Problem 1: Questions Based on Reading and Short Calculations

(a) (7 points) The following quantities all have a power law dependence on the cosmological scale factor $a(t)$. State the dependences (in the form of $\propto a^n$):

(i) The number density of baryons

(ii) The number density of photons

(iii) The energy density of baryons (protons and neutrons)

(iv) The energy density of photons

(v) The wavelength of photons

(vi) The temperature of a blackbody distribution of photons

(b) (5 points) On some problems you have been asked to assume that time dependence of the scale factor was that of a flat matter-dominated universe, $a(t) \propto t^{2/3}$. (As usual, “matter-dominated” is shorthand for dark matter + baryons.) What time dependence of the scale factor was that of a dark matter-dominated universe, $a(t) \propto t^{(3/2)}$. Argue from one or more of the above dependences why this assumption might not always be valid.

(c) (6 points) What is the temperature of the background radiation (i) today and (ii) when the universe first became optically transparent? Combine these numbers with your answer from (a)-(vii) and the relationship between $a$ and $z$ to estimate the redshift when the universe became transparent.

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Please answer part (a) directly next to each part, but answer parts (b) and (c) on the blank page at the right.

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Please turn over the right page.
Problem 1, Continued.

(d) (6 points) Assume for the moment (contrary to (b)) that the universe was always matter dominated, and take $H_0 = 72 \text{ km} \text{s}^{-1} \text{Mpc}^{-1}$. Using the matter-dominated form of $a(t)$, what is the age of the universe (i) today and (ii) when it became transparent? (If you did not remember the temperature at which the universe became transparent, you will get full credit here if you answer the question assuming 10,000K. You may find it useful to know that $1/10^{10} \text{ yr} = 97.8 \text{ km} \text{s}^{-1} \text{Mpc}^{-1}$.) If you are unclear about how to find the age of the universe in terms of $H_0$, note that you can first find $H_0$ in terms of $t$, and then invert the formula.

(e) (6 points) Write down a formula for the horizon distance $d_H(t)$ in terms of an arbitrary $a(t)$, and then evaluate it for the particular case of a matter-dominated universe. Taking the ages from (d), find the horizon distance (i) today and (ii) when the universe became transparent. (The horizon distance is defined as the present distance of objects that are so far away that light has just barely had time to reach us.)

Problem 2: A Two-Level High-Speed Merry-Go-Round

(a) (5 points) Suppose that cars 5–8 are all emitting light waves in all directions. If an observer in car 1 receives light waves from each of these cars, what redshift $z$ does she observe for each of the four signals?

(b) (10 points) Suppose that a spaceship is receding to the right at a relativistic speed $u$, along a line through the hub, as shown in the diagram. Suppose that an observer in car 6 receives a radio signal from the spaceship, at the time when the car is in the position shown in the diagram. What redshift $z$ is observed?
Consider a flat, matter-dominated universe.

(a) Suppose that there is another galaxy, galaxy $B$, also located at a physical distance $d$ from galaxy $A$, at time $t$. What is the maximum speed that light can reach galaxy $B$, and why can a light signal not arrive at galaxy $B$, even if the source moves at the speed of light? Is this the limiting value at arbitrarily late times?

(b) In this universe, the speed of light is measured relative to an observer moving in any direction. Let $v$ be the speed of light as measured in a nonrelativistic observer moving at relative velocity $u$ to another observer. Use this definition to show that the speed $v$ of light in this universe is the same as that in nonrelativistic, observer-moving coordinate systems. (You might be measured, for example, in watts per meter $J^2/\Omega$.)

(c) Suppose that there is a second galaxy, galaxy $C$, also located at a physical distance $d$ from galaxy $A$, but in a direction orthogonal to that of galaxy $B$. Let $t$ be the time of emission, $p$ and $s$ the time at which the light signal reaches galaxy $C$, and $q$, the time of reception. Show that the observed redshift $z$ is given by

$$z = \frac{t - p - s}{t - q}.$$