MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe December 23, 2009
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QUIZ 3 SOLUTIONS

Quiz Date: December 3, 2009

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

(a) *(10 points)* This question concerns some numbers related to the cosmic microwave background (CMB) that one should never forget. State the values of these numbers, to within an order of magnitude unless otherwise stated. In all cases the question refers to the present value of these quantities.

(i) The average temperature T of the CMB (to within 10%). $\boxed{2.725\,\text{K}}$

(ii) The speed of the Local Group with respect to the CMB, expressed as a fraction v/c of the speed of light. (The speed of the Local Group is found by measuring the dipole pattern of the CMB temperature to determine the velocity of the spacecraft with respect to the CMB, and then removing spacecraft motion, the orbital motion of the Earth about the Sun, the Sun about the galaxy, and the galaxy relative to the center of mass of the Local Group.)

The dipole anisotropy corresponds to a "peculiar velocity" (that is, velocity which is not due to the expansion of the universe) of $630 \pm 20\,\text{km}\,\text{s}^{-1}$*, or in terms of the speed of light,* $\boxed{v/c \approx 2 \times 10^{-3}}$.

(iii) The intrinsic relative temperature fluctuations $\Delta T/T$, after removing the dipole anisotropy corresponding to the motion of the observer relative to the CMB. $\boxed{1.1 \times 10^{-5}}$

(iv) The ratio of baryon number density to photon number density, $\eta = n_{\text{bary}}/n_\gamma$.

The WMAP 5-year value for $\eta = n_b/n_\gamma = \boxed{(6.225 \pm 0.170) \times 10^{-10}}$, *which to closest order of magnitude is* 10^{-9}.

(v) The angular size θ_H, in degrees, corresponding to what was the Hubble distance c/H at the surface of last scattering. This answer must be within a factor of 3 to be correct. $\boxed{\sim 1^\circ}$

(b) *(3 points)* Because photons outnumber baryons by so much, the exponential tail of the photon blackbody distribution is important in ionizing hydrogen well after kT_γ falls below $Q_H = 13.6$ eV. What is the ratio kT_γ/Q_H when the ionization fraction of the universe is 1/2?

(i) 1/5   **[(ii) 1/50]**   (iii) 10^{-3} (iv) 10^{-4} (v) 10^{-5}

This is not a number one has to commit to memory if one can remember the temperature of (re)combination in eV*, or if only in* K *along with the conversion factor ($k \approx 10^{-4}\,\text{eV}\,\text{K}^{-1}$). One can then calculate that near recombination, $kT_\gamma/Q_H \approx (10^{-4}\,\text{eV}\,\text{K}^{-1})(3000\,\text{K})/(13.6\,\text{eV}) \approx 1/45$.*

(c) *(2 points)* Which of the following describes the Sachs-Wolfe effect?

(i) Photons from fluid which had a velocity toward us along the line of sight appear redder because of the Doppler effect.

(ii) Photons from fluid which had a velocity toward us along the line of sight appear bluer because of the Doppler effect.

[(iii)] Photons from overdense regions at the surface of last scattering appear redder because they must climb out of the gravitational potential well.

(iv) Photons from overdense regions at the surface of last scattering appear bluer because they must climb out of the gravitational potential well.

(v) Photons traveling toward us from the surface of last scattering appear redder because of absorption in the intergalactic medium.

(vi) Photons traveling toward us from the surface of last scattering appear bluer because of absorption in the intergalactic medium.

Explanation: Denser regions have a deeper (more negative) gravitational potential. Photons which travel through a spatially varying potential acquire a redshift or blueshift depending on whether they are going up or down the potential, respectively. Photons originating in the denser regions start at a lower potential and must climb out, so they end up being redshifted relative to their original energies.

(d) *(10 points)* For each of the following statements, say whether it is true or false:

(i) Dark matter interacts through the gravitational, weak, and electromagnetic forces. T or **[F]** ?

(ii) The virial theorem can be applied to a cluster of galaxies to find its total mass, most of which is dark matter. **[T]** or F ?

(iii) Neutrinos are thought to comprise a significant fraction of the energy density of dark matter. T or **[F]** ?

(iv) Magnetic monopoles are thought to comprise a significant fraction of the energy density of dark matter. T or **[F]** ?

(v) Lensing observations have shown that MACHOs cannot account for the dark matter in galactic halos, but that as much as 20% of the halo mass could be in the form of MACHOs. **[T]** or F ?

PROBLEM 2: PRESSURE, ENERGY DENSITY, AND COSMIC EVOLUTION WITH MYSTERIOUS STUFF *(35 points)*

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} \ .$$

(The above expression is proportional to $1/\sqrt{V + \Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) The total energy is the energy density times the volume, so

$$U(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}}\,(V + \Delta V) = u_0 \sqrt{V(V + \Delta V)} \ .$$

Expanding to first order,

$$U(V + \Delta V) = U_0 \sqrt{1 + \frac{\Delta V}{V}} \simeq U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) \ ,$$

where $U_0 = u_0 V$ is the original energy inside the piston. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V}\, U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p\,\Delta V$. So

$$\boxed{\Delta W = -p\,\Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V}\, U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2}\, u_0 \ .}$$

(d) The critical density ρ_c is defined as that density for which $k = 0$, where the Friedmann equation from the front of the exam implies that

$$H^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ .$$

Thus the critical density today is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} \ .$$

The mass density today of any species X is then related to $\Omega_{X,0}$ by

$$\rho_{X,0} = \rho_c \Omega_{X,0} = \frac{3H_0^2 \Omega_{X,0}}{8\pi G} \ .$$

The total mass density today is then expressed in terms of its four components as

$$\rho_0 = \frac{3H_0^2}{8\pi G} \left[\Omega_{m,0} + \Omega_{r,0} + \Omega_{v,0} + \Omega_{\text{ms},0} \right] \ .$$

But we also know how each of these contributions to the mass density scales with $x(t)$: $\rho_m \propto 1/x^3$, $\rho_r \propto 1/x^4$, $\rho_v \propto 1$, and $\rho_{\text{ms}} \propto 1/\sqrt{V} \propto 1/x^{3/2}$. Inserting these factors,

$$\boxed{\rho(t) = \frac{3H_0^2}{8\pi G} \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} \right] \ .}$$

(e) The Friedmann equation then becomes

$$\left(\frac{\dot{x}}{x} \right)^2 = \frac{8\pi G}{3} \frac{3H_0^2}{8\pi G} \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} \right] - \frac{kc^2}{a^2} \ .$$

Defining

$$H_0^2 \Omega_{k,0} = -\frac{kc^2}{a^2(t_0)} \ ,$$

so

$$-\frac{kc^2}{a^2(t)} = -\frac{kc^2}{a^2(t_0)} \frac{1}{x^2} = \frac{H_0^2 \Omega_{k,0}}{x^2} \ ,$$

and then the Friedmann equation becomes

$$\left(\frac{\dot{x}}{x} \right)^2 = H_0^2 \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2} \right] \ .$$

Applying this equation today, when $\dot{x}/x = H_0$, one finds that

$$\boxed{\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{v,0} - \Omega_{\text{ms},0}\ .}$$

Rearranging the equation for $(\dot{x}/x)^2$ above,

$$H_0\,\mathrm{d}t = \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}}\ .$$

The age of the universe is found by integrating over the full range of x, which starts from 0 when the universe is born, and is equal to 1 today. So

$$\boxed{t_0 = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\text{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}}\ .}$$

PROBLEM 3: A NEW THEORY OF THE WEAK INTERACTIONS

(40 points)

(a) In the standard model, the black-body radiation at $kT \approx 200$ MeV contains the following contributions:

$$\left.\begin{array}{ll}
\text{Photons:} & g = 2 \\
e^+e^-: & g = 4\times\frac{7}{8} = 3\frac{1}{2} \\
\nu_e, \nu_\mu, \nu_\tau: & g = 6\times\frac{7}{8} = 5\frac{1}{4} \\
\mu^+\mu^-: & g = 4\times\frac{7}{8} = 3\frac{1}{2} \\
\pi^+\pi^-\pi^0 & g = 3
\end{array}\right\}\ g_{\text{TOT}} = 17\frac{1}{4}$$

The mass density is then given by

$$\rho = \frac{u}{c^2} = g_{\text{TOT}}\frac{\pi^2}{30}\,\frac{(kT)^4}{\hbar^3 c^5}\ .$$

In kg/m^3, one can evaluate this expression by

$$\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\,\frac{\left[200\times 10^6\,\text{eV}\times\dfrac{1.602\times10^{-19}\,\text{J}}{\text{eV}}\right]^4}{\left(1.055\times10^{-34}\,\text{J-s}\right)^3\left(2.998\times10^8\,\text{m/s}\right)^5}\ .$$

Checking the units,

$$[\rho] = \frac{\mathrm{J}^4}{\mathrm{J}^3\text{-}\mathrm{s}^3\text{-}\mathrm{m}^5\text{-}\mathrm{s}^{-5}} = \frac{\mathrm{J}\text{-}\mathrm{s}^2}{\mathrm{m}^5}$$

$$= \frac{\left(\mathrm{kg}\text{-}\mathrm{m}^2\text{-}\mathrm{s}^{-2}\right)\mathrm{s}^2}{\mathrm{m}^5} = \mathrm{kg/m}^3 \; .$$

So, the final answer would be

$$\boxed{\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6 \times 1.602\times 10^{-19}\right]^4}{\left(1.055\times 10^{-34}\right)^3\left(2.998\times 10^8\right)^5}\frac{\mathrm{kg}}{\mathrm{m}^3}\; .}$$

You were not expected to evaluate this, but with a calculator one would find

$$\rho = 2.10 \times 10^{18}\ \mathrm{kg/m}^3 \; .$$

In $\mathrm{g/cm}^3$, one would evaluate this expression by

$$\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6\,\mathrm{eV}\times\dfrac{1.602\times 10^{-12}\,\mathrm{erg}}{\mathrm{eV}}\right]^4}{\left(1.055\times 10^{-27}\,\mathrm{erg\text{-}s}\right)^3\left(2.998\times 10^{10}\,\mathrm{cm/s}\right)^5}\; .$$

Checking the units,

$$[\rho] = \frac{\mathrm{erg}^4}{\mathrm{erg}^3\text{-}\mathrm{s}^3\text{-}\mathrm{cm}^5\text{-}\mathrm{s}^{-5}} = \frac{\mathrm{erg}\text{-}\mathrm{s}^2}{\mathrm{cm}^5}$$

$$= \frac{\left(\mathrm{g}\text{-}\mathrm{cm}^2\text{-}\mathrm{s}^{-2}\right)\mathrm{s}^2}{\mathrm{cm}^5} = \mathrm{g/cm}^3 \; .$$

So, in this case the final answer would be

$$\boxed{\rho = \left(17\frac{1}{4}\right)\frac{\pi^2}{30}\frac{\left[200\times 10^6 \times 1.602\times 10^{-12}\right]^4}{\left(1.055\times 10^{-27}\right)^3\left(2.998\times 10^{10}\right)^5}\frac{\mathrm{g}}{\mathrm{cm}^3}\; .}$$

No evaluation was requested, but with a calculator you would find

$$\rho = 2.10 \times 10^{15}\ \mathrm{g/cm}^3 \; ,$$

which agrees with the answer above.

Note: A common mistake was to leave out the conversion factor 1.602×10^{-19} J/eV (or 1.602×10^{-12} erg/eV), and instead to use $\hbar = 6.582 \times 10^{-16}$ eV-s. But if one works out the units of this answer, they turn out to be eV-sec^2/m^5 (or eV-sec^2/cm^5), which is a most peculiar set of units to measure a mass density.

In the NTWI, we have in addition the contribution to the mass density from R^+-R^- pairs, which would act just like e^+-e^- pairs or μ^+-μ^- pairs, with $g = 3\frac{1}{2}$. Thus $g_{\rm TOT} = 20\frac{3}{4}$, so

$$\boxed{\rho = \left(20\frac{3}{4}\right)\frac{\pi^2}{30}\frac{\left[200 \times 10^6 \times 1.602 \times 10^{-19}\right]^4}{(1.055 \times 10^{-34})^3\,(2.998 \times 10^8)^5}\frac{\rm kg}{\rm m^3}}$$

or

$$\boxed{\rho = \left(20\frac{3}{4}\right)\frac{\pi^2}{30}\frac{\left[200 \times 10^6 \times 1.602 \times 10^{-12}\right]^4}{(1.055 \times 10^{-27})^3\,(2.998 \times 10^{10})^5}\frac{\rm g}{\rm cm^3}\ .}$$

Numerically, the answer in this case would be

$$\rho_{\rm NTWI} = 2.53 \times 10^{18}\ {\rm kg/m}^3 = 2.53 \times 10^{15}\ {\rm g/cm}^3\ .$$

(b) As long as the universe is in thermal equilibrium, entropy is conserved. The entropy in a given volume of the comoving coordinate system is

$$a^3(t)s\,V_{\rm coord}\quad ,$$

where s is the entropy density and $a^3 V_{\rm coord}$ is the physical volume. So

$$a^3(t)s$$

is conserved. After the neutrinos decouple,

$$a^3 s_\nu \quad \text{and} \quad a^3 s_{\rm other}$$

are separately conserved, where $s_{\rm other}$ is the entropy of everything except neutrinos.

Note that s can be written as

$$s = gAT^3\quad ,$$

where A is a constant. Before the disappearance of the e, μ, R, and π particles from the thermal equilibrium radiation,

$$s_\nu = \left(5\frac{1}{4}\right) AT^3$$

$$s_{\rm other} = \left(15\frac{1}{2}\right) AT^3 \quad .$$

So

$$\frac{s_\nu}{s_{\rm other}} = \frac{5\frac{1}{4}}{15\frac{1}{2}} \quad .$$

If $a^3 s_\nu$ and $a^3 s_{\rm other}$ are conserved, then so is $s_\nu / s_{\rm other}$. By today, the entropy previously shared among the various particles still in equilibrium after neutrino decoupling has been transfered to the photons so that

$$s_{\rm other} = s_{\rm photons} = 2AT_\gamma^3 \quad .$$

The entropy in neutrinos is still

$$s_\nu = \left(5\frac{1}{4}\right) AT_\nu^3 \quad .$$

Since $s_\nu / s_{\rm other}$ is constant we know that

$$\frac{\left(5\frac{1}{4}\right) T_\nu^3}{2T_\gamma^3} = \frac{s_\nu}{s_{\rm other}} = \frac{5\frac{1}{4}}{15\frac{1}{2}}$$

$$\Longrightarrow \quad \boxed{T_\nu = \left(\frac{4}{31}\right)^{1/3} T_\gamma} \quad .$$

(c) One can write

$$n = g^* BT^3 \quad ,$$

where B is a constant. Here $g_\gamma^* = 2$, and $g_\nu^* = 6 \times \frac{3}{4} = 4\frac{1}{2}$. In the standard model, one has today

$$\frac{n_\nu}{n_\gamma} = \frac{g_\nu^* T_\nu^3}{g_\gamma^* T_\gamma^3} = \frac{\left(4\frac{1}{2}\right)}{2}\frac{4}{11} = \boxed{\frac{9}{11}\ .}$$

In the NTWI,

$$\frac{n_\nu}{n_\gamma} = \frac{\left(4\frac{1}{2}\right)}{2}\,\frac{4}{31} = \boxed{\frac{9}{31}\ .}$$

(d) At $kT = 200$ MeV, the thermal equilibrium ratio of neutrons to protons is given by

$$\frac{n_{\rm n}}{n_{\rm p}} = e^{-1.29\,{\rm MeV}/200\,{\rm MeV}} \approx 1 \quad .$$

In the standard theory this ratio would decrease rapidly as the universe cooled and kT fell below the p-n mass difference of 1.29 MeV, but in the NTWI the ratio freezes out at the high temperature corresponding to $kT = 200$ MeV, when the ratio is about 1. When kT falls below 200 MeV in the NTWI, the neutrino interactions

$$n + \nu_e \leftrightarrow p + e^- \qquad \text{and} \qquad n + e^+ \leftrightarrow p + \bar{\nu}_e$$

that maintain the thermal equilibrium balance between protons and neutrons no longer occur at a significant rate, so the ratio $n_/n_p$ is no longer controlled by thermal equilibrium. After kT falls below 200 MeV, the only process that can convert neutrons to protons is the rather slow process of free neutron decay, with a decay time τ_d of about 890 s. Thus, when the deuterium bottleneck breaks at about 200 s, the number density of neutrons will be considerably higher than in the standard model. Since essentially all of these neutrons will become bound into He nuclei, the higher neutron abundance of the NTWI implies a

$$\boxed{\text{higher predicted He abundance.}}$$

To estimate the He abundance, note that if we temporarily ignore free neutron decay, then the neutron-proton ratio would be frozen at about 1 and would remain 1 until the time of nucleosynthesis. At the time of nucleosynthesis essentially all of these neutrons would be bound into He nuclei (each with 2 protons and 2 neutrons). For an initial 1:1 ratio of neutrons to protons, all the neutrons and protons can be bound into He nuclei, with no protons left over in the form of hydrogen, so Y would equal 1. However, the free neutron decay process will cause the ratio n_n/n_p to fall below 1 before the start of nucleosynthesis, so the predicted value of Y would be less than 1.

To calculate how much less, note that Ryden estimates the start of nucleosynthesis at the time when the temperature reaches $T_{\rm nuc}$, which is the temperature for which a thermal equilibrium calculation gives $n_D/n_n = 1$. This corresponds to what Weinberg refers to as the breaking of the deuterium bottleneck. The temperature $T_{\rm nuc}$ is calculated in terms of $\eta = n_B/n_\gamma$ and physical constants, so it would not be changed by the NTWI. The time when this temperature is reached, however, would be changed slightly by the change in the ratio T_ν/T_γ. Since this effect is rather subtle, no points will be taken off if you omitted it. However, to be as accurate as possible, one should recognize that nucleosynthesis occurs during the radiation-dominated era, but long after the e^+-e^- pairs have disappeared, so the black-body radiation consists of photons at temperature T_γ and neutrinos at a lower temperature T_ν. The energy density is given by

$$u = \frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3}\left[2 + \left(\frac{21}{4}\right)\left(\frac{T_\nu}{T_\gamma}\right)^4\right] \equiv g_{\rm eff}\frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3} \ ,$$

where

$$g_{\rm eff} = 2 + \left(\frac{21}{4}\right)\left(\frac{T_\nu}{T_\gamma}\right)^4 \ .$$

For the standard model

$$g_{\rm eff}^{\rm sm} = 2 + \left(\frac{21}{4}\right)\left(\frac{4}{11}\right)^{4/3} \ ,$$

and for the NTWI

$$g_{\rm eff}^{\rm NTWI} = 2 + \left(\frac{21}{4}\right)\left(\frac{4}{31}\right)^{4/3} \ .$$

The relation between time and temperature in a flat radiation-dominated universe is given in the formula sheets as

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}} \ .$$

Thus,

$$t \propto \frac{1}{g_{\rm eff}^{1/2} T^2} \ .$$

In the standard model Ryden estimates the time of nucleosynthesis as $t_{\rm nuc}^{\rm sm} \approx$ 200 s, so in the NTWI it would be longer by the factor

$$t_{\rm nuc}^{\rm NTWI} = \sqrt{\frac{g_{\rm eff}^{\rm sm}}{g_{\rm eff}^{\rm NTWI}}}\, t_{\rm nuc}^{\rm sm} \ .$$

While of coure you were not expected to work out the numerics, this gives

$$t_{\rm nuc}^{\rm NTWI} = 1.20\, t_{\rm nuc}^{\rm sm} \ .$$

Note that Ryden gives $t_{\rm nuc} \approx 200s$, while Weinberg places it at $3\frac{3}{4}$ minutes ≈ 225 s, which is close enough.

To follow the effect of this free decay, it is easiest to do it by considering the ratio neutrons to baryon number, n_n/n_B, since n_B does not change during this period. At freeze-out, when $kT \approx 200$ MeV,

$$\frac{n_n}{n_B} \approx \frac{1}{2} \ .$$

Just before nucleosynthesis, at time $t_{\rm nuc}$, the ratio will be

$$\frac{n_n}{n_B} \approx \frac{1}{2} e^{-t_{\rm nuc}/\tau_d} \ .$$

If free decay is ignored, we found $Y = 1$. Since all the surviving neutrons are bound into He, the corrected value of Y is simply deceased by multiplying by the fraction of neutrons that do not undergo decay. Thus, the prediction of NTWI is

$$\boxed{Y = e^{-t_{\rm nuc}/\tau_d} = \exp\left\{ -\frac{\sqrt{\frac{g_{\rm eff}^{\rm sm}}{g_{\rm eff}^{\rm NTWI}}}\, 200}{890} \right\} \ ,}$$

where $g_{\rm eff}^{\rm sm}$ and $g_{\rm eff}^{\rm NTWI}$ are given above. When evaluated numerically, this would give

$$Y = \text{Predicted He abundance by weight} \approx 0.76 \ .$$