from 3 minutes to 15 minutes.

(B) Important interactions and particle properties in nucleosynthesis:

(i) *(5 points)* Write down the two most important reactions which maintain a neutron-proton equilibrium in the very early universe.

The two most important interactions are:

$$\boxed{\begin{aligned} p + e^- &\longleftrightarrow n + \nu_e \\ p + \bar{\nu}_e &\longleftrightarrow n + e^+ \end{aligned}}$$

Note that both reactions are two-body interactions in both directions. A common error was to include the decay process $n \longrightarrow p + e^- + \bar{\nu}_e$. There are two problems with this interaction: (1) the forward reaction is slow, with a decay time of about 15 minutes, and (2) the reverse reaction is a three body interaction, making it unlikely. Since the neutron and proton are in equilibrium during the period before about 1 second, only fast interactions are important. Another common mistake was to exchange ν_e and $\bar{\nu}_e$ in the interactions. The electron neutrino is the neutrino with the same (electron) lepton number as the electron itself, and since lepton number is conserved in these interactions, the electron neutrino must appear on the opposite side of the interaction as the electron to balance it.

(ii) *(5 points)* Create a list of all the unique species in the above two reactions. For each of the species in your list, state the charge, baryon number, and lepton number.

Species	Symbol	Charge (q)	Baryon Number (B)	Lepton Number (L)
Neutron	n	0	+1	0
Proton	p	+1	+1	0
Electron	e^-	-1	0	+1
Positron	e^+	+1	0	-1
e Neutrino	ν_e	0	0	+1
e Anti-neutrino	$\bar{\nu}_e$	0	0	-1

(C) More on nucleosynthesis:

(i) *(5 points)* A theory of big bang nucleosynthesis was first worked out in the late 1940's by George Gamow, Ralph Alpher, and Robert Herman. This theory differed from the currently accepted theory in at least four significant ways. Name one.

(1) They assumed that the universe began in a state of all neutrons, rather the thermal equilibrium mix assumed in modern calculations.

(2) They took into account the conversion of neutrons to protons only by free decay of the neutrons. They ignored the reactions

$$
\begin{aligned}
n + e^+ &\longleftrightarrow p + \bar{\nu}_e \\
n + \nu_e &\longleftrightarrow p + e^- \ ,
\end{aligned}
$$

which play a very important role in modern calculations.

(3) They attempted (unsuccessfully) to account for all of nucleosynthesis — they did not realize that the nucleosynthesis of heavier elements takes place primarily in the interior of stars.

(4) They used fewer than the presently accepted number of neutrinos.

(ii) *(5 points)* Weinberg emphasizes that most of the detailed properties of the early universe are determined by the assumption that it was in a state of thermal equilibrium. Thermal equilibrium, however, cannot change a conserved quantity, so each conserved quantity must be specified. Weinberg mentions three conserved quantities whose densities must be specified in the recipe for the early universe. One is electric charge (which is specified to be zero or negligibly small). What are the other two?

The other two conserved quantities are $\boxed{\textbf{baryon number}}$ and $\boxed{\textbf{lepton number}.}$ (Weinberg also mentions that the electron lepton number and the muon lepton number appear to be separately conserved. Today we would have to add tau lepton number to this list. These conservation laws are still consistent with all known experiments, but there are theoretical reasons for doubting their exactness. We will talk about this later in the course.)

PROBLEM 2: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(33 points)**

(a) Since $\theta = \phi =$ constant, $d\theta = d\phi = 0$, and for light rays one always has $d\tau = 0$. The line element therefore reduces to

$$0 = -c^2\,dt^2 + a^2(t)d\psi^2\ .$$

Rearranging gives

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{c^2}{a^2(t)}\ ,$$

which implies that

$$\boxed{\frac{d\psi}{dt} = \pm\frac{c}{a(t)}\ .}$$

The plus sign describes outward radial motion, while the minus sign describes inward motion.

(b) The maximum value of the ψ coordinate that can be reached by time t is found by integrating its rate of change:

$$\psi_{\rm hor} = \int_0^t \frac{c}{a(t')}dt'\ .$$

The physical horizon distance is the proper length of the shortest line drawn at the time t from the origin to $\psi = \psi_{\rm hor}$, which according to the metric is given by

$$\ell_{\rm phys}(t) = \int_{\psi=0}^{\psi=\psi_{\rm hor}} ds = \int_0^{\psi_{\rm hor}} a(t)\,d\psi = \boxed{a(t)\int_0^t \frac{c}{a(t')}dt'\ .}$$

(c) From part (a),

$$\frac{d\psi}{dt} = \frac{c}{a(t)}\ .$$

By differentiating the equation $ct = \alpha(\theta - \sin\theta)$ stated in the problem, one finds

$$\frac{dt}{d\theta} = \frac{\alpha}{c}(1 - \cos\theta)\ .$$

Then

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{\alpha(1-\cos\theta)}{a(t)}\ .$$

Then using $a = \alpha(1 - \cos\theta)$, as stated in the problem, one has the very simple result

$$\boxed{\frac{d\psi}{d\theta} = 1\ .}$$

(d) This part is very simple if one knows that ψ must change by 2π before the photon returns to its starting point. Since $d\psi/d\theta = 1$, this means that θ must also change by 2π. From $a = \alpha(1 - \cos\theta)$, one can see that a returns to zero at $\theta = 2\pi$, so this is exactly the lifetime of the universe. So,

$$\boxed{\frac{\text{Time for photon to return}}{\text{Lifetime of universe}} = 1\ .}$$

If it is not clear why ψ must change by 2π for the photon to return to its starting point, then recall the construction of the closed universe that was used in Lecture Notes 6. The closed universe is described as the 3-dimensional surface of a sphere in a four-dimensional Euclidean space with coordinates (x, y, z, w):

$$x^2 + y^2 + z^2 + w^2 = a^2\ ,$$

where a is the radius of the sphere. The Robertson-Walker coordinate system is constructed on the 3-dimensional surface of the sphere, taking the point $(0, 0, 0, 1)$ as the center of the coordinate system. If we define the w-direction as "north," then the point $(0, 0, 0, 1)$ can be called the north pole. Each point (x, y, z, w) on the surface of the sphere is assigned a coordinate ψ, defined to be the angle between the positive w axis and the vector (x, y, z, w). Thus $\psi = 0$ at the north pole, and $\psi = \pi$ for the antipodal point, $(0, 0, 0, -1)$, which can be called the south pole. In making the round trip the photon must travel from the north pole to the south pole and back, for a total range of 2π.

Discussion: Some students answered that the photon would return in the lifetime of the universe, but reached this conclusion without considering the details of the motion. The argument was simply that, at the big crunch when the scale factor returns to zero, all distances would return to zero, including the distance between the photon and its starting place. This statement is correct, but it does not quite answer the question. First, the statement in no way rules out the possibility that the photon might return to its starting point before the big crunch. Second, if we use the delicate but well-motivated definitions that general relativists use, it is not necessarily true that the photon returns to its starting point at the big crunch. To be concrete, let me consider a radiation-dominated closed universe—a hypothetical universe for which the only "matter" present consists of massless particles such as photons or neutrinos. In that case (you can check my calculations) a photon that leaves the north pole at $t = 0$ just reaches the south pole at the big crunch. It might seem that reaching the south pole at the big crunch is not any different from completing the round trip back to the north pole, since the distance between the north pole and the south pole is zero at $t = t_{\rm Crunch}$, the time of the big crunch. However, suppose we adopt the principle that the instant of the initial singularity and the instant of the final crunch are both too singular to be considered part of the spacetime. We will allow ourselves to mathematically consider times ranging from $t = \epsilon$ to $t = t_{\rm Crunch} - \epsilon$, where ϵ is arbitrarily small, but we will not try to describe what happens exactly at $t = 0$ or $t = t_{\rm Crunch}$. Thus, we now consider a photon that starts its journey at $t = \epsilon$, and we follow it until $t = t_{\rm Crunch} - \epsilon$. For the case of the matter-dominated closed universe, such a photon would traverse a fraction of the full circle that would be almost 1, and would approach 1 as $\epsilon \to 0$. By contrast, for the radiation-dominated closed universe, the photon would traverse a fraction of the full circle that is almost 1/2, and it would approach 1/2 as $\epsilon \to 0$. Thus, from this point of view the two cases look very different. In the radiation-dominated case, one would say that the photon has come only half-way back to its starting point.

PROBLEM 3: EXAMINING A PECULIAR SPACETIME METRIC *(35 points)**

(a) The ruler extends only in the x direction, so $\mathrm{d}y = \mathrm{d}z = \mathrm{d}t = 0$. Then $\mathrm{d}s^2 = \mathrm{d}x^2$, or $\mathrm{d}s = |\mathrm{d}x|$. The physical length is then

$$\ell_{\rm phys} = \int \mathrm{d}s = \int_a^b \mathrm{d}x = \boxed{\; b - a \; .}$$

(b) Since the clock is stationary, $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = 0$, and then $-c^2\,\mathrm{d}\tau^2 = -(x^2/T_0^2)\mathrm{d}t^2$. So

$$\mathrm{d}\tau = \frac{x}{cT_0}\,\mathrm{d}t \;,$$

where $x = a$ is the position of the clock. Then

$$\tau = \int_0^\beta \frac{a}{cT_0}\,\mathrm{d}t = \boxed{\;\frac{a\beta}{cT_0}\; .}$$

(c) The geodesic equation has the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \;,$$

where the nonzero components of $g_{\mu\nu}$ are

$$g_{00} = -\frac{x^2}{T_0^2} \ , \qquad g_{11} = g_{22} = g_{33} = 1 \ .$$

On the left-hand side of the geodesic equation, $\mu = 0$ and ν is summed from 0 to 3 (using the Einstein summation convention). But the only value of ν for which $g_{0\nu}$ is nonzero is $\nu = 0$, so the left-hand side becomes

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{00}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right\} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left[-\frac{x^2}{T_0^2}\right]\frac{\mathrm{d}t}{\mathrm{d}\tau}\right\} .$$

The right-hand side is proportional to

$$\partial_0 g_{\lambda\sigma} \equiv \frac{\partial g_{\lambda\sigma}}{\partial t} = 0 \ ,$$

since none of the components of $g_{\mu\nu}$ depend on t.

Pedagogical Note: The derivative here is a partial derivative ($\partial/\partial t$) and not a total derivative ($\mathrm{d}/\mathrm{d}t$) like the derivative on the left-hand side. If we let X denote an arbitrary quantity, then the partial derivative $\partial X/\partial t$ is the derivative of X with respect to t, treating the other coordinates (x, y, and z) as constants. Since none of the components of $g_{\mu\nu}$ depend on t, the derivative $\partial g_{\mu\nu}/\partial t$ vanishes. The total derivative $\mathrm{d}X/\mathrm{d}t$, on the other hand, means to calculate the full change in X as t varies. When evaluating $\mathrm{d}X/\mathrm{d}t$, if X depends on x, y, or z, and they in turn depend on t, then this dependence would be taken into account through the chain rule:

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= \frac{\partial X}{\partial t} + \frac{\partial X}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial X}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial X}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \frac{\partial X}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t} \ .\end{aligned}$$

When comparing the two lines above, note that $\mathrm{d}x^0/\mathrm{d}t = \mathrm{d}t/\mathrm{d}t = 1$.

Putting together the two sides,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left[-\frac{x^2}{T_0^2}\right]\frac{\mathrm{d}t}{\mathrm{d}\tau}\right\} = 0 \ .}$$

Since T_0 is a constant, the equation is equivalent to

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{x^2\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right\} = 0\ .}$$

Either boxed equation is a perfectly acceptable answer. There was no need for you to have expanded this equation, but if you did you should have gotten

$$x^2\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} + 2x\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} = 0\ .$$

(d) With no motion in the y or z directions, the metric equation becomes

$$-c^2\,\mathrm{d}\tau^2 = -\frac{x^2}{T_0^2}\,\mathrm{d}t^2 + \mathrm{d}x^2\ .$$

Dividing by $\mathrm{d}\tau^2$,

$$-c^2 = -\frac{x^2}{T_0^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2\ .$$

Solving for $\dfrac{\mathrm{d}t}{\mathrm{d}\tau}$,

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{T_0}{x}\sqrt{c^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2}\ .}$$

(e) The geodesic equation implies that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{x^2\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right\} = 0 \quad\Longrightarrow\quad x^2\,\frac{\mathrm{d}t}{\mathrm{d}\tau} = \text{const}\ .$$

We can evaluate the constant by determining the value of $x^2\,\mathrm{d}t/\mathrm{d}\tau$ when the particle is released at $t = 0$. At this instant $x = a$ and $\mathrm{d}x/\mathrm{d}\tau = 0$, so according to the answer to (d), $\mathrm{d}t/\mathrm{d}\tau = cT_0/a$. Thus, at any time

$$x^2\,\frac{\mathrm{d}t}{\mathrm{d}\tau} = acT_0\ .$$

Replacing $\mathrm{d}t/\mathrm{d}\tau$ by the answer from (d),

$$xT_0\sqrt{c^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2} = acT_0\ .$$

Manipulating,

$$c^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 = c^2\,\frac{a^2}{x^2}$$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 = c^2\,\frac{a^2 - x^2}{x^2}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -c\,\frac{\sqrt{a^2 - x^2}}{x}\ .$$

Note that I used the negative square root in the last step, because we were given the hint that the particle moves to smaller x, so $\mathrm{d}x/\mathrm{d}\tau$ should be negative. If we were not given this hint, we could have inferred that the particle moves to smaller x, because otherwise the argument of the square root would be negative.

Rearranging,

$$\mathrm{d}\tau = -\frac{x\,\mathrm{d}x}{c\sqrt{a^2 - x^2}}\ .$$

To obtain a definite integral, we keep in mind that as τ changes from 0 to some final value τ_f, x changes from a to some final value x_f. Thus

$$\int_0^{\tau_f} \mathrm{d}\tau = -\int_a^{x_f} \frac{x\,\mathrm{d}x}{c\sqrt{a^2 - x^2}}\ ,$$

or

$$\boxed{\tau_f = \int_{x_f}^{a} \frac{x\,\mathrm{d}x}{c\sqrt{a^2 - x^2}}\ .}$$

Note that I removed the minus sign by reversing the limits of integration. Equivalently, one can drop the subscripts f and use x and τ to describe the position and proper time variables, but then one should give a different name (such as x') to the variable of integration:

$$\boxed{\tau = \int_{x}^{a} \frac{x'\,\mathrm{d}x'}{c\sqrt{a^2 - x'^2}}\ .}$$

You were not asked to carry out the integration, but you can do it by using the trigonometric substitution $x \equiv a \sin\theta$. Then $\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$, and

$$\int \frac{x\,\mathrm{d}x}{\sqrt{a^2 - x^2}} = \int \frac{a^2 \sin\theta\,\cos\theta\,\mathrm{d}\theta}{a\cos\theta} = \int a\,\sin\theta\,\mathrm{d}\theta = -a\,\cos\theta = -\sqrt{a^2 - x^2}\ .$$

Finally,

$$\tau = -\frac{1}{c}\sqrt{a^2 - x'^2}\,\Bigg|_x^a = \frac{1}{c}\sqrt{a^2 - x^2}\ .$$

Solving for x,

$$x(\tau) = \sqrt{a^2 - c^2\tau^2}\ .$$

Discussion: The metric discussed in this problem is called the Rindler metric, and it is actually a description of Minkowski space with peculiar coordinates. If we let X, Y, Z, and T denote the usual Minkowski space coordinates, the Rindler coordinates are related by

$$\begin{aligned} t &= T_0 \operatorname{arctanh}\left(\frac{cT}{X}\right) & \qquad\qquad & T = \frac{x}{c}\sinh\left(\frac{t}{T_0}\right) \\ x &= \sqrt{X^2 - c^2T^2} & \text{or} \qquad & X = x\cosh\left(\frac{t}{T_0}\right) \\ y &= Y & & Y = y \\ z &= Z & & Z = z \end{aligned}$$

The Rindler coordinate system (which is restricted to $x > 0$) actually covers only one quadrant of the Minkowski space, with $X > 0$ and $|T| < X/c$. When described in Minkowski coordinates, a particle that is stationary in the Rindler coordinates is undergoing uniform acceleration in its own rest frame, where the magnitude of the uniform acceleration depends on x. The particle described in this problem was actually standing still in the Minkowski coordinates. The Rindler coordinate system has a horizon at $x = 0$, which has many similarities to the horizon of a black hole, in spite of the fact that the spacetime is simply Minkowski space.

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† Solution written by Leo Stein.

* Solution written by Alan Guth.