MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 5, 2009
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QUIZ 2

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BLANK PAGES AND A SUMMARY OF USEFUL INFORMATION ARE AT THE END OF THE EXAM.

Problem	**Maximum**	**Score**
1	32	
2	33	
3	35	
TOTAL	100	

Your Name ______________________

PROBLEM 1: DID YOU DO THE READING? *(32 points)*

With the prevalence of Google today, a budding physicist may wonder "do I really need to memorize all these numbers?" This physicist has failed to consider the importance of impressing one's peers at cocktails or beer hour, when googling is generally considered a social faux pas.

(A) The following is a list of numbers you may need to have handy when doing a back-of-the-envelope calculation (BOTEC) in a bar (since these are going into BOTECs, knowing the answer to within an order of magnitude is good enough). (Reminder: 1 keV = 10^3 eV; 1 MeV = 10^6 eV; 1 GeV = 10^9 eV.) For each of the following, one and only one answer is correct:

(i) *(2 points)* Rest mass of the proton:
(a) 9.38 eV (b) 93.8 keV (c) 93.8 MeV (d) .938 GeV (e) 9.38×10^{13} eV .

(ii) *(4 points)* Mass difference between the proton and neutron:
(a) .129 eV (b) 12.9 keV (c) 1.29 MeV (d) 129 GeV (e) 1.29×10^{10} eV .
Circle the heavier species: proton or neutron?

(iii) *(2 points)* Rest mass of the electron:
(a) 0 eV (b) 0.511 eV (c) 511 keV (d) 51.1 MeV (e) 5.11 GeV .

(iv) *(2 points)* Baryon to photon ratio (i.e, the ratio of number densities):
(a) 10^1 (b) 10^0 (c) 10^{-3} (d) 10^{-5} (e) 10^{-9} .

(v) *(2 points)* Age of the universe when big bang nucleosynthesis is over:
(a) 1 nanosecond (b) 1 second (c) 10 minutes (d) 380,000 years
(e) 1.37×10^{10} years.

(B) Important interactions and particle properties in nucleosynthesis:

(i) *(5 points)* Write down the two most important reactions which maintain a neutron-proton equilibrium in the very early universe.

(ii) *(5 points)* Create a list of all the unique species in the above two reactions. For each of the species in your list, state the charge, baryon number, and lepton number.

(C) More on nucleosynthesis:

(i) *(5 points)* A theory of big bang nucleosynthesis was first worked out in the late 1940's by George Gamow, Ralph Alpher, and Robert Herman. This theory differed from the currently accepted theory in at least four significant ways. Name one.

(ii) *(5 points)* Weinberg emphasizes that most of the detailed properties of the early universe are determined by the assumption that it was in a state of thermal equilibrium. Thermal equilibrium, however, cannot change a conserved quantity, so each conserved quantity must be specified. Weinberg mentions three conserved quantities whose densities must be specified in the recipe for the early universe. One is electric charge (which is specified to be zero or negligibly small). What are the other two? (You need only give the names of the conserved quantities, not their values.)

PROBLEM 2: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE *(33 points)*

The following problem was Problem 3, Quiz 2, 1998. This year it was Problem 2 of the Review Problems for Quiz 2.

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

(a) *(8 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi =$ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(9 points)* Write an expression for the physical horizon distance $\ell_{\rm phys}$ at time t. You should leave your answer in the form of a definite integral.

The form of $a(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.,* dominated by nonrelativistic matter), then $a(t)$ is described by the parametric equations

$$ct = \alpha(\theta - \sin\theta) ,$$

$$a = \alpha(1 - \cos\theta) ,$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho a^3}{c^2} .$$

These equations are identical to those on the front of the exam, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where θ is the parameter used to describe the evolution.

(d) *(6 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

(Warning: The symbol θ is used for two things: before part (b), θ is an angular coordinate; after part (b), θ is the development angle.

PROBLEM 3: EXAMINING A PECULIAR SPACETIME METRIC *(35 points)*

Consider the spacetime metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -\frac{x^2}{T_0^2}\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 ,$$

where T_0 is a constant with the units of time. In this problem we will be considering only motion in the x direction, so the $\mathrm{d}y^2$ and $\mathrm{d}z^2$ terms will not be relevant.

(a) *(5 points)* Suppose that a ruler, which is static in these coordinates, stretches from $x = a$ to $x = b$, where a and b are constants, with $b > a$. What is the physical length of the ruler?

(b) *(5 points)* Suppose that a clock, which is also stationary in these coordinates, is located at $x = a$. How much time does the clock measure between $t = 0$ and $t = \beta$, where β is a constant.

(c) *(10 points)* Suppose a particle moving under the influence of gravity alone is released from rest at $x = a$, at time $t = 0$. Since gravity is the only force acting on the particle, its motion will be described by the geodesic equation,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} ,$$

as is also given in the formula sheet for this quiz. Expand the $\mu = 0$ (where $x^0 \equiv t$) geodesic equation. Your final expression should involve only the coordinates x and t, and their derivatives with respect to τ, and the parameters c and T_0.

(d) *(10 points)* The geodesic equation derived in part (c) is not enough to find the relation between x and τ, which is what we want to be able to understand the motion of the particle. Use the definition of the metric directly to obtain an expression for $\mathrm{d}t/\mathrm{d}\tau$ in terms of x, $\mathrm{d}x/\mathrm{d}\tau$, and the parameters c and T_0. You may use the fact that, for the case of interest, there is no motion in the y or z direction.

(e) *(5 points)* By using your answers from parts (c) and (d), find x as a function of τ, or τ as a function of x, where in either case the parameters c, T_0, and/or a may appear in your equations. You will get full credit for your answer if you write an expression for τ as a definite integral, which you need not evaluate. *(Hint: the particle moves toward smaller values of x.)*

USEFUL INFORMATION:

SPEED OF LIGHT IN COMOVING COORDINATES:

$$v_{\text{coord}} = \frac{c}{a(t)} \ .$$

DOPPLER SHIFT (For motion along a line):

$$z = v/u \quad \text{(nonrelativistic, source moving)}$$

$$z = \frac{v/u}{1 - v/u} \quad \text{(nonrelativistic, observer moving)}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \quad \text{(special relativity, with } \beta = v/c\text{)}$$

COSMOLOGICAL REDSHIFT:

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}$$

SPECIAL RELATIVITY:

Time Dilation Factor:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c$$

Lorentz-Fitzgerald Contraction Factor: γ

Relativity of Simultaneity:
Trailing clock reads later by an amount $\beta \ell_0 / c$.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \quad \ddot{a} = -\frac{4\pi}{3} G\rho a \ ,$$

$$\rho(t) = \frac{a^3(t_i)}{a^3(t)} \rho(t_i)$$

$$\Omega \equiv \rho/\rho_c \ , \quad \text{where} \quad \rho_c = \frac{3H^2}{8\pi G} \ .$$

Flat ($k = 0$): $a(t) \propto t^{2/3}$

$\Omega = 1$.

Closed ($k > 0$): $ct = \alpha(\theta - \sin\theta)$, $\dfrac{a}{\sqrt{k}} = \alpha(1 - \cos\theta)$,

$\Omega = \dfrac{2}{1 + \cos\theta} > 1$,

where $\alpha \equiv \dfrac{4\pi}{3} \dfrac{G\rho}{c^2} \left(\dfrac{a}{\sqrt{k}}\right)^3$.

Open ($k < 0$): $ct = \alpha(\sinh\theta - \theta)$, $\dfrac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1)$,

$\Omega = \dfrac{2}{1 + \cosh\theta} < 1$,

where $\alpha \equiv \dfrac{4\pi}{3} \dfrac{G\rho}{c^2} \left(\dfrac{a}{\sqrt{\kappa}}\right)^3$,

$\kappa \equiv -k > 0$.

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{ds}\left\{ g_{ij} \frac{dx^j}{ds} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell}\right) \frac{dx^k}{ds} \frac{dx^\ell}{ds}$$

or:

$$\frac{d}{d\tau}\left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma}\right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$